List of exercises (last modified 14.11.2018)

1. Let ξ_n be a Markov chain with 6 states $A_1, \dots, A_6$, where $A_1 = (1,0,0,0,0,0)$, $A_2 = (0,1,0,0,0,0)$, etc. Assume that it is given by the transition probability matrix
$$\Pi = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 1/4 & 1/2 & 1/4 & 0 & 0 & 0 \\ 1/16 & 1/4 & 1/4 & 1/4 & 1/16 & 1/8 \\ 0 & 0 & 1/4 & 1/2 & 1/4 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$
This Markov chain arises in the example from genetics that was discussed during one of the lectures. Because of the structure of the matrix it is known that with probability one there exists $N \in \mathbb{N}$ such that have either $\xi_n = A_1$ for any $n \geq N$ or $\xi_n = A_5$ for any $n \geq N$. We denote these events as $\xi_n \to A_1$ and $\xi_n \to A_5$ correspondingly. Compute the probabilities $\mathbb{P}(\xi_n \to A_i | \xi_0 = A_j)$ for $i = 1, 5$ and $j = 2, 3, 4$.

2. Let $\mathcal{P}$ be the space of (probability) distributions,
$$\mathcal{P} = \{\mu = (\mu_1, \dots, \mu_L) \in \mathbb{R}^L : \mu_i \geq 0 \quad \text{and} \quad \sum_{i=1}^{L} \mu_i = 1\},$$
and d be the variational distance on $\mathcal{P}$, i.e. $d(\mu, \nu) = \frac{1}{2}\sum_{i=1}^{L} |\mu_i - \nu_i|$, $\mu, \nu \in \mathcal{P}$. Check that $(\mathcal{P}, d)$ is a complete metric space.

3. Consider a random walk on the state space $\{1, \dots, L\}$ given by the transition probabilities $p_{ii+1} = p$ and $p_{ii-1} = 1 - p$ for $2 \leq i \leq L-1$, $p_{12} = a$, $p_{11} = 1 - a$ and $p_{LL-1} = b$, $p_{LL} = 1 - b$ for some $0 < p < 1$ and $0 < a, b \leq 1$, while for all other i, j we have $p_{ij} = 0$.

   a) Prove that the corresponding transition probability matrix is ergodic if and only if $a < 1$ or $b < 1$.

   b) For any a, b, p as above find a stationary state. Is it unique?

4. a) Is the transition probability matrix for the Ehrenfest model ergodic? b) Find a stationary state for the Ehrenfest model. Is it unique?

5. Prove that the convergence to the stationary state in the ergodic theorem is exponential. That is, consider a Markov chain with finite number of states and ergodic transition probability matrix. Ergodic theorem, in particular, states that the chain has a unique stationary state π and for any initial distribution $p^{(0)}$ we have $d(p^{(n)}, \pi) \to 0$ as $n \to \infty$, where d is the variational distance. Show that there exist constants $C > 0$ and $0 < \lambda < 1$ such that for any initial distribution $p^{(0)}$ we have $d(p^{(n)}, \pi) \leq C\lambda^n$.

6. Consider an ergodic Markov chain ξ_n with the state space $\{1, \ldots, L\}$, transition probability matrix (p_{ij}) and stationary state $\pi = (\pi_1, \ldots, \pi_L)$. For any $i, j \in \{1, \ldots, L\}$ consider the random variables $\nu_{ij}^n = \#\{1 \le k \le n : \xi_{k-1} = i, \xi_k = j\}$, where $\#A$ denotes the number of elements in a set A (its cardinality). Prove that $\nu_{ij}^n / n \to \pi_i p_{ij}$ as $n \to \infty$ in probability.

   Hint: follow the same strategy that was used when proving the law of large numbers. The proof is quite long.

7. Let $\Pi = (p_{ij})_{1 \le i,j \le L}$ be a stochastic matrix. During one of the lectures it was proven that 1 is its eigenvalue and all other eigenvalues λ satisfy $|\lambda| \le 1$. Assume now that Π is positive in the sense that $p_{ij} > 0$ for any i, j. Show that for any eigenvalue $\lambda \ne 1$ we have $|\lambda| < 1$.

8. Let Π be a stochastic matrix and Λ be its Jordan normal form. Show that for any eigenvalue λ of Π satisfying $|\lambda| = 1$, the corresponding block of Λ is of the size 1×1 (i.e. it consists of the unique element λ).

   Assume now that the matrix Π is positive (in the sense of the previous exercise). Compute the limit Λ^n as $n \to \infty$ and deduce that Π^n converges. Deduce from here the ergodic theorem for the Markov chain with the transition probability matrix Π.