Elliptic Functions

Complex Jacobian elliptic functions

§11.1 Definition of Jacobian elliptic functions in terms of θ.

Recall: Jacobi's $\mathrm{sn}(u,k)$ was defined as the inverse function of

$$u = \int_0^{\mathrm{sn}(u,k)} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Exercise of the last section: $\forall$ elliptic functions $=$ rational function of θ.

Question: *How can Jacobian functions be defined by θ?*

Answer: $v := \dfrac{u}{\pi\theta_{00}^2}$, $k := \dfrac{\theta_{10}^2}{\theta_{00}^2}$,

$$\mathrm{sn}(u,k) := -\frac{\theta_{00}}{\theta_{10}}\frac{\theta_{11}(v)}{\theta_{01}(v)},\ \mathrm{cn}(u,k) := \frac{\theta_{01}}{\theta_{10}}\frac{\theta_{10}(v)}{\theta_{01}(v)},\ \mathrm{dn}(u,k) := \frac{\theta_{01}}{\theta_{00}}\frac{\theta_{00}(v)}{\theta_{01}(v)}.$$

- $\mathrm{sn}(u)$ defined above has *two periods*: $2\pi\,\theta_{00}^2$, $\pi\,\theta_{00}^2\,\tau$.

  Exercise: Check this. (Hint: quasi-periodicity of θ-functions.)

- $\mathrm{sn}(u)$ is meromorphic. $\Longleftarrow$ θ's are entire.

$\Longrightarrow$ $\mathrm{sn}(u)$ is an elliptic funtion.

Similarly,

- $\mathrm{cn}(u)$ is an elliptic function with periods $2\pi\,\theta_{00}^2$, $\pi\,\theta_{00}^2(1+\tau)$.

- $\mathrm{dn}(u)$ is an elliptic function with periods $\pi\,\theta_{00}^2$, $2\pi\,\theta_{00}^2\tau$.

Let us check that they coincide what we defined before (on $\mathbb{R}$).

- $\mathrm{sn}(0) = 0$, $\mathrm{cn}(0) = \mathrm{dn}(0) = 1 \Leftarrow \theta_{11}(0) = 0$ & definitions.
- sn: odd, cn, dn: even $\Leftarrow \theta_{11}(u)$: odd, $\theta_{ab}(u)$: even $((a,b) \neq (1,1))$.
- $\mathrm{sn}^2 u + \mathrm{cn}^2 u = 1$, $k^2 \mathrm{sn}^2 u + \mathrm{dn}^2 u = 1$.

Proof:

$$\mathrm{sn}^2 u + \mathrm{cn}^2 u = \frac{\theta_{00}^2\, \theta_{11}(v)^2 + \theta_{01}^2\, \theta_{10}(v)^2}{\theta_{10}^2\, \theta_{01}(v)^2}.$$

Recall the addition formula (A1):

$$\begin{aligned}\theta_{00}(x+u)\,\theta_{00}(x-u)\,\theta_{00}^2 &= \theta_{01}(x)^2\,\theta_{01}(u)^2 + \theta_{10}(x)^2\,\theta_{10}(u)^2 \\ &= \theta_{00}(x)^2\,\theta_{00}(u)^2 + \theta_{11}(x)^2\,\theta_{11}(u)^2.\end{aligned}$$

$x = v$, $u = \frac{1+\tau}{2} \Longrightarrow \theta_{11}(v)^2\,\theta_{00}^2 = \theta_{01}(v)^2\,\theta_{10}^2 - \theta_{10}(v)^2\,\theta_{01}^2$.

$$\Longrightarrow \mathrm{sn}^2 u + \mathrm{cn}^2 u = 1.$$

$x = v$, $u = \frac{1}{2}$ in (A1): $\theta_{01}(v)^2\,\theta_{00}^2 = \theta_{00}(v)^2\,\theta_{01}^2 + \theta_{11}(v)^2\,\theta_{10}^2$.

$$\Longrightarrow \; \frac{\theta_{10}^4}{\theta_{00}^4}\frac{\theta_{00}^2\,\theta_{11}(v)^2}{\theta_{10}^2\,\theta_{01}(v)^2} + \frac{\theta_{01}^2\,\theta_{11}(v)^2}{\theta_{00}^2\,\theta_{01}(v)^2} = 1, \text{ i.e., } k^2\,\mathrm{sn}^2(u) + \mathrm{dn}^2(u) = 1.$$

□

- $\dfrac{d}{du}\,\mathrm{sn}(u) = \mathrm{cn}(u)\;\mathrm{dn}(u).$

<u>Proof</u>:

Chain rule & $v = \dfrac{u}{\pi\,\theta_{00}^2}$

$$\Longrightarrow \frac{d}{du}\,\mathrm{sn}(u) = \frac{dv}{du}\,\frac{d}{dv}\left(-\frac{\theta_{00}}{\theta_{10}}\,\frac{\theta_{11}(v)}{\theta_{01}(v)}\right)$$

$$= -\frac{1}{\pi\,\theta_{00}\,\theta_{10}}\,\frac{\theta_{11}'(v)\theta_{01}(v) - \theta_{11}(v)\theta_{01}'(v)}{\theta_{01}(v)^2}.$$

Recall the addition formula (A3):

$$\theta_{11}(x+u)\theta_{01}(x-u)\theta_{10}\,\theta_{00} = \theta_{00}(x)\theta_{10}(x)\theta_{01}(u)\theta_{11}(u)+\theta_{01}(x)\theta_{11}(x)\theta_{00}(u)\theta_{10}(u).$$

Expand around $u = 0$ and take the coefficients of u^1:

$$\big(\theta_{11}'(x)\,\theta_{01}(x) - \theta_{11}(x)\,\theta_{01}'(x)\big)\theta_{00}\,\theta_{10} = \theta_{00}(x)\,\theta_{10}(x)\,\theta_{01}\,\theta_{11}'.$$

Substitute this into the previous equation ($x \mapsto v$):

$$\begin{aligned}\frac{d}{du}\,\mathrm{sn}(u) &= -\frac{1}{\pi\,\theta_{00}\,\theta_{10}}\,\frac{\theta_{00}(v)\,\theta_{10}(v)\,\theta_{01}\,\theta_{11}'}{\theta_{00}\,\theta_{10}\,\theta_{01}(v)^2} \\ &= \frac{\theta_{01}^2}{\theta_{00}\,\theta_{10}}\,\frac{\theta_{00}(v)\,\theta_{10}(v)}{\theta_{01}(v)^2} \qquad \text{(Jacobi's derivative formula)} \\ &= \mathrm{cn}(u)\;\mathrm{dn}(u).\end{aligned}$$

$\square$

As we have seen in the real case, the above formulae lead to

$$\frac{d}{du}\,\mathrm{sn}(u) = \sqrt{(1-\mathrm{sn}^2(u))(1-k^2\,\mathrm{sn}^2(u))}.$$

$$\Longrightarrow u = \int_0^{\mathrm{sn}(u)} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Consistent with the previous definition.

§11.2 Properties of $\mathrm{sn}(u, k)$.

What we know about Jacobi's funciotns $/\mathbb{R}$:

- periodicity (e.g., period of $\mathrm{sn} = 4K(k)$).
- limits (e.g., $\mathrm{sn} \to \sin$ when $k \to 0$).
- addition formulae.

can be checked on the basis of the definition by θ.

Because of the lack of time, we prove only the addition formula for sn.

Proofs of other properties are only sketched.

• Addition formula of sn.

Recall addition formulae (A3) & (A2) of θ's:

$$\theta_{11}(x+y)\theta_{01}(x-y)\theta_{10}\theta_{00} = \theta_{00}(x)\theta_{10}(x)\theta_{01}(y)\theta_{11}(y) + \theta_{01}(x)\theta_{11}(x)\theta_{00}(y)\theta_{10}(y),$$

$$\theta_{01}(x+y)\theta_{01}(x-y)\theta_{01}^2 = \theta_{01}(x)^2\theta_{01}(y)^2 - \theta_{11}(x)^2\theta_{11}(y)^2.$$

Set $u = \pi\,\theta_{00}^2\,x$, $v = \pi\,\theta_{00}^2\,y$:

$$-(\text{ratio of LHS's}) \times \frac{\theta_{01}^2}{\theta_{10}^2} = -\frac{\theta_{00}}{\theta_{10}}\frac{\theta_{11}(x+y)}{\theta_{01}(x+y)} = \mathrm{sn}(u+v).$$

$$-(\text{ratio of RHS's}) \times \frac{\theta_{01}^2}{\theta_{10}^2} = \frac{\mathrm{sn}(u)\,cn(v)\ \mathrm{dn}(v) + \mathrm{sn}(v)\ \mathrm{cn}(u)\ \mathrm{dn}(u)}{1 - k^2\ \mathrm{sn}(u)^2\,\mathrm{sn}(v)^2},$$

as was proved before. □

- <u>Limits $k \to 0$, $k \to 1$</u>.

<u>$k = k(\tau) \to 0$</u> $\Longleftrightarrow \tau \to i\infty$.

In this limit: $\theta_{11}(u,\tau) \sim \sin u$, $\theta_{01}(u,\tau) \sim 1$, etc. $\Longrightarrow \mathrm{sn}(u,k) \to \sin(u)$.

<u>$k \to 1$</u> $\Longleftrightarrow k' \to 0$ $(k' := \sqrt{1-k^2})$.

Modular properties: relations of $\theta_{ab}(u,\tau)$ and $\theta_{a'b'}\left(\frac{u}{\tau}, -\frac{1}{\tau}\right)$.

$$\Longrightarrow \begin{cases} k' = k'(\tau) = k\left(-\frac{1}{\tau}\right), \\ \mathrm{sn}(iu,k) = \dfrac{i\ \mathrm{sn}(u,k')}{\mathrm{cn}(u,k')} \text{ etc.} \end{cases}$$

$$\Longrightarrow \lim_{k \to 1} \mathrm{sn}(u,k) = \tanh(u) \text{ etc.}$$

$\square$

• Periodicity.

Recall: for $k = \dfrac{\theta_{10}^2}{\theta_{00}^2}$,

$$x = \mathrm{sn}(u) = -\frac{\theta_{00}}{\theta_{10}} \frac{\theta_{11}(\frac{u}{\pi\theta_{00}^2}, \tau)}{\theta_{01}(\frac{u}{\pi\theta_{00}^2}, \tau)} \quad \overset{\text{inverse}}{\longleftrightarrow} \quad u = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

A- & B-periods of the elliptic integral (RHS):

$$4K(k) = 4\int_0^1 \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}},$$
$$2iK'(k) = 2\int_1^{1/k} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

Periods of $\mathrm{sn}(u)$ (defined by θ-functions; LHS): $2\pi\,\theta_{00}^2$, $\pi\,\theta_{00}^2\,\tau$.

$$\Longrightarrow \quad 4\mathbb{Z}\,K(k) + 2\mathbb{Z}\,iK'(k) = 2\mathbb{Z}\,\pi\,\theta_{00}^2 + \mathbb{Z}\,\pi\,\theta_{00}^2\tau.$$

Or, equivalently, $\exists\; m_1, m_2, n_1, n_2 \in \mathbb{Z}$,

$$4K(k) = 2m_1\pi\,\theta_{00}^2 + n_1\pi\,\theta_{00}^2\tau, \qquad 2iK'(k) = 2m_2\pi\,\theta_{00}^2 + n_2\pi\,\theta_{00}^2\tau.$$

Theorem:

$$K(k) = \frac{\pi}{2}\,\theta_{00}^2, \qquad K'(k) = \frac{\pi}{2i}\,\theta_{00}^2\tau.$$

Proof is not difficult but lengthy.

We omit it here because of the lack of time. □