Wave equation

List 6 (15.05.2018)

Deadline — 29.05.2018.

Solution of the Cauchy problem $u_{tt} = a^2 \Delta u$, $u|_{t=0} = \varphi(x)$, $u_t|_{t=0} = \psi(x)$, where Δ — is n-dimensional Laplace operator is given by the formula:

$$u(x,t) = \frac{1}{2\pi a} \int_{|\xi - x| < at} \frac{\psi(\xi) d\xi}{\sqrt{(at)^2 - |\xi - x|^2}} + \frac{\partial}{\partial t} \left[\frac{1}{2\pi a} \int_{|\xi - x| < at} \frac{\psi(\xi) d\xi}{\sqrt{(at)^2 - |\xi - x|^2}} \right] \quad \text{(Poisson Formula)},$$

for $n = 2$;

$$u(x,t) = \frac{1}{4\pi a^2 t} \int_{|\xi - x| = at} \psi(\xi) dS_\xi + \frac{\partial}{\partial t} \left[\frac{1}{4\pi a^2 t} \int_{|\xi - x| = at} \varphi(\xi) dS_\xi \right] \quad \text{(Kirchoff Formula)};$$

for $n = 3$.

1. Prove that the solution of the Cauchy problem

$$u_{tt} = a^2 \Delta u + g(t) f(x), \qquad u(x,0) = u_0(x), \quad u_t(x,0) = u_1(x)$$

is given by the formula $u(x,t) = u_0(x) + t u_1(x) + f(x) \int_0^t (t - \tau) g(\tau) d\tau$ if $f(x)$, $u_0(x)$, $u_1(x)$ are harmonic functions in $\mathbb{R}^n$, $g(t) \in C^1(t \geqslant 0)$.

2. Find a solution $u(x,t)$, $(x,t) = (x_1, x_2, x_3, t) \in \mathbb{R}^3 \times \mathbb{R}_+$ of the problem:

$$u_{tt} = \Delta u, \qquad u(x,0) = 0, \quad u_t(x,0) = \frac{1}{1 + (x_1 + x_2 + x_3)^2}.$$

3. Let $u(x,y,t)$ be a solution of the Cauchy problem:

$$u_{tt} = u_{xx} + u_{yy}, \qquad u(x,y,0) = 0, \quad u_t(x,y,0) = \psi(x,y) \in C^2(\mathbb{R}^2), \qquad t \geqslant 0$$

where $\psi(x,y) = 0$ for $(x,y) \in [0,1] \times [0,2]$, $\psi(x,y) > 0$ for all other (x,y).

a) Describe the set of values of $(x,y,t) \in \mathbb{R}^2 \times \overline{\mathbb{R}}_+$, such that $u(x,y,t) = 0$ by means enequatities.

b) Paint this set.

5. Let $u(x,t)$ be a solution of the Cauchy problem:

$$u_{tt} = \Delta u, \qquad u(x,0) = 0, \quad u_t(x,0) = \psi(x), \quad x = (x_1, x_2, x_3) \in \mathbb{R}^3, \quad t \geqslant 0$$

where $\psi(x) = 0$ for $0.9 \leqslant \|x\| \leqslant 1$, $\psi(x) > 0$ for all other x.

For what (x,t) is the function $u(x,t)$ equal to zero?