

# Elliptic Functions

## Abel-Jacobi theorem

## §7.1 Abel-Jacobi theorem

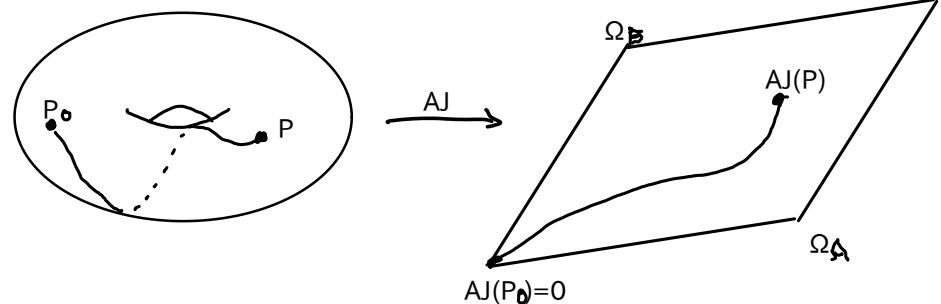
Recall: a period of  $\omega_1 = \frac{dz}{\sqrt{\varphi(z)}} = \frac{dz}{w}$  belongs to  $\Gamma := \mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B$ :

$$\Omega_A := \int_A \omega_1, \quad \Omega_B := \int_B \omega_1.$$

$\implies$  The *Abel-Jacobi map*:

$$AJ : \bar{\mathcal{R}} \ni P \mapsto \int_{P_0}^P \omega_1 \bmod \Gamma \in \mathbb{C}/\Gamma$$

is well-defined. ( $P_0$ : a fixed point in  $\bar{\mathcal{R}}$ .)



Remark: The Abel-Jacobi map is defined for any compact Riemann surface.

## Theorem (Abel-Jacobi theorem)

(i) The Abel-Jacobi map  $AJ$  is bijective.

(ii) It is an isomorphism of complex manifolds between  $\bar{\mathcal{R}}$  and  $\mathbb{C}/\Gamma$ .

### Proof of (ii) $\Leftarrow$ (i):

- $AJ$  is holomorphic ( $\Leftarrow$  definition).

- Complex analysis:

The inverse map of a holomorphic bijection is holomorphic.

□

## §7.2 Surjectivity of $AJ$ (Jacobi's theorem)

Recall:

- The image of a compact set by a continuous map is compact.
- A compact subset of a Hausdorff space is closed.

$AJ$ : holomorphic, in particular, continuous.

$\bar{\mathcal{R}}$ : compact.

$\implies AJ(\bar{\mathcal{R}})$  is *closed* in  $\mathbb{C}/\Gamma$ .

On the other hand,

- A holomorphic map is open, i.e., the image of an open set is open.

$\implies AJ(\bar{\mathcal{R}})$  is *open* in  $\mathbb{C}/\Gamma$ .

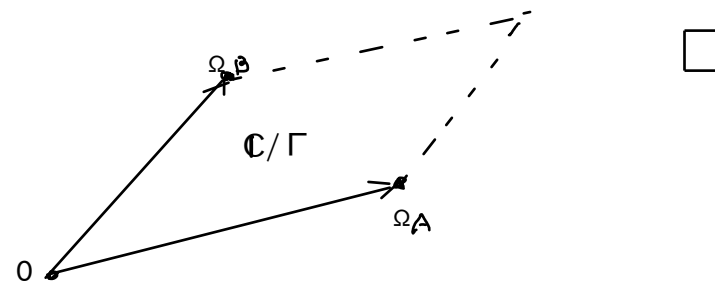
$AJ(\bar{\mathcal{R}})$  is closed & open in  $\mathbb{C}/\Gamma$ .

$\implies AJ(\bar{\mathcal{R}})$  is a connected component of  $\mathbb{C}/\Gamma$ .

*But  $\mathbb{C}/\Gamma$  is connected!*

Hence,

$$AJ(\bar{\mathcal{R}}) = \mathbb{C}/\Gamma.$$



Corollary:

$\Omega_A$  and  $\Omega_B$  are linearly independent over  $\mathbb{R}$ . In particular,  $\Omega_A, \Omega_B \neq 0$ .

Proof:  $\bar{\mathcal{R}}$ : compact  $\implies \mathbb{C}/\Gamma = AJ(\bar{\mathcal{R}})$ : compact.

$\Leftrightarrow$  If  $\Omega_A$  &  $\Omega_B$ : linearly dependent/ $\mathbb{R}$ ,  $\Gamma = \mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B \subset \mathbb{R}\Omega_A$  or  $\mathbb{R}\Omega_B$ .

$\implies \mathbb{C}/\Gamma$  is not compact. □

## §7.3 Injectivity of $AJ$ (Abel's theorem)

Assumption:  $AJ(P_1) = AJ(P_2)$ , but  $P_1 \neq P_2$ .

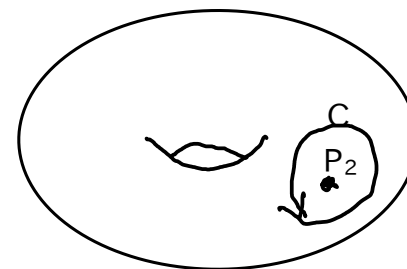
Goal: Construct a meromorphic function  $f(z)$  on  $\bar{\mathcal{R}}$  such that

- $f$  has a unique pole at  $P_2$ , which is simple.
- $f(z) = 0 \Leftrightarrow z = P_1$ . (This property will not be used.)

*But such  $f$  cannot exist!*

Because, as  $\omega_1$  is a holomorphic nowhere vanishing differential,

- $f(z)\omega_1$  has a simple pole at  $P_2$ .  $\implies \int_C f(z)\omega_1 \neq 0$ .
- $f(z)\omega_1$  is holomorphic elsewhere.  $\implies \int_C f(z)\omega_1 = 0$ .



( $C$ : a small circle around  $P_2$ ; Figure)

Contradiction  $\implies P_1 = P_2$ .

□

## Construction of $f(z)$ :

We define  $f(z)$  by

$$f(z) := \exp \left( \int_{Q_0}^z \omega_3(P_1, P_0) - \int_{Q_0}^z \omega_3(P_2, P_0) - \frac{2\pi i N}{\Omega_A} \int_{Q_0}^z \omega_1 \right).$$

Notations:

- $Q_0$ : a fixed point  $\neq P_0, P_1, P_2$ .
- $\omega_3(P, Q)$ : an normalised Abelian differential of the third kind with simple poles at  $P$  and  $Q$  normalised by
  - $\text{Res}_P \omega_3(P, Q) = 1, \text{Res}_Q \omega_3(P, Q) = -1$ .
  - $\int_A \omega_3(P, Q) = 0$ .

Existence of such  $\omega_3$  shall be proved later.

- $N$ : an integer determined later.

Need to show:

- $f(z)$  has a simple pole at  $P_2$  (and a simple zero at  $P_1$ ).
- $f(z)$  is a single-valued meromorphic function on  $\bar{\mathcal{R}}$ .

$f(z)$  has a simple pole at  $P_2$ . (The proof of  $f(P_1) = 0$  is similar.)

$$\omega_3(P_2, P_0) = \left( \frac{1}{z - P_2} + (\text{holomorphic function}) \right) dz \text{ at } P_2.$$

$$\implies \int_{Q_0}^z \omega_3(P_2, P_0) = \log(z - P_2) + (\text{holomorphic function}).$$

When  $z \rightarrow P_2$ , only this term in the definition of  $f(z)$  diverges.

$$\begin{aligned} \implies f(z) &\sim \exp(-\log(z - P_2) + (\text{holomorphic function})) \\ &= \frac{1}{z - P_2} \times (\text{non-zero holomorphic function}). \end{aligned}$$

□



## Single-valuedness of $f(z)$ .

Possible multi-valuedness  $\leftarrow$  ambiguity of integration contours.

Three types of contours should be checked.

- (i) contours around singularities of  $\omega_3(P_1, P_0)$  and  $\omega_3(P_2, P_0)$ .
- (ii) contours around the  $A$ -cycle.
- (iii) contours around the  $B$ -cycle.

### • Case (i).

When  $z$  goes around  $P_1$ : (The proofs for  $P_2$  and  $P_0$  are the same.)

$$\int_{Q_0}^{z \circlearrowleft P_1} \omega_3(P_1, P_0) = \int_{Q_0}^z \omega_3(P_1, P_0) + 2\pi i.$$

( $z \circlearrowleft P_1$  means that the contour additionally goes around  $P_1$ .)

$\implies f(z) \mapsto f(z) \times e^{2\pi i} = f(z)$ . OK!

- Case (ii).

Recall  $\int_A \omega_3(P, Q) = 0 \implies \int_{Q_0}^{z \circlearrowleft A} \omega_3(P_i, P_0) = \int_{Q_0}^z \omega_3(P_i, P_0).$

( $z \circlearrowleft A$ : the contour additionally goes around the  $A$ -cycle.)

On the other hand,  $\int_{Q_0}^{z \circlearrowleft A} \omega_1 = \int_{Q_0}^z \omega_1 + \Omega_A.$

$\implies f(z) \mapsto f(z) \times \exp\left(-\frac{2\pi i N}{\Omega_A} \Omega_A\right) = f(z). \text{ OK!}$

- Case (iii).

Lemma:  $\exists$  contour  $C : Q \rightarrow P$  such that

$$\int_B \omega_3(P, Q) = \frac{2\pi i}{\Omega_A} \int_C \omega_1.$$

□

We prove this lemma later.

By the lemma:

$$\begin{aligned}
& \left( \int_{Q_0}^{z \circlearrowleft B} \omega_3(P_1, P_0) - \int_{Q_0}^{z \circlearrowleft B} \omega_3(P_2, P_0) \right) \\
& - \left( \int_{Q_0}^z \omega_3(P_1, P_0) - \int_{Q_0}^z \omega_3(P_2, P_0) \right) \\
& = \int_B \omega_3(P_1, P_0) - \int_B \omega_3(P_2, P_0) = \frac{2\pi i}{\Omega_A} \left( \int_{P_0}^{P_1} \omega_1 - \int_{P_0}^{P_2} \omega_1 \right).
\end{aligned}$$

Assumption  $AJ(P_1) = AJ(P_2)$  means

$$\int_{P_0}^{P_1} \omega_1 - \int_{P_0}^{P_2} \omega_1 = M\Omega_A + N\Omega_B$$

for some  $M, N \in \mathbb{Z}$ . This is the “ $N$ ” in the definition of  $f(z)$ .

$$\begin{aligned}
\implies f(z) &\mapsto f(z) \exp \left( \frac{2\pi i}{\Omega_A} (M\Omega_A + N\Omega_B) - \frac{2\pi i N}{\Omega_A} \int_B \omega_1 \right) \\
&= f(z) \exp \left( 2\pi i M + \frac{2\pi i N \Omega_B}{\Omega_A} - \frac{2\pi i N}{\Omega_A} \Omega_B \right) \\
&= f(z).
\end{aligned}$$

Single-valuedness proved!! = End of the proof of the Abel-Jacobi theorem.

□

It remains to show:

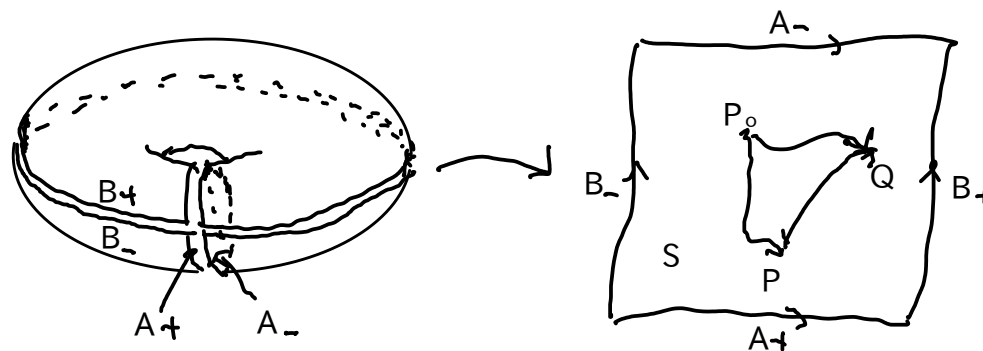
- Lemma.
- Existence of  $\omega_3(P, Q)$ .

- Proof of the lemma.

$$F(z) := \int_{P_0}^z \omega_1: \text{ multivalued holomorphic function on } \bar{\mathcal{R}}$$

(incomplete elliptic integral of the first kind).

Cut  $\bar{\mathcal{R}}$  along  $A$ - and  $B$ -cycles to a rectangle  $S$ : (Figure)



By the residue theorem,

$$\begin{aligned} \frac{1}{2\pi i} \int_{\partial S} F(z) \omega_3(P, Q) &= \text{Res}_P F(z) \omega_3(P, Q) + \text{Res}_Q F(z) \omega_3(P, Q) \\ &= F(P) - F(Q) = \int_{P_0}^P \omega_1 - \int_{P_0}^Q \omega_1 = \int_Q^P \omega_1. \end{aligned}$$

(All the contours are in  $S$ .)

On the other hand,

$$\int_{\partial S} F(z) \omega_3(P, Q) = \left( \int_{A_-} - \int_{A_+} + \int_{B_+} - \int_{B_-} \right) F(z) \omega_3(P, Q).$$

From the multi-valuedness of  $F(z)$ ,

$$\begin{aligned} & \int_{A_-} F(z) \omega_3(P, Q) - \int_{A_+} F(z) \omega_3(P, Q) \\ &= \int_A (F(z) - F(z \circlearrowleft_B)) \omega_3(P, Q) \\ &= \int_A \left( - \int_B \omega_1 \right) \omega_3(P, Q) = - \left( \int_B \omega_1 \right) \left( \int_A \omega_3(P, Q) \right) \\ &= 0. \quad \left( \text{Recall the normalisation : } \int_A \omega_3(P, Q) = 0. \right) \end{aligned}$$

Similarly,

$$\begin{aligned} & \int_{B_+} F(z) \omega_3(P, Q) - \int_{B_-} F(z) \omega_3(P, Q) \\ &= \left( \int_A \omega_1 \right) \left( \int_B \omega_3(P, Q) \right) = \Omega_A \int_B \omega_3(P, Q). \end{aligned}$$

As a result,

$$2\pi i \int_Q^P \omega_1 = \Omega_A \int_B \omega_3(P, Q).$$

□

- Proof of the existence of  $\omega_3(P, Q)$ .

We have only to show existence of  $\tilde{\omega}_3(P, Q)$  with simple poles at  $P$  and  $Q$ ,

$$\operatorname{Res}_P \tilde{\omega}_3(P, Q) = 1, \quad \operatorname{Res}_Q \tilde{\omega}_3(P, Q) = -1.$$

Because:

- $\tilde{\omega}_3(P, Q) + \lambda\omega_1$  has the same property for any  $\lambda \in \mathbb{C}$ .

- $\int_A \omega_1 = \Omega_A \neq 0$ .

$$\implies \text{If } \lambda = -\frac{1}{\Omega_A} \int_A \tilde{\omega}_3(P, Q),$$

$$\omega_3(P, Q) := \tilde{\omega}_3(P, Q) + \lambda\omega_1$$

satisfies all the conditions, including  $\int_A \omega_3(P, Q) = 0$ .



• Construction of  $\tilde{\omega}_3(P, Q)$ .

Recall:  $\bar{\mathcal{R}}$  = compactification of  $\mathcal{R} = \{(z, w) \mid w^2 = \varphi(z)\}$ ,

$$\varphi(z) = a(z - \alpha_0)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3).$$

Case I.  $P, Q \neq \infty_{\pm}$ .

Denote  $P = (z_1, w_1 = \sqrt{\varphi(z_1)})$ ,  $Q = (z_2, w_2 = \sqrt{\varphi(z_2)})$ .

(Branches of  $\sqrt{\phantom{x}}$  are defined appropriately.)

$$\tilde{\omega}_3(P, Q) := \frac{1}{2} \left( \frac{w + w_1}{z - z_1} - \frac{w + w_2}{z - z_2} \right) \frac{dz}{w}.$$

Exercise: Check that this  $\tilde{\omega}_3(P, Q)$  satisfies the required properties:

holomorphic on  $\bar{\mathcal{R}} \setminus \{P, Q\}$ , simple poles at  $P, Q$ ,  $\text{Res}_P = 1$ ,  $\text{Res}_Q = -1$ .

(Use an appropriate coordinate, especially at  $\infty_{\pm}$  and  $(z, w) = (\alpha_i, 0)$ !)

Case II.  $P = \infty_+, Q \neq \infty_{\pm}$ .

Case III.  $P = \infty_+, Q = \infty_-$ .

Exercise: Find  $\tilde{\omega}_3(P, Q)$  for the cases II and III.

(Hint: When  $z_1 \rightarrow \infty$ ,  $w_1 \sim \pm\sqrt{a} z_1^2$ .  $\implies \tilde{\omega}_3(P, Q)$  of Case I diverges.

Find an appropriate  $\lambda = \lambda(z_1)$  and take  $\lim_{z_1 \rightarrow \infty} (\tilde{\omega}_3(P, Q) - \lambda\omega_1)$ .

Exercise\*: Find  $\tilde{\omega}_3(P, Q)$  when  $\deg \varphi = 3$ .

Remark:

There is such  $\omega_3(P, Q)$  on any compact Riemann surface.

The proof requires much analysis!