Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

$$\text{(your final mark)} = \min\left\{\text{integer part of } \frac{3}{2}\text{(total points you get)}, 10\right\}$$

- This rule is subject to change and the latest rule applies.
- The deadline of **18** – **22**: 31 May 2018.

The periods of elliptic functions in this sheet is Ω_1 and Ω_2. We denote the period lattice $\mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2$ by Γ. The notations are the same as those in the lecture on 17 May 2018.

18. (1 pt.) (i) Show that $\wp'\left(\frac{\Omega_i}{2}\right) = 0$ ($i = 1, 2, 3$; $\Omega_3 := \Omega_1 + \Omega_2$).

(ii) Show that $e_i := \wp\left(\frac{\Omega_i}{2}\right)$ satisfy the following relations.

$$e_1 + e_2 + e_3 = 0, \qquad e_1 e_2 + e_2 e_3 + e_3 e_1 = -\frac{g_2}{4}, \qquad e_1 e_2 e_3 = \frac{g_3}{4}.$$

19. (1 pt.) Let $\bar{\mathcal{R}}$ be the elliptic curve, which is the compactification of $\{(z, w) \mid w^2 = 4z^3 - g_2\, z - g_3\}$. Prove that the map defined by

$$W : \mathbb{C}/\Gamma \ni u \mapsto (\wp(u), \wp'(u)) \in \bar{\mathcal{R}}$$

is an isomorphism of Riemann surfaces as follows. (In fact, this is the inverse of the Abel-Jacobi map AJ.)

(i) Show that W is *holomorphic* (even at $u = 0$) as a map to $\bar{\mathcal{R}}$. (Hint: In order to show that W is a holomorphic map in a neighbourhood of $u_0 \in \mathbb{C}/\Gamma$, one should use u as a local coordinate of $\mathbb{C}/\Gamma$ and choose an appropriate local coordinate of $\bar{\mathcal{R}}$ in a neighbourhood of $W(u_0)$.)

(ii) Show the bijectivity. (Hint: $\wp(u)$ is even and of order 2, i.e., takes any value $\in \mathbb{P}^1$ twice on $\mathbb{C}/\Gamma$. One also needs **18** (i) at several points.)

20. (1 pt.) Let $f(u)$ be an elliptic function.

(i) Suppose f is an even function and $\Omega \in \Gamma$. Show that, if $f(\Omega/2) = 0$ (resp. $\Omega/2$ is a pole of f), then $\Omega/2$ is a zero (resp. a pole) of even order.

(ii) Suppose f is an even function. Let $\{a_1, \dots, a_N\}$ be the set of all distinct zeros in the period parallelogram and $\{b_1, \dots, b_M\}$ be the set of all distinct poles. We denote the order of a_i (resp. b_j) by n_i (resp. k_j) and define the integers m_i and l_j as follows:

$$m_i := \begin{cases} n_i & (2a_i \notin \Gamma), \\ n_i/2 & (2a_i \in \Gamma), \end{cases} \qquad l_j := \begin{cases} k_j & (2b_j \notin \Gamma), \\ k_j/2 & (2b_j \in \Gamma). \end{cases}$$

Then there exists a complex number k such that

$$f(u) = k \frac{\prod_{i=1}^{N} (\wp(u) - \wp(a_i))^{m_i}}{\prod_{j=1}^{M} (\wp(u) - \wp(b_j))^{l_j}}.$$

(iii) Show that an odd elliptic function $f(u)$ is a product of $\wp'(u)$ with a rational function of $\wp(u)$. Combining this result with (ii), show that an arbitrary elliptic function $f(u)$ is expressed as

$$f(u) = R_1(\wp(u)) + R_2(\wp(u))\wp'(u),$$

where R_1 and R_2 are rational functions.

21. (1 pt.) Show the following addition formula, using the differential equation of $\wp(u)$ and the proof of the addition formula in the lecture:

$$\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \frac{1}{4}\left(\frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)}\right)^2.$$

(Hint: u_i's ($i = 1, 2, 3$; cf. the lecture for the definition of u_3) satisfy

$$\wp'(u_i)^2 = 4\wp(u_i)^3 - g_2\wp(u_i) - g_3, \qquad \wp'(u_i) = a\wp(u_i) + b.$$

Hence $\wp(u_i)$'s satisfy a cubic equation.)

22. (1 pt.) Re-interpreting the proof of the addtion formula of $\wp(u)$ in the lecture, show that one can define an abelian group structure of the elliptic curve $\bar{\mathcal{R}} := \overline{\{(z, w) \mid w^2 = 4z^3 - g_2 z - g_3\}}$, as follows:

(i) The unit element $\mathbf{O}$ is the point ∞ $(= [0 : 0 : 1] \in \mathbb{P}^2)$.

(ii) Three points P_1, P_2, P_3 on $\bar{\mathcal{R}}$ satisfy $P_1 + P_2 + P_3 = \mathbf{O}$. $\iff$ There exists a line passing through P_1, P_2 and P_3.