Elliptic Functions

Complex Jacobian elliptic functions
§11.1 Definition of Jacobian elliptic functions in terms of \( \theta \).

Recall: Jacobi’s \( \text{sn}(u, k) \) was defined as the inverse function of

\[
   u = \int_0^{\text{sn}(u,k)} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.
\]

Exercise of the last section: \( \forall \) elliptic functions = rational function of \( \theta \).

**Question:** How can Jacobian functions be defined by \( \theta \)?

**Answer:** \( v := \frac{u}{\pi \theta_{00}^2}, \; k := \frac{\theta_{10}^2}{\theta_{00}^2} \),

\[
   \text{sn}(u, k) := -\frac{\theta_{00} \theta_{11}(v)}{\theta_{10} \theta_{01}(v)}, \quad \text{cn}(u, k) := \frac{\theta_{01} \theta_{10}(v)}{\theta_{10} \theta_{01}(v)}, \quad \text{dn}(u, k) := \frac{\theta_{01} \theta_{00}(v)}{\theta_{00} \theta_{01}(v)}.
\]
• \( \text{sn}(u) \) defined above has two periods: \( 2\pi \theta^2_{00}, \pi \theta^2_{00} \tau \).
  
  Exercise: Check this. (Hint: quasi-periodicity of \( \theta \)-functions.)

• \( \text{sn}(u) \) is meromorphic. \( \iff \) \( \theta \)'s are entire.

\( \implies \) \( \text{sn}(u) \) is an elliptic function.

Similarly,

• \( \text{cn}(u) \) is an elliptic function with periods \( 2\pi \theta^2_{00}, \pi \theta^2_{00} (1 + \tau) \).

• \( \text{dn}(u) \) is an elliptic function with periods \( \pi \theta^2_{00}, 2\pi \theta^2_{00} \tau \).
Let us check that they coincide what we defined before (on $\mathbb{R}$).

- $\text{sn}(0) = 0$, $\text{cn}(0) = \text{dn}(0) = 1 \iff \theta_{11}(0) = 0$ & definitions.
- $\text{sn}$: odd, $\text{cn}$, $\text{dn}$: even $\iff \theta_{11}(u)$: odd, $\theta_{ab}(u)$: even $((a, b) \neq (1, 1))$.
- $\text{sn}^2 u + \text{cn}^2 u = 1$, $k^2 \text{sn}^2 u + \text{dn}^2 u = 1$.

**Proof:**

$$\text{sn}^2 u + \text{cn}^2 u = \frac{\theta_{00}^2 \theta_{11}(v)^2 + \theta_{01}^2 \theta_{10}(v)^2}{\theta_{10}^2 \theta_{01}(v)^2}.$$ 

Recall the addition formula (A1):

$$\theta_{00}(x + u) \theta_{00}(x - u) \theta_{00}^2 = \theta_{01}(x)^2 \theta_{01}(u)^2 + \theta_{10}(x)^2 \theta_{10}(u)^2$$

$$= \theta_{00}(x)^2 \theta_{00}(u)^2 + \theta_{11}(x)^2 \theta_{11}(u)^2.$$ 

$x = v$, $u = \frac{1+\tau}{2} \implies \theta_{11}(v)^2 \theta_{00}^2 = \theta_{01}(v)^2 \theta_{10}^2 - \theta_{10}(v)^2 \theta_{01}^2$.

$\implies \text{sn}^2 u + \text{cn}^2 u = 1.$
\[ x = v, \ u = \frac{1}{2} \text{ in (A1)}: \ \theta_{01}(v)^2 \theta_{00}^2 = \theta_{00}(v)^2 \theta_{01}^2 + \theta_{11}(v)^2 \theta_{10}^2. \]

\[ \implies \frac{\theta_{10}^4 \theta_{00}^2 \theta_{11}(v)^2}{\theta_{00}^4 \theta_{10}^2 \theta_{01}(v)^2} + \frac{\theta_{01}^2 \theta_{11}(v)^2}{\theta_{00}^2 \theta_{01}(v)^2} = 1, \ i.e., \ k^2 \text{sn}^2(u) + \text{dn}^2(u) = 1. \]

\[ \bullet \ \frac{d}{du} \text{sn}(u) = \text{cn}(u) \ \text{dn}(u). \]

**Proof:**

Chain rule & \( v = \frac{u}{\pi \theta_{00}^2} \)

\[ \implies \frac{d}{du} \text{sn}(u) = \frac{dv}{du} \frac{d}{dv} \left( -\frac{\theta_{00}}{\theta_{10}} \frac{\theta_{11}(v)}{\theta_{01}(v)} \right) \]

\[ = -\frac{1}{\pi \theta_{00} \theta_{10}} \frac{\theta_{11}(v)\theta_{01}(v) - \theta_{11}(v)\theta_{01}'(v)}{\theta_{01}(v)^2}. \]
Recall the addition formula (A3):

\[
\theta_{11}(x+u)\theta_{01}(x-u)\theta_{10}\theta_{00} = \theta_{00}(x)\theta_{10}(x)\theta_{01}(u)\theta_{11}(u) + \theta_{01}(x)\theta_{11}(x)\theta_{00}(u)\theta_{10}(u).
\]

Expand around \( u = 0 \) and take the coefficients of \( u^1 \):

\[
(\theta'_{11}(x) \theta_{01}(x) - \theta_{11}(x) \theta'_{01}(x)) \theta_{00} \theta_{10} = \theta_{00}(x) \theta_{10}(x) \theta_{01} \theta'_{11}.
\]

Substitute this into the previous equation \((x \mapsto v)\):

\[
\frac{d}{du} \text{sn}(u) = -\frac{1}{\pi \theta_{00} \theta_{10}} \frac{\theta_{00}(v) \theta_{10}(v) \theta_{01} \theta'_{11}}{\theta_{00} \theta_{10} \theta_{01}(v)^2}
\]

\[
= \frac{\theta_{01}^2}{\theta_{00} \theta_{10}} \frac{\theta_{00}(v) \theta_{10}(v)}{\theta_{01}(v)^2} \quad \text{(Jacobi's derivative formula)}
\]

\[
= \text{cn}(u) \ \text{dn}(u).
\]
As we have seen in the real case, the above formulae lead to

\[ \frac{d}{du} \text{sn}(u) = \sqrt{(1 - \text{sn}^2(u))(1 - k^2 \text{sn}^2(u))}. \]

\[ \implies u = \int_{0}^{|\text{sn}(u)|} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}. \]

Consistent with the previous definition.
§11.2 Properties of $\text{sn}(u, k)$.

What we know about Jacobi’s functions $/\mathbb{R}$:

- periodicity (e.g., period of $\text{sn} = 4K(k)$).
- limits (e.g., $\text{sn} \to \sin$ when $k \to 0$).
- addition formulae.

can be checked on the basis of the definition by $\theta$.

Because of the lack of time, we prove only the addition formula for $\text{sn}$.

Proofs of other properties are only sketched.
Addition formula of $\text{sn}$.

Recall addition formulae (A3) & (A2) of $\theta$’s:

\[
\theta_{11}(x + y)\theta_{01}(x - y)\theta_{10}\theta_{00} = \theta_{00}(x)\theta_{10}(x)\theta_{01}(y)\theta_{11}(y) \\
+ \theta_{01}(x)\theta_{11}(x)\theta_{00}(y)\theta_{10}(y), \\
\theta_{01}(x + y)\theta_{01}(x - y)\theta_{01}^2 = \theta_{01}(x)^2\theta_{01}(y)^2 - \theta_{11}(x)^2\theta_{11}(y)^2.
\]

Set $u = \pi \theta_{00}^2 x$, $v = \pi \theta_{00}^2 y$:

\[-(\text{ratio of LHS’s}) \times \frac{\theta_{01}^2}{\theta_{10}^2} = - \frac{\theta_{00} \theta_{11}(x + y)}{\theta_{10} \theta_{01}(x + y)} = \text{sn}(u + v).
\]

\[-(\text{ratio of RHS’s}) \times \frac{\theta_{01}^2}{\theta_{10}^2} = \frac{\text{sn}(u)\, \text{cn}(v)\, \text{dn}(v) + \text{sn}(v)\, \text{cn}(u)\, \text{dn}(u)}{1 - k^2\, \text{sn}(u)^2\, \text{sn}(v)^2},
\]

as was proved before. □
• **Limits** \( k \to 0, \ k \to 1. \)

\[
k = k(\tau) \to 0 \iff \tau \to i\infty.
\]

In this limit: \( \theta_{11}(u, \tau) \sim \sin u, \ \theta_{01}(u, \tau) \sim 1, \) etc. \( \implies \) \( \text{sn}(u, k) \to \sin(u). \)

\[
k \to 1 \iff k' \to 0 \ (k' := \sqrt{1 - k^2}).
\]

Modular properties: relations of \( \theta_{ab}(u, \tau) \) and \( \theta_{a'b'} \left( \frac{u}{\tau}, -\frac{1}{\tau} \right). \)

\[
\implies \begin{cases} 
k' = k'(\tau) = k \left( -\frac{1}{\tau} \right), \\
\text{sn}(iu, k) = \frac{i \ \text{sn}(u, k')}{\text{cn}(u, k')} \ 	ext{etc.}
\end{cases}
\]

\[
\implies \lim_{k \to 1} \ 	ext{sn}(u, k) = \tanh(u) \ 	ext{etc.}
\]
• Periodicity.

Recall: for \( k = \frac{\theta_{10}^2}{\theta_{00}^2} \),

\[
x = \text{sn}(u) = -\frac{\theta_{00}}{\theta_{10}} \frac{\theta_{11}(\frac{u}{\pi \theta_{00}^2}, \tau)}{\theta_{01}(\frac{u}{\pi \theta_{00}^2}, \tau)} \quad \text{inverse} \quad u = \int_0^x \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.
\]

\( A- \) & \( B \)-periods of the elliptic integral (RHS):

\[
4K(k) = 4 \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}},
\]

\[
2iK'(k) = 2 \int_1^{1/k} \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.
\]

Periods of \( \text{sn}(u) \) (defined by \( \theta \)-functions; LHS): \( 2\pi \theta_{00}^2, \pi \theta_{00}^2 \tau \).
\[
4\mathbb{Z} K(k) + 2\mathbb{Z} iK'(k) = 2\mathbb{Z} \pi \theta_{00}^2 + \mathbb{Z} \pi \theta_{00}^2 \tau.
\]

Or, equivalently, \(\exists m_1, m_2, n_1, n_2 \in \mathbb{Z},\)

\[
4K(k) = 2m_1 \pi \theta_{00}^2 + n_1 \pi \theta_{00}^2 \tau,
\]

\[
2iK'(k) = 2m_2 \pi \theta_{00}^2 + n_2 \pi \theta_{00}^2 \tau.
\]

**Theorem:**

\[
K(k) = \frac{\pi}{2} \theta_{00}^2, \quad K'(k) = \frac{\pi}{2i} \theta_{00}^2 \tau.
\]

**Proof** is not difficult but lengthy.

We omit it here because of the lack of time. \(\square\)