Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

$$\text{(your final mark)} = \min\left\{\text{integer part of } \frac{3}{2}\text{(total points you get)}, 10\right\}$$

- About twenty problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **14** – **15**: 17 April 2018.

14. (1 pt.) Check that the Abelian differential ω_1 defined in the lecture (5 April 2018) is holomorphic and nowhere vanishing on the ellipitc curve $\bar{\mathcal{R}}_\varphi$ for a polynomial $\varphi(z)$ of degree 3.

15. Let the A-cycle and the B-cycle on the elliptic curve $\bar{\mathcal{R}}_\varphi$ ($\varphi(z) = (1 - z^2)(1 - k^2 z^2)$) be those defined in the lecture on 5 April 2018 and $\omega_1 := \dfrac{dz}{w}$, $\omega_2 := \sqrt{\dfrac{1 - k^2 z^2}{1 - z^2}}dz$. We assume $0 < k < 1$.

(i) (1 pt.) Prove that the A-period of the Abelian differential ω_2 is equal to $4E(k)$, where $E(k)$ is the complete elliptic integral of the second kind.

(ii) (1 pt.) Show $d\left(\dfrac{zw}{1 - k^2 z^2}\right) = \dfrac{k^2 z^4 - 2z^2 + 1}{1 - k^2 z^2}\omega_1$. (Hint: Use $w^2 = \varphi(z)$ to compute dw.)

(iii) (2 pt.) Prove that $K(k)$ (the complete elliptic integral of the first kind) and $E(k)$ satisfy the following system of differential equations:

$$\frac{dE}{dk} = \frac{E}{k} - \frac{K}{k}, \qquad \frac{dK}{dk} = \frac{E}{kk'^2} - \frac{K}{k}.$$

(Hint: To obtain the first, differentiate ω_2 with respect to k and integrate it over $[0, 1]$. To obtain the second, compare $\dfrac{\partial}{\partial k}\omega_1 - \dfrac{1}{kk'^2}\omega_2 + \dfrac{1}{k}\omega_1$ with (ii) and consider the A-period. Note that the integral of an exact form over a cycle is zero.)