Mathematical methods of science
Fall 2018-2019. Lecturer A.S.Tikhomirov

Description:
The notion of vector, respectively, principal bundle over a smooth manifold is one of the central notions in modern mathematics and its applications to mathematical and theoretical physics. In particular, all known types of physical interations (gravitational, electromagnetic, etc.) are described in terms of connections and other geometric structures on vector/principal bundles on underlying manifolds. The properties of physical fields can be formulated in terms of geometric invariants of connections such as curvature and characteristic classes of corresponding vector/principal bundles. In this course we give an introduction to the differential geometry of vector and principal bundles and consider their metrics, connections, curvature and characteristic classes. Some applications to algebraic geometry, topology, and gauge theory of classical fields (in particular, Maxwell equations, Yang-Mills theory) are discussed.

Prerequisits:
first year of undergraduate study (standard courses in algebra, calculus, geometry, and topology).

Syllabus:

- **1. Manifolds.** Topological manifolds. Smooth manifolds. Examples of manifolds defined by atlases. Smooth maps between smooth manifolds. Vector bundles.
- **2. More Basics on Manifolds.** Tangent bundles and cotangent bundles. Action of smooth maps on vectors. Immersions, embeddings, and submanifolds. Submersions.
- **3. Tensors.** Some linear algebra. Operations on vector bundles; tensors. Index notation for tensors. The Lie bracket of vector fields. Exponentiating vector fields.
- **4. Exterior forms.** Exterior forms and the de Rham differential. Homology and cohomology. Examples.
- **5. Orientations and integration.** Orientations on real vector spaces. Orientations on manifolds and top degree forms. Integration on manifolds. Applications to de Rham cohomology. The classification of compact 2-manifolds.
- **6. Connections and curvature.** Differentiation in differential geometry. The definition of connections. Curvature of connections. Flat and locally trivial connections. Connections on TX and torsion.
- **7. Riemannian manifolds.** Riemannian metrics. The Levi-Civita connection. The Riemann curvature tensor. Volume forms and integrating functions.
- **8. More about Riemannian manifolds.** Examples: spheres and hyperbolic spaces. Riemannian 2-manifolds and surfaces in $\mathbb{R}^3$. Geodesics.
- **9. Lie groups and Lie algebras.** Lie groups. Examples of Lie groups. Lie algebras of Lie groups. Fundamental groups.
- **10. More about Lie groups and Lie algebras. Principal bundles.** Relating Lie algebras and Lie groups. The classification of complex Lie algebras. Real forms of Lie algebras. Principal bundles.
- **11. Vector-valued forms.** Vector-valued forms as sections of a vector bundle. Products of vector-valued forms. Directional derivative of a vector-valued function. Exterior derivative of a vector-valued form. Differential forms with values in a Lie algebra. Pullback of vector-valued forms. Forms with values in a vector bundle. Tensor fields on a manifold. The tensor criterion.
- **12. Connections and curvature again.** Connection and curvature matrices under a change of frame. Bianchi identities. The first Bianchi identity in vector form. Symmetry properties of the curvature tensor. Covariant derivative of tensor fields. The second Bianchi identity in vector form. Ricci curvature. Scalar curvature. Defining a connection using connection matrices. Induced connection on a pullback bundle.

- **13. Characteristic classes.** Invariant polynomials on $gl(r, \mathbb{R})$. The Chern-Weil homomorphism. Characteristic forms are closed. Differential forms depending on a real parameter. Independence of characteristic classes of a connection. Functorial definition of a characteristic class.
- **14. Pontrjagin classes. The Euler class and Chern classes.** Vanishing of characteristic classes. Pontrjagin classes. The Whitney product formula. Orientation on a vector bundle. Characteristic classes of an oriented vector bundle. The Pfaffian of a skew-symmetric matrix. The Euler class. Generalized Gauss-Bonnet theorem. Hermitian metrics. Connections and curvature on a complex vector bundle. Chern classes.
- **15. Some applications of characteristic classes.** The generalized Gauss-Bonnet theorem. Characteristic numbers. The cobordism problem. The embedding problem. The Hirzebruch signature formula. The Riemann-Roch.
- **16. More on principal bundles.** The frame bundle of a vector bundle. Fundamental vector fields of a right action. Integral curves of a fundamental vector field. Vertical subbundle of the tangent bundle TP. Horizontal distributions on a principal bundle.
- **17. Connections on a principal bundle.** Connections on a principal bundle. Vertical and horizontal components of a tangent vector. The horizontal distribution of an Ehresmann connection. Horizontal lift of a vector field to a principal bundle. Lie bracket of a fundamental vector field. Horizontal distributions on a frame bundle. Parallel translation in a vector bundle. Horizontal vectors on a frame bundle. Horizontal lift of a vector field to a frame bundle. Pullback of a connection on a frame bundle under a section.
- **18. Curvature on a principal bundle.** Curvature form on a principal bundle. Properties of the curvature form. The associated bundle. The fiber of the associated bundle. Tensorial forms on a principal bundle. Covariant derivative. A formula for the covariant derivative of a tensorial form.
- **19. Characteristic classes of principal bundles.** Invariant polynomials on a Lie algebra. The Chern-Weil homomorphism.
- **20. Applications to gauge theory of classical fields.** The Yang-Mills functional. Maxwell equations, rank two Euclidean Yang-Mills theory: instantons.

Textbooks:

1. L. Tu. Differential Geometry: Connections, Curvature, and Characteristic Classes. Springer, 2017.
2. P. Petersen. Riemannian geometry. Springer, 2006.
3. J. Milnor, J. Stasheff. Characteristic classes. Princeton, 1974.

Additional textbooks:

4. C. H. Taubes. Differential geometry: bundles, connections, metrics and curvature. Oxford, 2011.
5. T. Aubin. Nonlinear analysis on manifolds, Monge-Ampere equations. Springer, 1982.
6. C. Nash, S. Sen. Topology and geometry for physicists. Academic Press, 1987.
7. Palais. The geometrization of physics. 1981.
8. K. Nomizu. Lie Groups and differential geometry. 1956.