Basic algebraic topology

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Contents

Topology studies the most simple, whence the most fundamental properties of geometric objects. Its results are widely used in all the other domains of mathematics, in contemporary physics, and have applications to chemistry and biology. The following topics are to be considered in the lecture course:

- Topological spaces, continuous mappings, connectedness, compactness
- Fundamental groups and coverings
- Homology of simplicial complexes
- Morse theory
- Poincaré duality and multiplication in cohomology

Application of topological results require a lot of computations, and many problems will be discussed on the seminars and in the course of homework.

Chapter 1

Reminder: topological spaces, continuous mappings, examples. Properties: connectedness, compactness

In this chapter we recall main definitions and give examples of topological spaces we are going to work with during the lecture course.

1.1 Topological spaces

Definition 1.1 Let X be a set. A *topology* $\mathcal{T}$ on X is a subset $\mathcal{T} \subset 2^X$ of the set of all subsets of X satisfying the following axioms:

- the empty set $\emptyset$ and X itself belong to $\mathcal{T}$;
- the intersection $U \cap V$ of any two sets $U, V \in \mathcal{T}$ also is an element of $\mathcal{T}$;
- the union $\cup_{i \in I} U_i$ of any family I of sets in $\mathcal{T}$ also is an element of $\mathcal{T}$.

Such a pair $(X, \mathcal{T})$ is called a *topological space*.

A given set X can have many different topologies (finitely many, in fact, if X is finite). The elements of a topology $\mathcal{T}$ on a set X are also called *open subsets* of X (for a given topology $\mathcal{T}$). This allows one to rewrite the axioms of topology in the following way:

- the empty set $\emptyset$ and X itself are open;
- the intersection of any two open sets also is open;
- the union of any family of open sets is open.

For a point $x \in X$, any open set U containing x is called a *neighborhood* of x.

Complements of open sets in a given topological space $(X, \mathcal{T})$ are called *closed sets*. Hence, there is a one-to-one correspondence between the set of open and the set of closed sets in a given topological space $(X, \mathcal{T})$.

Exercise 1.2 *Duality between open and closed sets in a topological space gives a hint that axioms of topology can be written down in terms of the set of closed sets. Do that.*

Definition 1.3 A subset $\mathcal{B} \subset \mathcal{T}$ of a topology $\mathcal{T} \subset 2^X$ is called a *base* of the topology $\mathcal{T}$ if any open set can be represented as the union of a family of subsets in $\mathcal{B}$.

The notion of base of a topology simplifies defining topologies: instead of describing all open sets, it suffices to describe some base. Note, however, that not each subset $\mathcal{B}$ of subsets of X can serve as a base of a topology.

Exercise 1.4 *Prove that a set $\mathcal{B}$ of subsets in X is a base of a topology on X iff the union of the sets in $\mathcal{B}$ coincides with X (that is, any point of X belongs to some element of B), and the intersection of any two elements in $\mathcal{B}$ admits a representation as a union of subsets in $\mathcal{B}$.*

1.2 Examples

Topological spaces can be very complicated. However, as usual, simple topological spaces are most useful ones, and in our lectures we will restrict ourselves with rather simple instances of topological spaces. Nevertheless, examples given in the present section by no means exhaust the complete list of examples.

Example 1.5

1. If X consists of a single point, then, because of the first axiom, there is only one possible topology on X.
2. For any finite set X, we can introduce the *discrete topology* on X. This topology consists of *all* subsets in X.
3. The subsets $\{\emptyset, \{1\}, \{1,2\}\}$ form a topology on $X = \{1,2\}$.

Exercise 1.6

1. *Enumerate all topologies on a* 2*-point set.*
2. *Enumerate all topologies on a* 3*-point set.*

Example 1.7

1. The line $\mathbb{R}$, the base of a topology given by all open intervals, is a topological space.
2. The plane $\mathbb{R}^2$, the base of a topology given by all open discs, is a topological space.
3. The Euclidean space $\mathbb{R}^n$ of arbitrary dimension n, the base of a topology given by all open balls, is a topological space.
4. Any metric space, the base of a topology given by all open balls, is a topological space.

5. The *Zariski topology* in $\mathbb{R}^n$ is defined in the following way: the closed sets are common zeroes of finite tuples of polynomials, $p_1(x_1, \ldots, x_n) = 0, \ldots, p_N(x_1, \ldots, x_n) = 0$.

Recall that a metric space (X, ρ) is a set X endowed with a metric ρ, that is, a function $\rho : X \times X \to \mathbb{R}$ satisfying the *axioms of a metric*:

- $\rho(x, y) \geq 0$ for all $x, y \in X$, and $\rho(x, y) = 0$ iff $x = y$;
- $\rho(x, y) = \rho(y, x)$ for all $x, y \in X$;
- $\rho(x, y) + \rho(y, z) \geq \rho(x, z)$ for all $x, y, z \in X$ (the *triangle inequality*).

The *open ball* of radius r centered at $x_0 \in X$ is the set of points $y \in X$ such that $\rho(x_0, y) < r$.

Statement 1.8 *Let $(X, \mathcal{T})$ be a topological space, and let $Y \subset X$ be a subset. Define the set $\mathcal{T}|_Y \subset 2^Y$ as the set consisting of intersections of open sets in X with Y, $\mathcal{T}|_Y = \{U \cap Y | U \in \mathcal{T}\}$. Then the pair $(Y, \mathcal{T}|_Y)$ is a topological space.*

The topology $\mathcal{T}|_Y$ is called the *induced topology* on the subspace Y of the topological space $(X, \mathcal{T})$. All subsets of Euclidean spaces are endowed with induced topology. In particular, this is true for algebraic (given by polynomial equations) or smooth (given by smooth independent equations) submanifolds. A standard example of such submanifolds are *unit spheres* $S^{n-1} \subset \mathbb{R}^n$ given by the equation $x_1^2 + \cdots + x_n^2 = 1$. Below, we will not specify explicitly topology induced by Euclidean spaces on their subspaces.

Exercise 1.9 *Let X be a metric space, and let Y be a subset in it. Prove that the topology on Y induced by the topology in X coincides with the topology associated to the metric on Y induced from that on X.*

Exercise 1.10 *Prove that the Zariski topology in $\mathbb{R}^1$ indeed is a topology. Show that it does not coincide with the Euclidean topology.*

Example 1.11 A *finite graph* is a pair (V, E) of finite sets V, E (the set of *vertices* and the set of *edges*) together with a mapping taking each edge to a pair of (not necessarily distinct) vertices (the *ends* of the edge). Each graph can be made into a topological space (the *topological graph*) obtained by taking a segment for each edge and gluing together the ends of segments with a common end into a single point (a vertex). This description uniquely describes the topological graph up to homeomorphism.

We will also need infinite graphs, where either the set of vertices, or the set of edges, or both are infinite.

Exercise 1.12 *Describe formally the topology of a finite topological graph (meaning that you must say what is the underlying point set and what are the open subsets in it).*

Example 1.13 In addition to real submanifolds in Euclidean spaces we will also consider complex submanifolds (those given by equations with complex coefficients) in complex vector spaces. They can be considered as special cases of real submanifolds, but is is more convenient to think of them as about a separate species.

1.3 Continuous mappings

Definition 1.14 A mapping $f : X \to Y$ of a topological space $(X, \mathcal{T}_X)$ to a topological space $(Y, \mathcal{T}_Y)$ is said to be *continuous* (with respect to the topologies $\mathcal{T}_X, \mathcal{T}_Y$) if the preimage $f^{-1}(W) \subset X$ of any open subset $W \in \mathcal{T}_Y$ in Y is an open subset in X, $f^{-1}(W) \in \mathcal{T}_X$.

A mapping is a *homeomorphism* if it is continuous and its inverse also is continuous.

Note that the notion of continuity of a mapping depends on the choice of topologies on both its source and its target. A mapping can be a homeomorphism only if it is one-to-one (but this requirement is not sufficient).

Exercise 1.15

1. *Enumerate all topologies on a 2-point set up to homeomorphism (that is, we consider two topologies different iff the corresponding topological spaces are not homeomorphic to one another).*
2. *Enumerate all topologies on a 3-point set up to homeomorphism (that is, we consider two topologies different iff the corresponding topological spaces are not homeomorphic to one another).*

Exercise 1.16 *Prove that any two intervals (a, b) in $\mathbb{R}$ are homeomorphic to one another. Prove that any two segments $[a, b]$ in $\mathbb{R}$ are homeomorphic to one another.*

1.4 Properties of topological spaces

Definition 1.17 A topological space X is said to be *connected* if it cannot be represented as a disjoint union of two open subsets, none of which is empty.

Exercise 1.18 *Prove that any topological space X admits a representation as a disjoint union of maximal connected subspaces, and this representation is unique.* (*A connected subspace of X is maximal if any subspace containing it is not connected.*)

A connected subset of X entering such a representation, that is, a maximal connected subset of X, is called a *connected component* of X.

Exercise 1.19 *Prove that any connected component of a topological space is closed.*

Definition 1.20 A topological space X is said to be *path connected* if for any two points $x_0, x_1 \in X$ there is a continuous mapping $\gamma : [0,1] \to X$ such that $\gamma(0) = x_0$, $\gamma(1) = x_1$.

Exercise 1.21 *Prove that if a topological space X is path connected, then it is connected.*

Exercise 1.22 *Prove that any topological space X admits a representation as a disjoint union of maximal path connected subspaces, and this representation is unique.* (*A path connected subspace of X is maximal if any subspace containing it is not path connected.*)

A subset of X entering such a representation, that is, a maximal path connected subset of X, is called a *path connected component* of X.

Exercise 1.23 *Give an example of a topological space having a path connected component of a topological space that is not closed.*

Exercise 1.24 *Using the notion of connectedness, prove that the interval* $(0,1)$ *is not homeomorphic to any ball in $\mathbb{R}^n$, for $n \geq 2$.*

Definition 1.25 A topological space X is said to be *compact* if any covering of X by open sets, $X = \cup_{i \in I} U_i$, admits a finite subcovering.

Exercise 1.26 *Prove that*

1. *the closed segment* $[0, 1]$ *is compact;*
2. *the open interval* $(0, 1)$ *is not compact;*
3. *a subset of an Euclidean space is compact iff it is bounded and closed;*
4. *a finite topological graph is compact.*

Exercise 1.27 *Prove that the image of a compact topological space under a continuous mapping is compact.*

Chapter 2

Quotient spaces modulo equivalence relations, quotient spaces modulo group actions. Product, cone, suspension, join.

In this chapter we recall ways of constructing new topological spaces from given ones.

2.1 Cartesian product

Let $(X, \mathcal{T}_X)$ and $(Y, \mathcal{T}_Y)$ be two topological spaces. The *product topology* on the Cartesian product $X \times Y$ is given by its base, which consists of the products $U \times V$, where $U \in \mathcal{T}_X$ and $V \in \mathcal{T}_Y$ are open subsets in the factors.

Exercise 2.1 *Show that the topology on the Euclidean space $\mathbb{R}^n$ coincides with the product topology in $\mathbb{R} \times \cdots \times \mathbb{R}$ (that is, the base of coordinate parallelepipeds produces the same topology as the base of balls).*

Exercise 2.2 *Prove that the product of two compact topological spaces is compact.*

2.2 Quotient spaces

Definition 2.3 Let X be a topological space and let $A \subset X$ be its subset. The topological space X/A obtained from X by *contracting A to a point* is constructed as follows:

- the points in X/A are the points $X \setminus A$ and an additional point, which we denote by a, $X/A = (X \setminus A) \sqcup \{a\}$;
- the open sets in X/A are (i) those open sets in X that do not intersect A; (ii) for each open set U in X containing A, the set $(U \setminus A) \cup \{a\}$.

This topology is called the *quotient topology* on X/A.

Exercise 2.4 *Show that the quotient topology is indeed a topology on X/A.*

Example 2.5
- Setting $A = \{0, 1\}$ in $X = [0, 1]$ we obtain $X/A = S^1$, the circle.
- Take several circles S^1, $X = \sqcup S^1$, and pick a point in each circle. Contracting these points to a single point we obtain a *bucket of circles*. A bucket of circles can be considered as a topological graph with a single vertex.

- Take the sphere S^2 in $\mathbb{R}^3$ and contract the North and the South pole to a single point.

Exercise 2.6 *Prove that contracting the boundary sphere of the closed unit ball $B^n \subset \mathbb{R}^n$ to a point we obtain the sphere S^n.*

Contracting a subspace to a point can be easily extended to a more general notion of quotient topology modulo arbitrary equivalence relation.

Definition 2.7 Let $(X, \mathcal{T})$ be a topological space and let $\sim$ be an arbitrary equivalence relation on X. Denote by $X/\sim$ the set of equivalence classes of $\sim$ on X. Introduce topology $\mathcal{T}/\sim$ on $X/\sim$ in the following way: a set $U \subset (X/\sim)$ is said to be open if the union of the equivalence classes belonging to U is an element of $\mathcal{T}$. This topology is called the *quotient topology* on $X/\sim$.

Theorem 2.8 *For any equivalence relation $\sim$ on a topological space $(X, \mathcal{T})$ the quotient topology $\mathcal{T}/\sim$ indeed is a topology on $X/\sim$. The mapping $X \to X/\sim$ taking each element of X to its equivalence class modulo $\sim$ is continuous.*

The quotient space modulo an equivalence relation is often referred to as the result of contracting each equivalence class to a point. Below, when describing equivalence relations, we will specify only those equivalence classes that differ from a single point.

Example 2.9
- Let X be a topological space, and let $A_1, A_2, \dots, A_n$ be a tuple of pairwise disjoint subsets in X, $A_i \subset X$, $A_i \cap A_j = \emptyset$ for $i \neq j$. Then we can introduce the equivalence relation $\sim$ on X by setting $x \sim y$ iff $y = x$ or x and y both belong to the same set A_i. Then $X/\sim$ is the topological space obtained by contracting each subset A_i to a point, $X/\sim = (\dots(X/A_1)/\dots A_n)$.
- A topological graph can be interpreted as an instance of the above construction. Indeed, take for X a disjoint union of segments $[0, 1]$, one for each edge of the graph. For each vertex v of the graph, take a subset A_v in X consisting of the ends of the segments incident to the vertex v, one end for each incidence. Then the quotient topological space is the desired topological graph.

- Take the square $X = [0, 1] \times [0, 1]$ and introduce the equivalence relation by identifying $[0, y] \sim [1, y], y \in [0, 1]$. The quotient space $X/\sim$ is the (closed) *cylinder*.

- The equivalence relation $[0, y] \sim [1, 1 - y], y \in [0, 1]$ on the square $X = [0, 1] \times [0, 1]$ produces the (closed) *Möbius band* as the quotient space $X/\sim$.

- The equivalence relation $[0, y] \sim [1, y], [x, 0] \sim [x, 1], x \in [0, 1], y \in [0, 1]$ on the square $X = [0, 1] \times [0, 1]$ produces the *torus*.

The last example can be generalized as follows. Consider a finite set of polygons, which can be assumed to be regular polygons with edges of length 1. Identify each edge of each polygon with the segment $[0, 1]$ in either of the two ways. Splitting edges of the polygons in pairs and gluing together edges of the same pair (identifying a point x on one edge to the same point on the other edge), we obtain a *two-dimensional surface*. This surface is *closed* if all the edges are split in pairs, and it is a *surface with boundary* provided some of the edges remain unpaired.

Exercise 2.10 *Prove that the quotient of a compact topological space with respect to an arbitrary equivalence relation is compact.*

2.3 Group actions

Let G be a group acting on a topological space $(X, \mathcal{T})$ by homeomorphisms. This means that a homomorphism $G \to \mathrm{Homeo}(X)$ of G to the group of homeomorphisms of X is given. Such an action defines an equivalence relation on X: two points $x_1, x_2 \in X$ are equivalent, $x_1 \sim x_2$, if there is an element $g \in G$ such that $g(x_1) = x_2$, that is, if they belong to the same orbit of the G-action. The quotient space modulo this equivalence relation, which in this case is denoted by X/G, is endowed with the quotient topology $\mathcal{T}/G$. The points of X/G are the orbits of the action of G.

Example 2.11 1. Let the group $\mathbb{Z}_2$ of residues modulo 2 act on the circle $X = S^1$ (thought of as the unit circle on the Euclidean xy-plane) so that the generator of the group acts as the reflection through the x-axis. Then the quotient $S^1/\mathbb{Z}^2$ is the closed semicircle (homeomorphic to the segment $[0, 1]$).

2. A cyclic group $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$ acts on the circle S^1 so that the generator acts by rotating the circle by the angle $2\pi/n$. The quotient space $S^1/\mathbb{Z}_n$ is homeomorphic to the circle once again.

3. The group of integers $\mathbb{Z}$ acts on the line $\mathbb{R}^1$: the generator shifts it by 1, $x \mapsto x+1$. The quotient space is the circle, $S^1 = \mathbb{R}^1/\mathbb{Z}$.

4. The group $\mathbb{Z}\times\mathbb{Z}$ acts on the plane $\mathbb{R}^2$: the two generators shift it by 1 along the x- and the y-axes, $x \mapsto x+1$, $y \mapsto y+1$. The quotient space is the product of two circles, $S^1 \times S^1 = \mathbb{R}^2/\mathbb{Z}\times\mathbb{Z}$, i.e. the torus.

5. For any integer n the group $\mathbb{Z}_2$ acts on the unit sphere S^{n-1}; the action is generated by the central reflection $(x_1, \ldots, x_n) \mapsto (-x_1, \ldots, -x_n)$ of the ambient space $\mathbb{R}^n$. The quotient space $S^{n-1}/\mathbb{Z}_2$ modulo this action is called the *real projective space* and is denoted by $\mathbb{R}\mathrm{P}^{n-1}$.

6. The group S^1 acts on the plane $\mathbb{R}^2$ by rotations. The quotient space $\mathbb{R}^2/S^1$ is the half-line $\mathbb{R}_{\geq 0} = \{x | x \geq 0\} \subset \mathbb{R}$.

7. The $(2n-1)$-dimensional unit sphere $S^{2n-1} \subset \mathbb{R}^{2n}$ can be considered as a subspace in the complex vector space $\mathbb{C}^n = \mathbb{R}^{2n}$. The circle group S^1 acts on this vector space by $\varphi : (z_1, \ldots, z_n) \mapsto (e^{2i\varphi}z_1, \ldots, e^{2i\varphi}z_n)$, and this action preserves the sphere. The quotient space S^{2n-1}/S^1 is called the *complex projective space* and is denoted by $\mathbb{C}\mathrm{P}^{n-1}$. In particular, for $n = 2$, we have the quotient space $S^2 = \mathbb{C}\mathrm{P}^1 = S^3/S^1$.

8. A similar action of the group S^3 of unit vectors in the space of quaternions on the sphere S^7 considered as the unit sphere in the plane over quaternions leads to the quotient space $S^4 = S^7/S^3$.

9. A *hyperelliptic curve* is a subset in $\mathbb{C}\mathrm{P}^1 \times \mathbb{C}\mathrm{P}^1$ given by an equation of the form $y^2 = P(x)$, for a polynomial P (of degree at least 5). The *hyperelliptic involution* acts on this curve by taking a point with coordinates (x, y) to $(x, -y)$. This action generates an action of the group $\mathbb{Z}_2$ on the hyperelliptic curve. The quotient modulo this action is naturally identified with the complex projective line $\mathbb{C}\mathrm{P}^1 = S^2$ of coordinate x.

Exercise 2.12 *1. The multiplicative group $\mathbb{R}^*$ of nonzero real numbers acts on the Euclidean space $\mathbb{R}^n$ by multiplying coordinates by a constant. Show that the quotient space of the complement to the origin $(\mathbb{R}^n \setminus \{0\})/\mathbb{R}^*$ is homemorphic to the projective space $\mathbb{R}\mathrm{P}^{n-1}$.*

2. *The multiplicative group $\mathbb{C}^*$ of nonzero complex numbers acts on the Euclidean space $\mathbb{C}^n$ by multiplying coordinates by a constant. Show that the quotient space of the complement to the origin $(\mathbb{C}^n \setminus \{0\})/\mathbb{C}^*$ is homemorphic to the projective space $\mathbb{C}\mathrm{P}^{n-1}$.*

3. *What will be the quotient topological space in the previous cases if we do not puncture the origin from the Euclidean space before taking the quotient?*

2.4 Cone, suspension, join, and gluing of topological spaces by a mapping

Definition 2.13 Let X be a topological space. Its *cone* CX is defined as the quotient space $(X\times[0,1])/\sim$, where the equivalence relation $\sim$ identifies all the points on the top of the product, $(x_1,1)\sim(x_2,1)$ for all $x_1,x_2\in X$.

Definition 2.14 Let X be a topological space. Its *suspension* SX is defined as the quotient space $(X\times[0,1])/\sim$, where the equivalence relation $\sim$ identifies all the points on the top, and all the points on the bottom of the product $(x_1,0)\sim(x_2,0)$ and $(x_1,1)\sim(x_2,1)$ for all $x_1,x_2\in X$.

Example 2.15 The suspension of a sphere is a sphere, $SS^{n-1}=S^n$.

Definition 2.16 Let X,Y be topological spaces. Their *join* $X\star Y$ is defined as the quotient space $(X\times[0,1]\times Y)/\sim$, where the equivalence relation $\sim$ identifies points on the top, and points on the bottom of the product in the following way: $(x,0,y_1)\sim(x,0,y_2)$ and $(x_1,1,y)\sim(x_2,1,y)$ for all $x,x_1,x_2\in X, y,y_1,y_2\in Y$.

Example 2.17 The join $[0,1]\star[0,1]$ of two segments is the 3-simplex.

Definition 2.18 Let X,Y be topological spaces, let $A\subset X$ be a subspace in X, and let $f:A\to Y$ be a continuous mapping. The result of *gluing* of X to Y along f is defined as the quotient space $(X\sqcup Y)/\sim$, where the equivalence relation $\sim$ identifies each point $a\in A$ with $f(a)\in Y$ (which means, in particular, that all the points in A that are taken by f to the same point are identified with one another).

Example 2.19 If f is a homeomorphism of the boundary circle S^1 of the unit closed disk $X = B^2$ to the boundary circle of another copy $Y = B^2$ of the disk, then the result of gluing of X and Y along this homeomorphism is the 2-sphere S^2.

Chapter 3

Graphs, surfaces, simplicial complexes. Simplicial divisions of spheres, tori, surfaces, projective spaces

In this chapter we introduce the notion of simplicial complex and give representations of certain topological spaces as simplicial complexes.

3.1 Simplices and simplicial complexes

The *n-dimensional simplex* Δ^n is the subspace in $\mathbb{R}^{n+1}$ consisting of points with all nonnegative coordinates $(x_1, \ldots, x_{n+1})$ satisfying the linear equation $x_1 + \cdots + x_{n+1} = 1$. Thus, the 1-dimensional simplex is a segment, the 2-dimensional segment is a triangle, the 3-dimensional simplex is a tetrahedron, and so on. By setting some of the coordinates $x_i = 0$, we obtain a *face* of the simplex. Each face of the simplex is a simplex itself.

A *simplicial complex* is a topological space obtained from several simplices by identifying some of their faces of the same dimension. If the vertices of each simplex are numbered, then the identification map is uniquely specified by a one-to-one correspondence between the sets of numbers of the corresponding faces. Moreover, a mapping from a set of vertices of a simplicial complex to the set of vertices of another simplicial complex determines a continuous mapping of two complexes provided the images of the vertices of any simplex in the preimage span a simplex in the image: one just extends such a mapping to each simplex linearly. Such a mapping is called a *simplicial mapping*.

We will usually consider finite simplicial complexes, but at some point infinite complexes will become necessary. A simplicial complex homeomorphic to a given topological space X is called a *simplicial decomposition* of X.

Example 3.1 The sphere S^2 can be represented as a simplicial complex in many different ways. Thus, the surface of the tetrahedron is a simplicial complex homeomorphic to the sphere. This complex consists of four 2-simplices (triangles), each glued to the other three triangles along a single edge.

Replacing the tetrahedron with the octahedron or the icosahedron we obtain different simplicial decompositions of the sphere.

Of course, simplicial decompositions of the sphere do not come necessarily from regular polyhedra.

Example 3.2 Any graph naturally is a simplicial complex.

The *dimension* of a simplicial complex is the largest dimension of the simplices forming this complex. Graphs are simplicial complexes of dimension 1, the simplicial complexes homeomorphic to the sphere S^2 have dimension 2.

Exercise 3.3 *Find a simplicial decomposition of*

1. *the projective plane* $\mathbb{R}\mathrm{P}^2$*;*
2. *the complex projective plane* $\mathbb{C}\mathrm{P}^2$*;*
3. *the* 3*-dimensional sphere* S^3

Exercise 3.4 *Construct a simplicial decomposition of a product* $\Delta^i \times \Delta^j$ *of two simplices. Use this construction to construct simplicial decompositions of tori* $\mathbb{T}^n = (S^1)^n$*.*

3.2 Two-dimensional surfaces

The definition of a 2-dimensional surface as the result of gluing regular polygons along mappings identifying certain pairs of their edges gives an immediate tool for constructing simplicial decompositions of the surfaces. Indeed, for this purpose it suffices to pick a triangulation of each polygon by pairwise nonintersecting diagonals.

Exercise 3.5 *Prove that if* $f : S_1 \to S_2$ *is a homeomorphism of surfaces, then* f *takes boundary points of* S_1 *to boundary points of* S_2*.*

Exercise 3.6 *Let* X *be the topological space obtained by identifying the North and the South pole of the* 2*-sphere* S^2*. Find a simplicial decomposition for* X*.*

Exercise 3.7 *Take a square. There are two ways to glue in pair two opposite edges of the square, one producing the cylinder, the other one the Möbius band. Construct a simplicial decomposition for both. Prove that the cylinder and the Möbius band are not homeomorphic to one another.*

Definition 3.8 A 2-dimensional surface is said to be *nonorientable* if it contains a subspace homeomorphic to the Möbius band. Otherwise it is said to be *orientable*.

Exercise 3.9 *How many different closed surfaces can be obtained from the square by gluing its edges in pairs? How many of them are nonorientable?*

3.3 Standard models for closed 2-dimensional surfaces

In this section we classify, up to homeomorphism, 2-dimensional surfaces that can be glued out of finitely many polygons. We define a standard model for such a surface, and show that each surface is homeomorphic to some standard model. Then we show that surfaces with different standard models are not homeomorphic to one another.

If the result of gluing a tuple of polygons is a connected surface, then the same surface can be obtained by gluing in pairs edges of a single polygon. This can be easily seen by successively attaching the polygons to the original one. Therefore, when solving the topological classification of closed surfaces problem we can assume without loss of generality that each such surface is glued out of a polygon with an even number of edges. The resulting surface is nonorientable iff there is a pair of edges in the polygon glued "with the reversed orientation".

Theorem 3.10 *Each closed orientable 2-dimensional surface can be obtained from a single $4g$-gon by gluing its edges in the order $aba^{-1}b^{-1}cdc^{-1}d^{-1}\dots$. In other words, each closed orientable surface is either a sphere or a connected sum of several copies of tori.*

Suppose we have an orientable gluing scheme for edges of a single polygon. Then

- either there exists a pair aa^{-1} of neighboring edges glued together;
- or there is a pair of alternating pairs of edges glued together, $a\dots b\dots a^{-1}\dots b^{-1}\dots$.

Indeed, take for a the edge belonging to a "shortest" pair of corresponding edges (the one with the lowest number of edges between a and a^{-1}). Then, if the edges a and a^{-1} are not neighboring, then any other edge b in between a and a^{-1} and its pair b^{-1} form an alternating pair with a, a^{-1}. Figure 3.3 shows how the alternating pair $a\dots b\dots a^{-1}\dots b^{-1}\dots$ can be replaced by an alternating neighboring pair $ABA^{-1}B^{-1}$.

Figure 3.1: Collecting together alternating pairs of edges

A pair of neighboring edges glued together can be glued without changing the topology of the resulting surface, and collecting alternating pairs of edges also does not increase the number of edges in the polygon. Therefore, repeating these steps we prove the theorem by induction.

Theorem 3.11 *Two standard models for gluing connected 2-dimensonal surfaces from a $4g_1$-gon and a $4g_2$-gon produce nonhomeomorphic surfaces provided $g_1 \neq g_2$.*

Suppose a surface S is the result of gluing together a finite set of polygons that has V vertices (the images of the vertices of the polygons), E edges (the images of the edges of the polygons), and F faces (the polygons themselves). The expression

$$\chi(S) = V - E + F$$

is called the *Euler characteristic* of the surface S.

Theorem 3.12 *All simplicial deccompositions of the same surface S give one and the same value of the Euler characteristic.*

In other words, $\chi(S)$ is a topological invariant of the surface.

In order to prove the theorem, note that a simplicial subdivision of a triangle does not change its Euler characteristic. Given two simplicial decompositions of a given surface, consider their common subdivision: the corresponding Euler characteristic will be the same.

Now it suffices to note that the standard model of a closed orientable surface on a $4g$-gon has Euler characteristic $2 - 2g$ (there are 1 vertex, $2g$ edges and 1 face).

Exercise 3.13 *What is the standard model for the surface with the model $a_1 a_2 \dots a_n a_1^{-1} a_2^{-1} \dots a_n^{-1}$?*

Exercise 3.14 *Prove that two models of the form $aabcb^{-1}c^{-1}$ and $aabbcc$ produce homeomorphic surfaces.*

Exercise 3.15 *Using a simplicial subdivision of the real projective plane, compute its Euler characteristic.*

Exercise 3.16 *Using a simplicial subdivision of the 3-sphere and the real projective space $\mathbb{R}P^3$, compute their Euler characteristics. Verify that the Euler characteristic does not allow one to distinguish between these two spaces.*

Exercise 3.17 *Prove that any nonorientable surface admits a standard model of the form $aabb\ldots$ on a $2g$-gon. By computing the Euler characteristics of these surfaces show that they are not homeomorphic for different values of g.*

Exercise 3.18 *Find standard models for orientable surfaces with boundary.*

Exercise 3.19 *Find standard models for nonorientable surfaces with boundary.*

Chapter 4

Paths, loops, and their homotopies. Homotopy of continuous mappings. Homotopy equivalence of topological spaces.

In this chapter we start to study the notion of homotopy. This notion formalizes the idea of continuous deformation of either a topological space, or a mapping. Homotopy is an equivalence relation on the set of topological spaces. This relation is weaker than that of homeomorphism, but homotopy equivalent spaces share many common properties. In particular, they have isomorphic homotopy groups.

4.1 Paths, loops, and their homotopies

Let X be a topological space. A continuous mapping $\gamma : [0,1] \to X$ is called a *path* in X. We say that the path γ *connects the point* $\gamma(0)$ *to the point* $\gamma(1)$ (the *beginning* of the path to its *end*). A path γ is called a *loop* with the initial point $x_0 \in X$ if $\gamma(0) = \gamma(1) = x_0$.

Two paths $\gamma_0, \gamma_1 : [0,1] \to X$ having coinciding ends, $\gamma_0(0) = \gamma_1(0), \gamma_0(1) = \gamma_1(1)$, are said to be *homotopic* if there is a continuous mapping $\Gamma : [0,1] \times [0,1]$ such that $\Gamma(t,0) = \gamma_0(t), \Gamma(t,1) = \gamma_1(t)$ and $\Gamma(0,s) \equiv \gamma_0(0)$, $\Gamma(1,s) \equiv \gamma_0(1)$. This definition means that we do not move the ends of the path and allow to deform continuously its interior part.

Note that for any $y \in [0,1]$, the restriction γ_y of the mapping Γ to the segments $[0,1] \times \{y\}$ is a continuous path in X having the same beginning and the same end as γ_0 and γ_1.

Exercise 4.1 *Prove that any loop at a point x_0 in the ball D^n is homotopic to the constant map $t \mapsto x_0$.*

Exercise 4.2 *Prove that any loop at a point x_0 in a bounded star-like domain in $\mathbb{R}^n$ is homotopic to the constant map $t \mapsto x_0$.*

Path connected topological spaces such that all the loops in them are homotopic to the constant map are said to be *simply connected*.

Exercise 4.3 *Prove that two paths $\gamma_0 : x \mapsto e^{\pi i x}$ and $\gamma_1 : x \mapsto e^{-\pi i x}$ are homotopic to one another if considered as paths in $\mathbb{C} = \mathbb{R}^2$ and are not homotopic to one another if considered as paths in the punctured plane $\mathbb{C} \setminus \{0\} = \mathbb{R}^2 \setminus \{0\}$.*

4.2 Homotopic continuous maps

The notion of homotopy extends to continuous mappings of arbitrary topological spaces, not necessarily the segment.

Let M, X be two topological spaces, and let $f_0, f_1 : M \to X$ be two continuous maps. These two maps are said to be *freely homotopic* if there is a continuous mapping $F : M \times [0,1] \to X$ such that $F|_{M\times\{0\}} = f_0$ and $F|_{M\times\{1\}} = f_1$. Free homotopy is an equivalence relation on the space of continuous mappings $M \to X$.

The notion of free homotopy is not of much use, however. For example, any two paths in a path connected topological space are freely homotopic. That is why we usually consider homotopy of mappings satisfying certain restrictions, like in the case of paths in X we consider only those paths that have a given beginning and a given end. See, however, the next section.

4.3 Homotopy equivalence of topological spaces

Let X and Y be two topological spaces. These spaces are said to be *homotopy equivalent* if there is a continuous mapping $f : X \to Y$ and a continuous mapping $g : Y \to X$ such that their composition $g \circ f : X \to X$ is freely homotopic to the identity mapping $\mathrm{id}_X : X \to X$ and their composition $f \circ g : Y \to Y$ is freely homotopic to the identity mapping $\mathrm{id}_Y : Y \to Y$. Two homotopy equivalent spaces can have very different topological properties (like two balls D^n and D^m of different dimensions), but some of their properties are very close in nature.

Exercise 4.4 *Prove that homotopy equivalence indeed is an equivalence relation on the set of topological spaces.*

A topological space homotopy equivalent to a point is said to be *contractible.*

Exercise 4.5 *Prove that any two homeomorphic topological spaces are homotopy equivalent.*

Exercise 4.6 *Prove that if Y is contractible, then $X \times Y$ is homotopy equivalent to X, for any topological space X.*

The statement of this exercise can be extended to a more general situation. Call a continuous mapping $\pi : Y \to X$ a *locally trivial fibration* with a fiber Z if any point $x \in X$ possesses a neighborhood $U \ni x$ such that the restriction of π to the preimage $\pi^{-1}(U)$ coincides with the projection of the direct product $U \times Z$ to the first factor.

Exercise 4.7 *Prove that if $f : X \to S^1$ is a locally trivial fibration over the circle S^1 with the fiber $[0, 1]$, then X is homeomorphic to either the cylinder $S^1 \times [0, 1]$ or to the Möbius band.*

Exercise 4.8 *Prove that if $\pi : Y \to X$ is a locally trivial fibration with contractible fiber Z, and X is a simplicial complex, then Y is homotopy equivalent to X.*

Exercise 4.9 *Prove that the cylinder $S^1 \times [0, 1]$ and the Möbius band both are homotopy equivalent to the circle S^1.*

Exercise 4.10 *Prove that if a topological space is contractible, then it is path connected.*

Exercise 4.11 *Let X, Y be topological spaces, $A \subset X$, and let $\psi_1, \psi_2 : A \to Y$ be two continuous mappings that are freely homotopic. Denote by Z_1, Z_2 the topological spaces obtained by gluing X and Y along the mappings ψ_1 and ψ_2. Is it true that Z_1 and Z_2 necessarily are homotopy equivalent?*

Exercise 4.12 *Let X be a topological space, $A \subset X$. Denote by Y the result of gluing the cylinder over A to X, so that $Y = (X \sqcup A \times [0, 1])/(a \sim (a, 0))$. Prove that X and Y are homotopy equivalent.*

Below, we consider the following list of topological spaces:

$$[0, 1], (0, 1), \mathbb{R}^2, \mathbb{R}^n, S^n, S^1 \times (0, 1), S^1 \times [0, 1], \text{ Möbius band}, \mathbb{RP}^2,$$

as well as the same spaces with punctured at several points and graphs.

Exercise 4.13
- *Which of these spaces are contractible (do not prove noncontractibility!)?*
- *Which of these spaces are simply connected (do not prove nonsimply-connectedness)?*
- *Which pairs of these spaces are homotopy equivalent (do not prove homotopy nonequivalence)?*
- *For each topological space from the list, find a compact topological space homotopy equivalent to it;*
- *Which of these spaces (including punctured ones) are homotopy equivalent to a graph?*

Exercise 4.14 *Prove that contracting an edge in a graph that is not a loop we obtain a homotopy equivalent graph.*

Chapter 5

Fundamental group, invariance under homotopy equivalence. Fundamental groups of certain spaces.

In this chapter we define one more invariant of topological spaces, namely, the fundamental group.

5.1 Fundamental group

Define the *concatenation* of two paths γ_1, γ_2 such that the end of the first path coincides with the beginning of the second one, $\gamma_1(1) = \gamma_2(0)$, by

$$\gamma_2 \# \gamma_1(t) = \begin{cases} \gamma_1(2t) & \text{for} \quad t \in [0, \frac{1}{2}], \\ \gamma_2(2t-1) & \text{for} \quad t \in [\frac{1}{2}, 1]. \end{cases}$$

Recall that a loop in a topological space X with the starting point $x_0 \in X$ is a continuous path $\gamma : [0,1] \to X$ such that $\gamma(0) = \gamma(1)$, and, therefore, the concatenation of any two loops with the same starting point is well defined. We will refer to it as the product of two loops.

In spite of the fact that this operation is well defined, it is not associative: generally speaking, the product $(\gamma_3 \# \gamma_2) \# \gamma_1$ does not coincide with $\gamma_3 \# (\gamma_2 \# \gamma_1)$. However, concatenation becomes associative when considered on homotopy classes of loops rather than on loops themselves.

Let $\pi_1(X, x_0)$ denote the set of classes of homotopy equivalence of loops in X starting and ending at x_0. Define multiplication on the set $\pi_1(X, x_0)$ as $[\gamma_2][\gamma_1] = [\gamma_2 \# \gamma_1]$.

Theorem 5.1 *This multiplication makes $\pi_1(X, x_0)$ into a group.*

This group is called the *fundamental group of the topological space X with the base point x_0*. Note that computing a fundamental group is not an easy task in many cases, and even when it is computed, it is not always easy to compare results of two computations: whether they produce isomorphic groups or not.

In order to prove the theorem, we have to prove the following:

- multiplication is associative;
- there is a neutral element;
- each element of $\pi_1(X, x_0)$ has an inverse.

A homotopy between the paths $(\gamma_3 \# \gamma_2) \# \gamma_1$ and $\gamma_3 \# (\gamma_2 \# \gamma_1)$ can be given by the parameter change

$$\gamma(t,s) = \begin{cases} \gamma_1(2t) & \text{for} \quad t \in [0, \frac{1}{2}], \\ \gamma_2(t - \frac{1}{2}) & \text{for} \quad t \in [\frac{1}{2}, \frac{3}{4}] \\ \gamma_3(2t-1) & \text{for} \quad t \in [\frac{3}{4}, 1]. \end{cases},$$

which gives $\gamma(t,0) = (\gamma_3\#\gamma_2)\#\gamma_1$, while $\gamma(t,1) = \gamma_3\#(\gamma_2\#\gamma_1)$.

The neutral element of the fundamental group is given by the constant mapping $\gamma(t) \equiv x_0$, and the inverse element is given by the same loop, but passed in the opposite direction.

5.2 Dependence on the base point

If two points $x_0 \in X, x_0' \in X$ belong to different path connected components of the topological space X, then the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_0')$ are not related to one another. However, if x_0 and x_0' belong to the same path connected component of X, then the two groups are isomorphic. Therefore, one can talk about the isomorphism type of the fundamental group $\pi_1(X)$ of a path connected topological space X.

Theorem 5.2 *Let X be a topological space, and suppose x_0, x_0' are two points belonging to the same path connected component of X. Then the fundamental groups $\pi_1(X, x_0)$ and $\pi_1(X, x_0')$ are isomorphic.*

Indeed, let $\gamma' : [0,1] \to X$, $\gamma'(0) = x_0, \gamma'(1) = x_0'$ be a path in X connecting x_0 and x_0'. Define a mapping $\pi_1(X, x_0') \to \pi_1(X, x_0)$ by $[\gamma] \mapsto [\gamma'^{-1}\#\gamma\#\gamma']$ where we denote by γ'^{-1} the path γ' passed in the opposite direction, $\gamma'^{-1}(t) = \gamma'(1-t)$. The fact that this mapping establishes an isomorphism between the two groups is obvious. This isomorphism depends on the homotopy class $[\gamma']$ of the path γ'.

5.3 Fundamental groups of certain spaces.

Theorem 5.3 *If a topological space X is contractible, then the fundamental group $\pi_1(X)$ is trivial (consists only of the neutral element).*

In particular, the trivial group is the fundamental group of any ball of dimension ≥ 1, of the Euclidean spaces $\mathbb{R}^n$ and $\mathbb{C}^n$, as well as any simplex.

Also a simple theorem describes the fundamental group of a product of topological spaces.

Theorem 5.4 *For two topological spaces X, Y and points $x_0 \in X$, $y_0 \in Y$, we have $\pi_1(X \times Y, (x_0, y_0)) \equiv \pi_1(X, x_0) \times \pi_1(Y, y_0)$.*

Theorem 5.5 *The fundamental group of the n-dimensional sphere, for $n \geq 2$, is trivial.*

Note that in contrast to balls, spheres, although their fundamental groups are trivial, are not contractible.

Indeed, if $\gamma : [0,1] \to S^n$ is a loop in the sphere such that its image does not contain a point $x \in S^n$, then we can consider this loop as a path in the space $S^n \setminus \{x\}$, which is homeomorphic to the contractible space $\mathbb{R}^n$. Hence, the loop is contractible, whence trivial. However, we must overcome the difficulty related to existence of loops passing through each point of the sphere (a *Peano curve* is an example of a loop of this type).

Now suppose the image of γ is the whole sphere S^n. Let $A \in S^n$ be a point different from x_0. Define the function $f_{\gamma,A} : [0,1] \to \mathbb{R}$ by the formula

$$f_{\gamma,A}(x) = \rho(\gamma(x), x_0) + \rho(\gamma(x), A),$$

where ρ is the standard distance on S^n. Note that, because of the triangle inequality, $f_{\gamma,A}(x) \geq \rho(x_0, A)$ for any $x \in [0,1]$.

Lemma 5.6 *There are finitely many points* $0 = t_0, t_1, \ldots, t_N = 1$ *in the segment* $[0,1]$*,* $t_0 < t_1 < t_2 < \cdots < t_N$*, such that the image of each segment* $[t_i, t_{i+1}]$ *does not contain either* x_0 *or* A*.*

Indeed, for each point $t \in [0,1]$ choose an interval containing t and such that $|f_{\gamma,A}(t) - f_{\gamma,A}(s)| \leq \frac{1}{2}\rho(x_0, A)$ for all s in the interval. The image of such an interval under γ does not contain either x_0, or A (or both). Since the segment is compact, we can select finitely many intervals covering it, which proves the lemma.

Using the lemma, we can homotopy the curve γ on each segment $[t_i, t_{i+1}]$, preserving its ends, in such a way that the modified curve does not contain A in the image of the interior of each segment. Then, shifting from A finitely many points (if necessary), we obtain a loop homotopic to the original one and not passing through the point A. Hence, it is contractible to x_0.

5.4 Fundamental groups of homotopy equivalent spaces

Theorem 5.7 *Let* X, Y *be two homotopy equivalent path connected spaces,* $x_0 \in X$*,* $y_0 \in Y$*. Then the fundamental groups* $\pi_1(X, x_0)$ *and* $\pi_1(Y, y_0)$ *are isomorphic.*

Proof. Let X, Y be two topological spaces, $x_0 \in X$, and let $f : X \to Y$ be a continuous map. Any loop $\gamma : [0,1] \to X$ with the beginning and the end at x_0, $\gamma(0) = \gamma(1) = x_0$, defines a loop in Y with the beginning and the end at $f(x_0) \in Y$. This loop is nothing but the composition $f \circ \gamma : [0,1] \to Y$.

Clearly, if two loops $\gamma_1, \gamma_2 : [0,1] \to X$ are homotopic, then the loops $f \circ \gamma_1, f \circ \gamma_2 : [0,1] \to Y$ are homotopic as well. Therefore, composition with f defines a mapping $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$. It is easy to see that this mapping is a group homomorphism.

If $y_0 \in Y$ is a point and $\zeta : [0,1] \to Y$ is a path connecting $f(x_0)$ to y_0, then the mapping $[\gamma] \mapsto [\zeta^{-1} \# (f \circ \gamma) \# \zeta]$ defines a homomorphism $\pi_1(X, x_0) \to \pi_1(Y, y)$. (We do not require that $y_0 \neq f(x_0)$; if the two points coincide, then the homomorphism could still be different from f_*). This homomorphism depends on the homotopy class of the path ζ.

Now, if X, Y are path connected and homotopy equivalent, then take two continuous maps $f : X \to Y$ and $g : Y \to X$ such that the composite maps $g \circ f : X \to X$ and $f \circ g : Y \to Y$ are homotopic to the identity maps of the corresponding spaces. Let $x_0 \in X$. Choose an arbitrary continuous path $\zeta : [0,1] \to X$ connecting x_0 and $f \circ g(x_0)$, that is, $\zeta(0) = x_0, \zeta(1) = f \circ g(x_0)$. Then the mapping $[\gamma] \mapsto [\zeta^{-1} \# (f \circ g \circ \gamma) \# \zeta]$ is an isomorphism between $\pi_1(X, x_0)$ and $\pi_1(X, f \circ g(x_0))$. To prove this statement, it suffices to consider the homotopy between $f \circ g$ and the identity map of X.

Hence, we have two homomorphisms $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ and $g_* : \pi_1(Y, f(x_0)) \to \pi_1(X, g \circ f(x_0))$ whose composition $g_* \circ f_*$ is an isomorphism. This means, in particular, that f_* has no kernel.

Exercise 5.8 *Prove that, for a continuous mapping $f : X \to Y$ of topological spaces, the mapping $f_* : \pi_1(X, x_0) \to \pi_1(Y, f(x_0))$ is indeed a group homomorphism.*

Corollary 5.9 *The fundamental group of any graph is isomorphic to the fundamental group of a bouquet of circles.*

Exercise 5.10 *Prove that a connected graph (not necessarily simple, loops and multiple edges are allowed) with V vertices and E edges is homotopy equivalent to the bouquet of $E - V + 1$ circles.*

Corollary 5.11 *The fundamental group of any surface punctured at finitely many points is isomorphic to the fundamental group of a bouquet of circles.*

Exercise 5.12 *Find the number of circles in the bouquet homotopy equivalent to an orientable surface of genus g punctured at n points. Find the number of circles in the bouquet homotopy equivalent to a nonorientable surface of genus g punctured at n points.*

5.5 Edge-path group of a simplicial complex

Let Δ^2 be a 2-dimensional simplex, that is, a triangle. Two vertices of the triangle can be connected by its edges in two different ways: either along the edge of which they are the ends, or along the other pair of edges. These two paths in Δ^2 are homotopic (since the triangle is contractible).

Now let X be a simplicial complex, and let $x_0 \in X$ be one of its vertices (a simplex of dimension 0). An *edge-loop* in X starting and ending at x_0 is a sequence of oriented edges (1-dimensional simplices) $(e_1, \dots, e_k)$ such that the beginning of e_1 as well as the end of e_k is x_0, while, for $i > 1$, the beginning of e_i coincides with the end of e_{i-1}.

Note that an edge-loop can be made into an ordinary loop by choosing its arbitrary parametrization (that can well be a piecewise linear mapping of the segment $[0, 1]$ to the sequence of edges).

Two edge-loops can be multiplied by concatenating them. On the other hand, a replacement of an edge e_i in an edge-loop by two other edges of a triangle containing e_i provides an edge-loop homotopic to the original one, and the same is true for the inverse operation. We also allow to annihilate two consecutive edges in an edge-loop that are the same edge passed in the opposite directions, or to insert a pair of such edges. The equivalence classes of edge-loops modulo chains of these operations form a group with respect to the multiplication above. We call this group the *edge-loop group* of the simplicial complex X with the base vertex x_0 and denote it by $E(X, x_0)$.

Theorem 5.13 *The edge-loop group $E(X, x_0)$ of a connected simplicial complex X with a base vertex $x_0 \in X$ depends only on the 2-skeleton of X. It is naturally isomorphic to the fundamental group $\pi_1(X, x_0)$.*

Pick a numbering of the vertices of X by numbers $1, \dots, N$. Let T be a spanning tree of the 1-skeleton of X (that is, T is a tree containing all vertices of X). Associate to an edge connecting the vertices i and j, $i < j$, the edge loop in X in the following way, which depends on whether the edge ij belongs to the tree T. If the edge ij belongs to the tree T, then connect the vertex i with x_0 by a path in T. The loop in question goes along this path, then along the chosen edge ij, then back from j to x_0. Obviously, this loop is contractible along T. If the edge ij, $i < j$, does not belong to the tree T, then the loop goes from x_0 to i along T, then along the edge ij, then along the path in T connecting j and x_0. Denote the constructed loops e_{ij}, for all ij in the 1-skeleton of X.

Theorem 5.14 *For a given spanning tree T of the 1-skeleton of X, the fundamental group $\pi_1(X, x_0)$ admits a representation of the form*

$$\pi_1(X, x_0) = \langle e_{ij}, ij \subset X^{(1)} | e_{ij} = 1, \text{ for } ij \text{ in } T, e_{ik} = e_{ij} e_{jk} \rangle$$

where the generators $e_{ij}, i < j$, correspond to the 1-dimensional simplices in X, a and the triple relations $e_{ik} = e_{ij} e_{jk}$ correspond to all the 2-dimensional simplices, with vertices i, j, k, $i < j < k$.

Note that a spanning tree T in the theorem can be replaced by an arbitrary simply connected subcomplex of X containing all vertices. In applications, this remark could prove to be useful.

Exercise 5.15 *Prove that the fundamental group of a compact orientable 2-dimensional surface of genus g admits a representation with a single relation of the form $\langle a_1, b_1, \ldots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \rangle$.*

Prove that the fundamental group of a compact nonorientable 2-dimensional surface of genus g admits a representation with a single relation of the form $\langle a_1, \ldots, a_g | a_1^2 \ldots a_g^2 \rangle$.

Chapter 6

Coverings, monodromy. Universal covering

Coverings are an efficient and an effective tool for computing fundamental groups. Moreover, coverings are intimately related to fundamental groups: there is a natural one-to-one correspondence between subgroups of the fundamental group of a path connected topological space and coverings of this space.

6.1 Fundamental groups of projective spaces

Let $\mathbb{R}\mathrm{P}^n$ be the n-dimensional real projective space, and let $\pi : S^n \to \mathbb{R}\mathrm{P}^n$ be the mapping identifying pairs of opposite points of the sphere. Pick an arbitrary point $x_0 \in \mathbb{R}\mathrm{P}^n$. Consider a loop $\gamma : [0,1] \to P^n$, $\gamma(0) = \gamma(1) = x_0$, and let $\pi^{-1}(\gamma([0,1])) \subset S^n$ be the total preimage of this loop in the sphere. Each point of the loop has two preimages under π. Let y_0 be one of the two preimages $\pi^{-1}(x_0)$ of the base point $x_0 \in \mathbb{R}\mathrm{P}^n$, and denote by y_0' the second preimage. We are going to show that there is a unique way to chose one of the two preimages of each point $\gamma(x)$ so that these preimages together form a continuous path $\tilde{\gamma} : [0,1] \to S^n$, $\tilde{\gamma}(0) = y_0$. The end of this path can be either the point y_0, or the point y_0'.

For each point $\gamma(x) \in \mathbb{R}\mathrm{P}^2$ choose a small open ball $U_x \ni \gamma(x)$ such that its preimage $\pi^{-1}(U_x)$ in S^n is a disjoint union of two balls. The preimages $\gamma^{-1}(U_x)$ of such balls form an open cover of $[0,1]$. Each open subset in $[0,1]$ is a disjoint union of intervals, and we choose a finite subcover of $[0,1]$ by intervals. For an interval I in this subcover, the preimage $\pi^{-1}(\gamma(I))$ consists of two disjoint paths. Indeed, the path $\gamma(I)$ belongs to some ball U_x, and the preimage $\pi^{-1}(U_x)$ is the disjoint union of two balls.

Now order the intervals covering the segment $[0,1]$ by increasing coordinates of their left ends (without loss of generality one may suppose that there are no two intervals with coinciding left ends, since it is possible to leave only the largest of the two in the cover). Choose the connected component of the preimage $\pi^{-1}(\gamma(I_1))$ of the interval I_1 that starts at the point y_0. Then chose the connected component of the preimage $\pi^{-1}(\gamma(I_2))$ of the interval I_2 that coincides with the previously chosen preimage of I_1 on their intersection, and so on. In this way we obtain a connected path $\tilde{\gamma} : [0,1] \to S^n$, $\tilde{\gamma}(0) = y_0$.

Theorem 6.1 *For $n \geq 2$, we have $\pi_1(\mathbb{R}\mathrm{P}^n) = \mathbb{Z}_2$.*

Indeed, the mapping π takes any loop in S^n with the base point y_0, as well as any path connecting y_0 with y_0' to a loop in $\mathbb{R}\mathrm{P}^2$ with the base point x_0. These kinds of loops represent different elements of the fundamental group $\pi_1(\mathbb{RP}^n, x_0)$: homotopy of loops in $\mathbb{RP}^n$ preserve the ends of the the path in S^n. On the other hand, any loop in S^n with the base point y_0 is contractible, whence represents the neutral element of the fundamental group $\pi_1(\mathbb{RP}^n, x_0)$. And if the path $\tilde{\gamma}$ connects two points y_0 and y_0', then the path $\widetilde{\gamma\#\gamma}$ for γ passed twice is a loop with the base point y_0, whence contractible.

6.2 Coverings, their degrees and pullbacks of paths

The example studied in the previous section is the simplest example of a covering.

A continuous mapping $f : Y \to X$ of two topological spaces is called a *covering* if any point $x \in X$ has a neighborhood U such that its preimage under f is a disjoint union $f^{-1}(U) = V_1 \sqcup V_2 \sqcup \dots$ of open subsets in Y such that the restriction of f to each subset V_i is a homeomorphism.

Lemma 6.2 *Let $f : Y \to X$ be a covering, and suppose X is path connected. Then if some point $x \in X$ has finitely many preimages under f, then the same is true for all the points in X, and the number of the points in the preimage of each point is the same.*

This common number of preimages is called the *degree* of the covering f. If the set of preimages of each point in X is infinite, then we say that the covering has *infinite degree*. Below, we will always assume that the space X is path connected, so that the degree is always well defined. It is denoted by $\deg\ f$.

Proof. Let x_1, x_2 be two points in X, and suppose that the number of points in the preimage $f^{-1}(x_2)$ differs from that in $f^{-1}(x_1)$. Connect the points x_1, x_2 by a continuous path $\gamma : [0,1] \to X$, $\gamma(0) = x_1, \gamma(1) = x_2$. Each point $\gamma(t)$ of the path, $t \in [0,1]$, admits a neighborhood $U_t \subset X$ such that the restriction of f to $f^{-1}(U_t)$ is a collection of homeomorphisms. Since the segment is compact, one can choose finitely many neighborhoods U_t covering the image $\gamma([0,1])$ of the path. Let these be neighborhoods $U_{t_1}, \dots, U_{t_k}$, $t_1 < t_2 < \dots < t_k$. All the points in one neighborhood U_{t_i} have the same number of preimages under f. Split the neighborhoods U_{t_i} into two subsets: one, where the number of preimages coincides with that for x_1, and its complement. Then, since the segment is connected, the unions of the two subsets must intersect, and we arrive at a contradiction.

Exercise 6.3 *Construct a covering of the circle of a given degree n.*

Exercise 6.4 *Prove that the mapping $\mathbb{C}\backslash\{0\} \to \mathbb{C}\backslash\{0\}$ given by the formula $z \to z^n$ is a covering of degree n.*

Exercise 6.5 *Prove that the mapping $z \mapsto P(z)$ given by a polynomial $P(z)$ of degree n is a covering over the complement in $\mathbb{C}$ to the set of critical values of P (a critical value of a polynomial is its value at a point where $dP = 0$). What is the degree of this covering?*

Exercise 6.6 *Prove that the mapping $z \mapsto e^z$ is a covering $\mathbb{C} \to \mathbb{C} \setminus \{0\}$. What is the degree of this covering?*

6.3 Coverings and subgroups of the fundamental group

Let $f : Y \to X$ be a covering, $y_0 \in Y$, and $x_0 = f(y_0)$. Since f is a continuous mapping, it defines a homomorphism $f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$.

Lemma 6.7 *For a covering $f : Y \to X$, $f(y_0) = x_0$, the homomorphism $f_* : \pi_1(Y, y_0) \to \pi_1(X, x_0)$ is a monomorphism.*

The proof of the lemma requires a more general statement. Recall that a *pullback* $f^*\gamma : [0,1] \to Y$ of a path $\gamma : [0,1] \to X$ is a path in Y such that $f \circ f^*\gamma : [0,1] \to X$ coincides with γ.

Theorem 6.8 (Covering homotopy for loops) *Let $f : Y \to X$ be a covering, $y_0 \in Y$, $x_0 = f(y_0)$. Then any loop $\gamma : [0,1] \to X$, $\gamma(0) = \gamma(1) = x_0$ admits a unique pullback $f^*\gamma : [0,1] \to Y$ such that $f^*\gamma(0) = y_0$.*

The proof proceeds in the same way as the above proof of the fact that the degree of a covering is well-defined. We cover the image of the loop γ by open sets as in the definition of a covering, take their preimages under γ, choose a finite subcovering of $[0,1]$ by intervals, and for each such interval pick an appropriate connected component of the preimage under f of the loop γ restricted to this interval, so that the chosen preimages glue together into a continuous path.

Proof of Lemma 6.7. We want to prove that if a loop $\gamma : [0,1] \to X$, $\gamma(0) = \gamma(1) = x_0$ is homotopic to the constant loop, then the path $f^*\gamma$ is in fact a loop and is homotopic to the constant one. Take a homotopy $\Gamma : [0,1] \times [0,1] \to X$, $\Gamma(0,s) \equiv \Gamma(1,s) \equiv x_0$, $\Gamma(t,0) \equiv \gamma(t)$, $\Gamma(t,1) \equiv x_0$, between γ and the constant loop. For each $s \in [0,1]$ the mapping $\gamma_s : [0,1] \to X$, $\gamma_s(t) = \Gamma(t,s)$ is a loop in X starting and ending at x_0, and it has a unique pullback $f^*\gamma_s : [0,1] \to Y$ starting at y_0, $\gamma_s(0) = y_0$, the pullback $f^*\gamma_1$ being the constant map.

Denote by s_0 the supremum of the set of values of s such that all the points $f^*\gamma_s(1)$ coincide with the end $f^*\gamma(1)$ of the pullback of γ. Then $s_0 = 1$. Indeed, suppose the converse. Take, as above, an open covering of the loop $\gamma_{s_0}([0,1])$ by open subsets of X, choose a finite subcovering $U_1, \dots, U_k$, and the ends $f^*\gamma_s(1)$ of the pullbacks of the paths γ_s must belong to one and the same copy of the preimage $f^{-1}(U_k)$ of the last neighborhood U_k under f. Hence, we cannot have $s_0 < 1$.

Now it is easy to see that the pullbacks $f^*\gamma_s$ of the paths γ_s form a homotopy between the paths $f^*\gamma_0 = f^*\gamma$ and $f^*\gamma_1$.

Exercise 6.9 *State and prove the analogue of the Covering homotopy theorem for arbitrary paths in X, not necessarily loops, beginning at x_0.*

Exercise 6.10 *Consider all paths in Y starting at some point in $f^{-1}(x_0)$ and covering the loop γ. Show that*

- *the mapping taking a point in $f^{-1}(x_0)$ to the end of the corresponding path is a permutation of the set $f^{-1}(x_0)$;*
- *this permutation depends only on the homotopy type $[\gamma] \in \pi_1(X, x_0)$;*
- *the mapping associating to an element in $\pi_1(X, x_0)$ the corresponding permutation of $f^{-1}(x_0)$ is a group homomorphism.*

This homomorphism is called the *monodromy* of the covering $f : Y \to X$ (associated with the base point $x_0 \in X$).

6.4 Bijection between path connected coverings and subgroups of the fundamental group

Theorem 6.11 *Let X be a path connected simplicial complex, and let $x_0 \in X$. Then there is a one-to-one correspondence between subgroups* *of* $\pi_1(X, x_0)$ *and coverings* $(Y, y_0) \to (X, x_0)$ *such that* Y *is path connected, considered up to equivalence.*

Here two coverings $f : (Y, y_0) \to (X, x_0)$ and $f' : (Y', y_0') \to (X, x_0)$ are said to be *equivalent* if there exists a homeomorphism $h : (Y, y_0) \to (Y', y_0')$ such that $f' \circ h = f$.

Proof. The mapping f_* associates to a covering $f : (Y, y_0) \to (X, x_0)$ by a connected topological space Y the subgroup $f_*\pi_1(Y, y_0)$ in $\pi_1(X, x_0)$. Let $f' : (Y', y_0') \to (X, x_0)$ be another covering, and suppose that the two subgroups $f_*\pi_1(Y, y_0)$, $f'_*\pi_1(Y', y_0')$ in $\pi_1(X, x_0)$ coincide. Let us construct an equivalence $h : (Y, y_0) \to (Y', y_0')$ between the two coverings.

Each path $\gamma : [0, 1] \to X$ starting at x_0 has unique pullbacks $f^*\gamma$ starting at y_0 and $f'^*\gamma$ starting at y_0' to Y and Y', respectively. Define the mapping $h : Y \to Y'$ in the following way. Let $y \in Y$, and pick a path $\zeta : [0, 1] \to Y$ in Y connecting y_0 to y. Then we set $h(y)$ to be equal to the end of the path $f'^*\zeta$, which starts at y_0'. Since the two subgroups $f_*\pi_1(Y, y_0)$, $f'_*\pi_1(Y', y_0')$ in $\pi_1(X, x_0)$ coincide, the mapping h is well defined: the value $h(y)$ does not depend on the choice of the path ζ connecting y_0 to y. Clearly, h is a homeomorphism and, together with f and f', forms a commutative triangle.

Now, for each subgroup in $\pi_1(X, x_0)$, we must construct a covering $f : (Y, y_0) \to (X, x_0)$. Consider the topological space $\mathcal{P}(X, x_0)$ whose elements are equivalence classes of paths $\gamma : [0, 1] \to X$ starting at x_0. Two paths γ_1, γ_2 are considered as being equivalent if their ends coincide and the loop $\gamma_2^{-1}\gamma_1$ is contractible. The topology on the space $\mathcal{P}(X, x_0)$ is introduced by means of the following base. For an open set $U \subset X$, $x \in U$, and a path γ starting at x_0 and ending at x, we define the subset $U_\gamma \subset \mathcal{P}(X, x_0)$ as the set of equivalence classes of paths whose beginning coincides with γ, and the rest of the path is contained in U. The sets U_γ for various paths γ form a base of a topology in $\mathcal{P}(X, x_0)$.

For a subgroup $G \subset \pi_1(X, x_0)$, consider the equivalence relation $\sim_G$ on $\mathcal{P}(X, x_0)$ defined by $\gamma_1 \sim_G \gamma_2$ iff $\gamma_1(1) = \gamma_2(1)$ and $[\gamma_2^{-1}\gamma_1] \in G$. Clearly, the quotient space $\mathcal{P}(X, x_0)/\sim_G$ together with the natural mapping $(\mathcal{P}(X, x_0)/\sim_G, (x_0, \mathrm{id})) \to (X, x_0)$ taking the equivalence class of a path γ to x is a covering, and the image of $\pi_1((\mathcal{P}(X, x_0)/\sim_G, (x_0, \mathrm{id}))$ under this covering is G.

6.5 Universal coverings and group actions

A covering $f : (Y, y_0) \to (X, x_0)$ is called a *universal covering* if the topological space Y is simply connected, $\pi_1(Y, y_0) = \{e\}$. The name is justified by the *universality property* of the universal covering: if $f : (Y, y_0) \to (X, x_0)$ is a universal covering and $g : (Z, z_0) \to (X, x_0)$ is any covering of X, then there is a covering $h : (Y, y_0) \to (Z, z_0)$ such that the composition $g \circ h$ coincides with f.

Because of the uniqueness of a covering corresponding to a given subgroup of $\pi_1(X, x_0)$, any path connected simplicial complex admits a unique up to equivalence universal covering.

Suppose a group G acts on a topological space Y by homeomorphisms. We say that G acts *freely* if no element of g other than unity has fixed points. We say that G acts *properly discontinuously* if for any element $g \in G$, $g \neq \mathrm{id}$, and any point $y \in Y$ there is a neighborhood U of y such that $U \cap gU = \emptyset$.

Theorem 6.12 *Let Y be a path connected topological space. If a group G that is at most countable acts on Y freely and properly discontinuously and $y_0 \in Y$, then the quotient space Y/G is path connected and the factorization mapping $(Y, y_0) \to (Y/G, [y_0])$ is a covering. If, in addition, Y is simply connected, then the factorization mapping is a universal covering and $\pi_1(Y/G, [y_0])$ is naturally isomorphic to G.*

Exercise 6.13 *Show that both requirements are necessary: the factorization mapping can prove to be not a covering provided G acts either not freely or not properly discontinuously.*

Exercise 6.14 *Construct the following topological spaces as the quotient spaces of their universal coverings modulo a free properly discontinuous group action: the circle S^1, the bouquet $\vee_{i=1}^n$ of n circles, the punctured complex line $\mathbb{C} \setminus \{0\}$, the cylinder, the Möbius band, the torus $\mathbb{T}^2$, the projective plane $\mathbb{R}\mathrm{P}^2$, the Klein bottle, the group $\mathrm{SO}(3)$ of rotations of $\mathbb{R}^3$, the projective space $\mathbb{R}\mathrm{P}^n$ of arbitrary dimension. Using these constructions find the fundamental groups of these spaces.*

Exercise 6.15 *For each of the topological spaces mentioned in the previous exercise construct its covering for each subgroup of its fundamental group.*

Exercise 6.16 *Check that the fundamental groups of the torus, projective plane and Klein bottle computed with the help of Theorem 6.12 coincide with those computed through the edge-loop group.*

Exercise 6.17 *Prove that any finitely generated Abelian group is realizable as the fundamental group of some topological space.*

Exercise 6.18 *Give an example of a topological space with a noncommutative finite fundamental group.*

Chapter 7

Tools for computing fundamental groups

In this chapter, we will discuss additional tools for computing fundamental groups.

7.1 Van Kampen theorem

It was given as an exercise that the fundamental group of the connected sum of two path connected spaces coincides with the free product of the fundamental groups of the summands. To be more precise, if (Y, y_0) and (Z, z_0) are two path connected topological spaces, with basepoints $y_0 \in Y$, $z_0 \in Z$, then their connected sum is $(X, x_0) = (Y, y_0) \vee (Z, z_0) = (X \sqcup Y)/(y_0 \sim z_0)$, with the base point x_0 formed by the identified base points of the two spaces. We have

$$\pi_1(X, x_0) = \pi_1(Y, y_0) * \pi_1(Z, z_0).$$

Here, the star on the right denotes the free product of groups.

7.1.1 Recollection: free product of groups

Let us recall what is the free product of groups. Let H, G be two groups. A *reduced word in the alphabet* $G \sqcup H$ is a word of the form

$$h_1 g_1 h_2 g_2 \dots h_k g_k,$$

for some positive integer k, such that

- $h_i \in H$, $g_i \in G$, for $i = 1, \dots, k$;
- none of the internal letters $g_1, h_2, \dots, h_k$ is the unit element of the corresponding group;

(note that either h_1, or g_k, or both, are allowed be unit elements). The *free product* $G * H$ of the groups G and H consists of reduced words in the alphabet $G \sqcup H$.

The product of two elements in $G * H$ is defined as concatenation of two reduced words and applying repeatedly to the resulting word the following reduction steps, until obtaining a reduced word:

1. erase all internal unit elements in the word;
2. replace any sequence of consecutive elements belonging to one of the groups by their product in this group.

Exercise 7.1 *Prove that if the group G has presentation $\langle g_1, \dots, g_m | u_1, \dots, u_M\rangle$, and the group H has presentation $\langle h_1, \dots, h_n | v_1, \dots, v_N\rangle$, then their free product $H*G$ admits presentation $\langle h_1, \dots, h_n, g_1, \dots, g_m | u_1, \dots, u_M, v_1, \dots, v_N\rangle$.*

Note that the free product of two groups, both of which are nontrivial, is necessarily non-commutative.

7.1.2 Van Kampen's theorem

The statement about the fundamental groups of the connected sum of two topological spaces extends without changes to the case when a path connected topological space X is represented as a union of two path connected topological spaces, $X = Y \cup Z$, provided the intersection $Y \cap Z$ is contractible and contains the base point of X. Van Kampen's theorem generalizes this statement to a more general situation, where the intersection $Y \cap Z$ is not necessarily contractible, and even can have a nontrivial fundamental group.

Theorem 7.2 *Suppose a path connected topological space X is represented as a union of two path connected topological spaces, $X = Y \cup Z$, and the intersection $Y \cap Z$ is path connected and contains the base point x_0. Denote by $\varphi : Y \cap Z \to Y$, $\psi : Y \cap Z \to Z$ the inclusion mappings. Suppose also that the fundamental group $\pi_1(Y, x_0)$ admits a presentation $\langle g_1, \dots, g_m | u_1, \dots, u_M\rangle$, the fundamental group $\pi_1(Z, x_0)$ admits a presentation $\langle h_1, \dots, h_n | v_1, \dots, v_N\rangle$, and the fundamental group $\pi_1(Y \cap Z, x_0)$ is generated by elements $q_1, \dots, q_\ell$. Then the fundamental group $\pi_1(X, x_0)$ admits presentation*

$$\begin{aligned}\pi_1(X, x_0) \quad = \quad & \langle h_1, \dots, h_n, g_1, \dots, g_m | \\ & v_1, \dots, v_N, u_1, \dots, u_M, \varphi_*(q_1)\psi_*^{-1}(q_1), \dots, \varphi_*(q_\ell)\psi_*^{-1}(q_\ell)\rangle.\end{aligned}$$

In other words, the fundamental group $\pi_1(X, x_0)$ is the *amalgamated product* of the groups $\pi_1(Y, x_0)$ and $\pi_1(Z, x_0)$.

Exercise 7.3 *Use Van Kampen's theorem to compute*

- *the fundamental group of the sphere S^n, $n \geq 2$;*
- *the fundamental group of a 2-dimensional surface.*

Corollary 7.4 *If, in the assumptions of the previous theorem, the intersection $Y \cap Z$ is simply connected, then the fundamental group $\pi_1(X, x_0)$ is the free product of the fundamental groups $\pi_1(Y, y_0)$ and $\pi_1(Z, z_0)$.*

7.1.3 Proof of Van Kampen's theorem

Let $X = Y \cup Z$ be a representation of a path connected topological space X as a union of two open path connected subsets $Y, Z \subset X$, and suppose $x_0 \in Y \cap Z$.

Firstly, the homotopy class of any loop in X can be represented as a composition of homotopy classes of loops contained either in Y, or in Z. Indeed, let $\gamma : [0,1] \to (X, x_0)$ be a loop. Each point $t \in [0,1]$ has an interval neighborhood whose image under γ is contained either in Y or in Z. Pick such an interval neighborhood for each point, and choose a finite subcovering of the covering of the segment by these intervals. Without loss of generality we may suppose that none of these intervals is contained in the other one, and the intervals intersect in pairs only. Then choosing a point in the intersection of each two consecutive intervals we cut the segment $[0,1]$ into finitely many segments such that the image of each segment under γ is contained either in Y or in Z (below, we call them small segments). By replacing consecutive small segments by a single small segment if necessary, we may also assume that the small segments alternate: if the image of a small segment is contained in Y, then the image of the next one is contained in Z, and vice versa. The common boundary point of two consecutive small segments belongs to $Y \cap Z$. For each such common boundary point, pick a path connecting it to x_0 inside $Y \cap Z$. The homotopy class of γ is then represented as a product of the homotopy classes of loops corresponding to each small segment, each loop being the composition of the inverse to the path in $Y \cap Z$ chosen for the beginning of the small segment, then the restriction of γ to the small segment, and then the path in $Y \cap Z$ chosen for the end of the small segment. Each such loop is totally contained either in Y or in Z.

Therefore, the generators of $\pi_1(Y, x_0)$ and $\pi_1(Z, x_0)$ generate together $\pi_1(X, x_0)$.

Now we are going to show that each homotopy of a loop in X with the base point x_0 can be decomposed into a sequence of homotopies, each totally contained either in Y, or in Z.

Let $\gamma_0, \gamma_1 : [0,1] \to X$ be two homotopic loops in (X, x_0), and let $\Gamma : [0,1] \times [0,1] \to X$ be a homotopy between them. Then each point in the square $[0,1] \times [0,1]$ admits an open neighborhood such that the image of the neighborhood under Γ is contained either in Y or in Z. Choose a coordinate rectangular subneighborhood in each such neighborhood. These coordinate rectangular neighborhoods cover the square; pick a finite subcovering from this covering. The sides of these finitely many rectangles, extended to the sides of the square, cut the square into finitely many rectangles with sides parallel to the coordinate axes. The image of each of these rectangles under the homotopy Γ is contained totally either in Y or in Z. Number the rectangles consecutively from left to right in the first row, then in the second row, and so on.

Note that any path in $[0, 1] \times [0, 1]$ starting at the left side of the square and ending at its right side parameterizes a loop in (X, x_0). We will consider paths going along the sides of the rectangular cells in the square. The homotopy Γ can be represented as a composition of a finite sequence of homotopies, numbered by the number of the corresponding square. In each homotopy, a part of the path represented by two sides of a rectangle is replaced by the part represented by the other two sides (a *switch*), while the rest of the path remains unchanged. Each such switch makes a homotopy inside either Y or Z, and after making all these switches, we obtain a homotopy between γ_0 and γ_1.

Hence, there are no relations in $\pi_1(X, x_0)$ but those coming from $\pi_1(Y, x_0)$ and $\pi_1(Z, x_0)$ and the identification of the elements of both groups lying inside $\pi_1(Y \cap Z, x_0)$, and the theorem is proved.

7.2 Knot and link groups

A *knot* in the 3-space is a smooth nondegenerate embedding $S^1 \to \mathbb{R}^3$ of a circle. Similarly, a *link* is a smooth nondegenerate embedding $S^1 \sqcup \cdots \sqcup S^1 \to \mathbb{R}^3$ of several disjoint circles. Below, we will restrict ourselves with giving definitions for knots only, while problems and examples will touch links as well.

A *knot invariant* is a function on knots whose values coincide on any two knots belonging to the same ambient isotopy class. An *ambient isotopy* (that is, an isotopy of the ambient space of the knot) is a 1-parameter family of diffeomorphisms $\mathbb{R}^3 \times [0, 1] \to \mathbb{R}^3$. The isomorphism class of the fundamental group of the complement to the image of a knot is an important knot invariant.

The complement to a point in the 3-sphere S^3 is homeomorphic to $\mathbb{R}^3$. This means that any knot in $\mathbb{R}^3$ can be considered as a knot in S^3 as well. The fundamental group of the complement to the image of a knot in $\mathbb{R}^3$ is isomorphic to the fundamental group of its complement in S^3. Indeed, if we represent the complement to the image of a knot in S^3 as the union of a small ball centered at the puncture and the complement to the puncture, then the intersection of these two open sets is homotopy equivalent to the 2-sphere

S^2, whence simply connected. Corollary 7.4 now asserts the isomorphism of the two fundamental groups. Considering S^3 instead of $\mathbb{R}^3$ will be sometimes more convenient for us.

7.2.1 Fundamental groups of the complements to simplest knots and links

An *unknotted* closed curve in $\mathbb{R}^3$ is the knot isotopic to the standard embedding of S^1 into the horizontal plane in $\mathbb{R}^3$. The fundamental group of its complement is isomorphic to $\mathbb{Z}$. Indeed, surround the circle by a sufficiently large 2-sphere. Then the exterior of the sphere can be contracted to the sphere, while the complement to the knot inside the sphere can be contracted to the union of the sphere and the vertical diameter in it. Hence, the complement to the circle is homotopy equivalent to the connected sum of the sphere S^2 and the circle S^1, and its fundamental group is $\mathbb{Z}$.

Exercise 7.5 *Show that the fundamental group of the complement in $\mathbb{R}^3$ of two circles in the horizontal plane is $\mathbb{Z} * \mathbb{Z}$.*

In contrast, if we consider a horizontal and a vertical circle, each passing through the center of the other one, then their complement in $\mathbb{R}^3$ is contractible to the torus $S^1 \times S^1$. The fundamental group of the complement is then $\mathbb{Z} \oplus \mathbb{Z}$, which proves that these two links are not ambient isotopic.

Another class of knots with easily computable fundamental groups is formed by so-called torus knots. Let's think of S^3 as of the unit sphere in $\mathbb{C}^2$. Let p, q be relatively prime positive integers. The (p, q)-torus knot in S^3 is the mapping $S^1 \to S^3$ given by

$$\varphi \mapsto (z_1, z_2) = (e^{ip\varphi}, e^{iq\varphi}), \qquad \varphi \in [0, 2\pi].$$

The image of this mapping belongs to the torus $S^1 \times S^1$ given by the equations $|z_1| = |z_2| = 1$. It winds p times around the first ("parallel") circle and q times around the second ("meridian") circle.

Example 7.6 *For any positive integer q, the $(1, q)$-torus knot is an unknot.*

Example 7.7 *The $(2, 3)$-torus knot is the trefoil.*

Theorem 7.8 *The fundamental group of the complement to the (p, q)-torus knot in S^3 admits the presentation $\langle a, b | a^p = b^q \rangle$.*

Proof. The torus $|z_1| = |z_2| = 1$ in S^3 splits the 3-sphere into two solid tori $D^2 \times S^1$. Let's contract the complement to the (p, q)-torus knot in S^3 to the image of an embedding of the surface $X_{p,q}$ constructed as follows.

Let X_m, $m = 1, 2, \dots$, denote the quotient of the cylinder $S^1 \times [0, 1]$ modulo the group $\mathbb{Z}/m\mathbb{Z}$-action, where the action is trivial on all points but the lower boundary $S^1 \times \{0\}$, where it is generated by rotation by the angle $2\pi/m$. The topological space X_m can be thought of as the result of identifying the boundary stars $\{0\} \times S_m$ and $\{1\} \times S_m$ in the product $[0, 1] \times S_m$ after rotating a star S_m by the angle $2\pi/m$; here S_m denotes the m-star graph, which consists of a central vertex connected by m edges with m other vertices. The noncentral vertices of the star graph S_m, after multiplication by $[0, 1]$, rotation and gluing, form the *boundary* circle ∂X_m.

The fundamental group $\pi_1(X_m)$ coincides with that of the cylinder and is isomorphic to $\mathbb{Z}$.

The topological space $X_{p,q}$ is obtained from X_p and X_q by gluing them together along a homeomorphism of their boundary circles $\partial X_p = S^1$ and $\partial X_q = S^1$.

The fundamental group $\pi_1(X_{p,q})$ admits a presentation of the form $\langle a, b | a^p = b^q \rangle$. Indeed, let a be a generator of the fundamental group $\pi_1(X_p)$, and let b be a generator of the fundamental group $\pi_1(X_q)$. The topological space $X_{p,q}$ can be naturally represented as the union of an open neighborhood of X_p and an open neighborhood of X_q in such a way that the intersection of the two neighborhoods is a cylinder. The p-times multiple of a is homotopic to the boundary of X_p, which coincides with boundary of X_q, homotopic, in turn, to the q-times multiple of b (taken with an appropriate orientation). Now Van Kampen's theorem gives the answer.

The topological space $X_{p,q}$ can be embedded into S^3 in the following way. Its X_p-part is embedded into one of the two solid tori, so that the central circle of X_p is taken to the central circle of the solid torus, while its X_q-part is embedded into the other solid torus, its central circle taken to the central circle of the solid torus. Under such an embedding, the common boundary $\partial X_p = \partial X_q$ is taken to the torus $S^1 \times S^1$, which is the common boundary of the two solid tori. We require this embedding being the half-twisted original (p, q)-torus knot, that is, the mapping $\varphi \mapsto (e^{pi\varphi + \pi i/p}, e^{qi\varphi})$.

The complement to the original torus knot can be contracted to the image of $X_{p,q}$ under this embedding. The contraction goes on inside each of the two solid tori, while the surface of the boundary torus with the torus knot eliminated is contracted to the image of $\partial X_p = \partial X_q$. This proves the theorem.

7.2.2 Wirtinger presentations

The Wirtinger presentation of the fundamental group of the complement to a knot expresses it in terms of a knot diagram. A plane knot diagram splits into connected intervals, each contained between two consecutive undercrossings. We consider these connected intervals as generators of the fundamental group. To each such interval a loop in the complement to the image of the knot in $\mathbb{R}^3$ with the base point x_0 above the plane of the diagram is associated in a natural way, as shown in the picture.

Relations are associated to the crossings of the diagram. If a is the overcrossing interval at an intersection point, and b, c are the two undercrossing intervals situated and oriented as shown in the picture. Then it is easy to check the relation

$$aba^{-1} = c$$

in $\mathbb{R}^3 \setminus K$, K being the image of the knot. Wirtinger's theorem states that the fundamental group $\pi_1(\mathbb{R}^3 \setminus K, x_0)$ is generated by the homotopy classes of loops associated to the intervals, while the relations associated to the crossings form a complete set of relations.

Example 7.9 *For two linked circles, the Wirtinger presentation has the form*

$$\pi_1(\mathbb{R}^3 \setminus L) = \langle a, b | aba^{-1} = b, bab^{-1} = a \rangle.$$

Either of the two relations means that a and b commute, so that the fundamental group is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$.

Example 7.10 *For the trefoil, its standard plane diagram has three intervals, and the three crossings of the diagram produce the three relations, so that,*

$$\pi_1(\mathbb{R}^3 \setminus K, x_0) = \langle a, b, c | aba^{-1} = c, bcb^{-1} = a, cac^{-1} = b \rangle.$$

Exercise 7.11 *Show that in the above presentation of the fundamental group of the complement to the trefoil knot in $\mathbb{R}^3$ each of the three relations is a corollary of the other two.*

Prove that any relation in the Wirtinger presentation of the fundamental group to the complement of a knot can be expressed in terms of the other relations, so that any relation in the Wirtinger presentation can be omitted.

Show that the Wirtinger presentation of the fundamental group of the complement to the trefoil gives a group isomorphic to $\langle a, b | a^2 = b^3 \rangle$.

Exercise 7.12 *By constructing, for a given knot diagram, a 2-dimensional simplex to which the complement to the knot can be contracted, prove that the Wirtinger presentation indeed is a presentation of the fundamental group of the complement to the knot.*

Chapter 8

Homology of chain complexes

The fundamental group of a topological space is a rather fine homotopy invariant. However, it possesses two major weaknesses. Firstly, as we have seen for simplicial complexes, it reflects the structure of the 2-skeleton of the complex only. Simplexes of dimension greater than 2 remain invisible for the fundamental group. Another weakness comes from the fact that it is not an easy task to compare fundamental groups of two topological spaces. If the two groups are given by their presentations in terms of generators and relations, then establishing their isomorphism or nonisomorphism becomes a nontrivial algorithmic problem.

Homology, which we are starting to study, provide a homotopy invariant that takes into account simplexes of all dimensions. On the other hand, comparing homology of two spaces also is easy. In the simplest case of homology with coefficients in the field of real numbers $\mathbb{R}$, this comparison reduces to that of the dimensions of homology vector spaces.

The idea of homology is justified by a natural philosophy argument:

No boundary has a boundary.

Any political map confirms this statement: the boundary of a state has no boundary even in the case when the state is not connected and contains enclaves.

8.1 Chain complexes

We start with the simplest case of introducing homology, namely, with homology with coefficients in a field of characteristic 0. Real numbers will do, and we consider below vector spaces over $\mathbb{R}$ without mentioning this fact explicitly. Homology appeared first as a characteristic of topological spaces, but their definition has nothing to do with topology.

Let $V_0, V_1, V_2, \dots$ be a sequence of vector spaces, and let $\partial_1 : V_1 \to V_0$, $\partial_2 : V_2 \to V_1, \dots$, $\partial_i : V_i \to V_{i-1}$, $\dots$ be a sequence of linear operators. Whenever possible, we assume that the vector spaces V_i are finite dimensional, but we will not be able to follow this restriction everywhere. We say that the sequence of spaces and operators

$$\dots \xrightarrow{\partial_{i+1}} V_i \xrightarrow{\partial_i} V_{i-1} \xrightarrow{\partial_{i-1}} \dots \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \tag{8.1}$$

is a *chain complex* if for all i we have $\partial_{i-1} \circ \partial_i = 0$ for the composition operator $\partial_{i-1} \circ \partial_i : V_i \to V_{i-2}$. In other words, for each i, the subspace $\operatorname{Im} \partial_{i+1} \subset V_i$ is contained in the subspace $\operatorname{Ker} \partial_i \subset V_i$: $\operatorname{Im} \partial_{i+1} \subset \operatorname{Ker} \partial_i \subset V_i$.

Be careful: chain complexes have nothing in common with simplicial complexes, at least, formally. However, simplicial complexes generate a huge series of important examples of chain complexes.

It is convenient to complete each chain complex with the mapping $\partial_0 : V_0 \to 0$ and, if the complex is finite and V_n is the space with the largest index, then to add the mapping $\partial_{n+1} : 0 \to V_n$, so that a finite chain complex will look like

$$0 \xrightarrow{\partial_{n+1}} V_n \xrightarrow{\partial_n} V_{n-1} \ldots \xrightarrow{\partial_{i+1}} V_i \xrightarrow{\partial_i} V_{i-1} \xrightarrow{\partial_i} \ldots \xrightarrow{\partial_2} V_1 \xrightarrow{\partial_1} V_0 \xrightarrow{\partial_0} 0 \quad (8.2)$$

(zeroes denote the 0-dimensional vector space $\mathbb{R}^0$). Note that such a completion preserves the property of a sequence to be a chain complex.

Example 8.1 *An exact sequence is an example of a complex. Recall that a sequence (8.1) of vector spaces and their linear maps is said to be* exact *if* $\operatorname{Im} \partial_{i+1} = \operatorname{Ker} \partial_i$ *for each* i.

Exercise 8.2 *Prove that any sequence of vector spaces and their linear maps of the form*

$$0 \longrightarrow V_1 \xrightarrow{d} V_0 \longrightarrow 0$$

is a complex.

Example 8.3 *Let* $\mathbb{R}[x]$ *be the ring of polynomials with real coefficients in one variable. Set* $V_1 = \mathbb{R}[x]$ *and* $V_0 = \mathbb{R}[x]dx$*, the vector space of* 1*-forms on the line with polynomial coefficients. Then the differential operator* $d : p(x) \mapsto p'(x)dx$ *takes* V_1 *to* V_0*, and the sequence*

$$0 \longrightarrow V_1 \xrightarrow{d} V_0 \longrightarrow 0$$

is a chain complex.

This example admits an easy generalization to spaces of polynomials in any number n *of variables and spaces of differential forms of degrees* $0, 1, 2, \ldots, n$*. The corresponding boundary operators are presented by the operator* d *taking a differential* k*-form to a differential* $(k+1)$*-form.*

If, instead of polynomials and differential forms with polynomial coefficients on $\mathbb{R}^n$*, we consider smooth functions on a manifold* X *and smooth differential forms, then the resulting complex is called the* de Rham complex *of* X.

Example 8.4 *The following version of the previous example is useful. Set* $V_1 = \mathbb{R}[x]e^{-x^2/2}$ *and* $V_0 = \mathbb{R}[x]e^{-x^2/2}dx$. *Then the differential operator* $d : p(x)e^{-x^2/2} \mapsto d(p(x)e^{-x^2/2}) = (p'(x) - xp(x))e^{-x^2/2}dx$ *takes* V_1 *to* V_0, *and the sequence*

$$0 \longrightarrow V_1 \xrightarrow{d} V_0 \longrightarrow 0$$

is a complex.

This example can be generalized in many directions. Firstly, the polynomial $-x^2/2$ *in the exponent can be replaced by an arbitrary polynomial. Secondly, we can consider complexes of differential forms in several variables,* $x \in \mathbb{R}^n$, *having coefficients* $p(x)e^{f(x)}$, *where* f *is a given polynomial, and* p *is an arbitrary polynomial. Moreover, for an arbitrary smooth function* $f : X \to \mathbb{R}$ *on a manifold* X *we can consider the chain complex of differential forms whose coefficients are* $p(x)e^{f(x)}$, *for arbitrary smooth functions* p.

Exercise 8.5 *Prove that the above examples indeed provide complexes.*

A huge series of examples of complexes is provided by simplicial complexes, and we devote a special section to the corresponding definition.

8.2 The chain complex of a simplicial complex

Let Δ^n be a simplex of dimension $n = \dim \Delta$. If n is greater than 0, then there are two ways to pick an *orientation* of Δ. Namely, pick two numberings of the vertices of Δ by numbers from 0 to n. We say that these two numberings *define the same orientation* if the permutation of $\{0, 1, \dots, n\}$ taking the first numbering to the second one is even; otherwise, we say that the two numberings define opposite orientations. (Recall that a permutation is *even* if it can be represented as a product of an even number of transpositions, and it is *odd* otherwise. For $n > 1$, half of the permutations in the symmetric group S_n are even; they form a subgroup $A_n \subset S_n$).

For the future purposes, we also extend in an obvious way the notion of orientation to the case when the vertices of the simplex are numbered by arbitrary pairwise distinct nonnegative integers, not necessarily varying from 0 to n.

Now associate a chain complex to the simplex Δ^n in the following way. Pick an orientation of each face of Δ^n. An oriented simplex of dimension k with vertices numbered $a_0, \dots, a_k$ will be denoted by $[a_0, \dots, a_k]$. Denote by $C_k(\Delta^n)$, $k > 0$, the vector space over $\mathbb{R}$ freely spanned by all oriented k-dimensional faces of Δ^n, and let $C_0(\Delta^n) \equiv \mathbb{R}^{n+1}$ be freely spanned by the vertices of Δ^n. For each simplex δ with a chosen orientation, we denote by $-\delta$ the same simplex with the opposite orientation, so that the sum of the two oriented simplexes is 0. For a simplex δ of dimension 0, $-\delta$ is just the formal opposite of the corresponding element of $C_0(\Delta^n)$.

Define the boundary operator $\partial_n : C_n(\Delta^n) \to C_{n-1}(\Delta^n)$ as follows: the boundary of the n-dimensional simplex is the sum of its faces of dimension $n-1$, taken with the signs $+$ or $-$:

$$\partial_n : \Delta_n = [0, 1, 2, \ldots, n] \mapsto \sum_{i=0}^{n} (-1)^i [0, 1, \ldots, \hat{\imath}, \ldots, n],$$

where notation $\hat{\imath}$ symbolizes the absence of the number i. Similarly, we define the boundary operator $\partial_k : C_k(\Delta^n) \to C_{k-1}(\Delta^n)$ by the equation

$$\partial_k : [v_0, v_1, \ldots, v_k] \mapsto \sum_{i=0}^{n} (-1)^i [v_0, v_1, \ldots, \hat{v}_i, \ldots, v_k]. \tag{8.3}$$

Example 8.6 *Let $n = 2$, and consider the simplex Δ^2 (a triangle) endowed with an orientation given by a numbering of its vertices with numbers $0, 1, 2$. Pick the orientations of the edges of Δ^2 orienting each of them from the smaller to the bigger number, so that $C_1(\Delta^2) = \langle [0,1], [1,2], [0,2] \rangle$. Then we have*

$$\partial_2 [0, 1, 2] = [1, 2] - [0, 2] + [0, 1].$$

It is easy to see that

$$\partial_1 \circ \partial_2 [0, 1, 2] = \partial_1([1, 2] - [0, 2] + [0, 1]) = ([2] - [1]) - ([2] - [0]) + ([1] - [0])$$

is zero, and hence the sequence of vector spaces and linear operators

$$0 \longrightarrow C_2(\Delta^2) \xrightarrow{\partial_2} C_1(\Delta^2) \xrightarrow{\partial_1} C_0(\Delta^2) \longrightarrow 0$$

indeed is a chain complex.

Exercise 8.7 *Prove that for any $n = 0, 1, 2, \ldots$ the sequence of vector spaces and linear operators $C_k(\Delta^n)$, $\partial_k : C_k(\Delta^n) \to C_{k-1}(\Delta^n)$ is a chain complex.*

The above notions can be extended to arbitrary simplicial complexes. Namely, for a simplicial complex X of dimension n, choose a numbering of its vertices and pick an orientation of each of its simplexes of positive dimension. Define $C_k(X)$ as the vector space freely spanned by all the oriented simplexes of dimension k. Define the boundary operator ∂_k on a simplex $[v_0, v_1, \dots, v_k]$ as in Eq. (8.3), and extend this operator to the whole vector space $C_k(X)$ by linearity. Then we obtain a sequence of vector spaces and linear operators

$$0 \xrightarrow{\partial_{n+1}} C_n(X) \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{i+1}} C_i(X) \xrightarrow{\partial_i} \dots \xrightarrow{\partial_2} C_1(X) \xrightarrow{\partial_1} C_0(X) \xrightarrow{\partial_0} 0 \tag{8.4}$$

that is a chain complex, by the previous exercise. We denote this chain complex by $\mathcal{C}(X)$.

8.3 Chains, cycles, boundaries, and homology

For a simplicial complex (8.1), the quotient vector space $\operatorname{Ker}\partial_k / \operatorname{Im}\partial_{k+1}$ is called the k th *homology vector space* and denoted by H_k. In particular, for the chain complex of a simplicial complex X (8.2) it is denoted by $H_k(X)$. The direct sum $\oplus_{k\geq 0} H_k(X)$ is denoted by $H_*(X)$.

In addition, the vector space V_k (resp., $C_k(X)$) is called the *space of k-chains*, the vector space $\operatorname{Ker}\partial_k \subset V_k$ is the *space of k-cycles*, and the vector space $\operatorname{Im}\partial_{k+1} \subset V_k$ is the *space of k-boundaries.*

Example 8.8 *Show that, for an n-dimensional simplex Δ^n, $n \geq 0$, we have*

$$H_k(\Delta^n) = \begin{cases} \mathbb{R} & k = 0 \\ 0 & k > 0. \end{cases}$$

Now, let's compute the homology of the boundary $\partial\Delta^2$ of the simplex Δ^2. This boundary consists of three vectors $[0,1], [1,2], [0,2]$, so that we have $C_1[\partial\Delta^2] = \langle [0,1], [1,2], [0,2] \rangle$, $C_0[\partial\Delta^2] = \langle [0], [1], [2] \rangle$. The boundary operator $\partial_1 : C_1[\partial\Delta^2] \to C_0[\partial\Delta^2]$ acts as follows:

$$\partial_1 : x[0,1] + y[1,2] + z[0,2] \mapsto -(x+z)[0] + (x-y)[1] + (z+y)[2].$$

Its image is 2-dimensional and coincides with the hyperplane in $C_0[\partial\Delta^2]$ consisting of the vectors with zero sum of the coordinates. Its kernel is 1-dimensional and is spanned by the vector $[0,1] + [1,2] - [0,2]$, a cycle. Therefore, $H_1[\partial\Delta^2] \equiv \mathbb{R}^1$ and $H_0[\partial\Delta^2] \equiv \mathbb{R}^1$.

Exercise 8.9 *Verify that any two simplicial decompositions of the circle* S^1 *produce the same homology.*

Exercise 8.10 *Compute the homology of the graph*

- *consisting of a single vertex and two loops;*
- *consisting of two vertices and three edges connecting them.*

Exercise 8.11 *Compute the homology of*

- *the boundary of the simplex* Δ^3*;*
- *the boundary of the simplex* Δ^n *of arbitrary dimension* $n \geq 0$.

An important number characterizing a finite complex consisting of finite dimensional vector spaces is its Euler characteristic. By definition, the *Euler characteristic* of a complex (8.2) is the alternating sum of dimensions of the vector spaces entering it:

$$\chi = \dim V_0 - \dim V_1 + \dim V_2 - \cdots + (-1)^n \dim V_n.$$

This definition can also be extended to chain complexes allowing infinite dimensional vector spaces, but with finite dimensional homology. Namely, we set

$$\chi = \dim H_0 - \dim H_1 + \dim H_2 - \cdots + (-1)^n \dim H_n.$$

Exercise 8.12 *Prove that for a finite chain complex consisting of finite dimensional vector spaces the two definitions of the Euler characteristic coincide.*

Exercise 8.13 *Find the Euler characteristics of the chain complexes in the above examples.*

Exercise 8.14 *Prove that a differential* 1*-form* $q(x)e^{-x^2/2}dx$ *is the differential of a function* $p(x)e^{-x^2/2}$ *if and only if*

$$\int_{-\infty}^{\infty} q(x)e^{-x^2/2}dx = 0.$$

Using this statement, find the homology of the chain complexes in Example 8.4 on the line for the polynomial $f(x) = -x^2/2$*;*

Exercise 8.15 *Find the homology of the chain complexes in Example 8.4 on the line for*

- *the polynomial* $f(x) = x^3/3$;
- *the polynomial* $f(x) = (x^3 - x)/3$;
- *arbitrary polynomial* $f(x)$.

Chapter 9

Homology of simplicial complexes

Homology is a way to replace complicated nonlinear topological objects (which are usually manifolds or algebraic varieties) with much simpler linear algebraic ones (vector spaces). Nonlinear continuous mappings induce linear mappings of homology. In order to prove that homology depends on the topological type rather than on the chosen simplicial subdivisions, we must prove that homology of two simplicial subdivisions of one and the same topological space are the same. Moreover, homology is the same for homotopy equivalent topological spaces.

9.1 Action of continuous mappings on homology of simplicial complexes

Recall that a simplicial map $f : X \to Y$ is defined by a mapping from the set of 0-simplices in X to the set of 0-simplices in Y such that the set of vertices of any simplex in X is taken to a set of vertices of some simplex in Y. For each $k = 0, 1, 2 \dots$, a simplicial map f defines a linear mapping $f_* : C_k(X) \to C_k(Y)$ of the vector space of k-chains in X to the vector space of k-chains in Y. The mapping takes a generator $[v_0, \dots, v_k]$ of $C_k(X)$ to 0 if the 0-simplices $f(v_0), \dots, f(v_k)$ are vertices of a simplex of dimension less than k, and to the simplex $[f(v_0), \dots, f(v_k)]$ provided the latter is a simplex of dimension k. Note that the latter simplex may be either a basic simplex in $C_k(Y)$, or a negative basic simplex, depending on whether the orientation $[f(v_0), \dots, f(v_k)]$ coincides with the chosen for the basic one. The mapping f_* is extended to linear combinations of basic simplices by linearity.

Lemma 9.1 *The mapping f_* commutes with the boundary operators in $\mathcal{C}(X)$ and $\mathcal{C}(Y)$, respectively.*

Indeed, for a k-simplex whose vertices are taken by f to the vertices of a simplex of dimension less than $k - 1$ this statement is obvious: the mapping f_* takes such a simplex, as well as its boundary, to 0. If the vertices of a k-simplex are taken by f to the vertices of a simplex of the same dimension, then the same is true for its face of arbitrary dimension, and in particular, for the boundary faces of the simplex. Finally, suppose the vertices $\{0, 1, \dots, k\}$ of a k-simplex are taken by f to the vertices of a $(k-1)$-simplex, so that $f(0) = f(1)$. Then

$$\begin{aligned} f_*\partial_k[0,1,\ldots,k] &= f_*\sum_{i=0}^{k}(-1)^i[0,1,\ldots,\hat{i},\ldots,k] \\ &= [f(0),f(2),\ldots,f(k)] - [f(1),f(2),\ldots,f(k)] \\ &= 0 \end{aligned}$$

as desired.

Corollary 9.2 *A simplicial map $f : X \to Y$ of simplicial complexes descends to a linear mapping $f_* : H_k(X) \to H_k(Y)$ in homology.*

9.2 Topological invariance of homology of simplicial complexes

Theorem 9.3 *Suppose two simplicial complexes X, Y are simplicial divisions of one and the same topological space. Then they have the same homology.*

In order to prove this theorem, we first prove the following

Lemma 9.4 *Let X' be the barycentric subdivision of a simplicial complex X. Then the homology of the two complexes coincide, $H_k(X') = H_k(X)$ for $k = 0, 1, 2, \ldots$.*

Recall that the barycentric subdivision of a simplicial complex consists of barycentric subdivisions of its simplices. The *barycentric subdivision* of a k-simplex is its splitting into $(k+1)!$ k-simplices constructed as follows. The new vertices of the barycentric subdivision are the barycenters of all the faces of positive dimension of the simplex, and there are $2^{k+1} - k - 2$ new vertices. Each barycenter belongs to the corresponding face. The k-simplices of the barycentric subdivision are in one-to-one correspondence with the set of permutations of the set $\{v_0, \ldots, v_k\}$ of vertices of the initial simplex. Namely, a permutation $(v_0', \ldots, v_k')$ defines the k-simplex whose first vertex is v_0', the second vertex is the barycenter of the face $[v_0', v_1']$, the third vertex is the barycenter of the face $[v_0', v_1', v_2']$, and so on.

Example 9.5 *The barycentric subdivision of a 0-simplex is the simplex itself. The barycentric subdivision of a 1-simplex consists of two line segments, each connecting a vertex of the initial simplex with its middle point.*

The barycentric subdivision of a triangle consists of 6 *triangles formed by connecting the barycenter of the initial triangle to its vertices and the middle points of the edges.*

In order to prove Lemma 9.4, we construct a simplicial mapping $i : X' \to X$ of the barycentric subdivision X' of X to X and show that the homology mapping i_* is invertible. Pick a numbering of the vertices of X. Take a simplex Δ^k in X and pick the number of each additional vertex in its barycentric subdivision $(\Delta^k)'$ in such a way that this number belongs to the set of numbers of the minimal face of Δ^k containing this additional vertex. By definition, the simplicial mapping i takes each vertex of the barycentric subdivision X' to the vertex of X having the same number.

Among the k-dimensional simplexes of the barycentric subdivision $(\Delta^k)'$ of a given k-simplex Δ^k in X, there is a unique simplex having the same set of vertex numbers as Δ^k itself. Indeed, in order to find such a simplex take the barycenter of Δ^k for the first vertex, then take the $(k-1)$-face of Δ^k not containing the vertex with the number chosen for the barycenter of Δ^k, and so on.

Now it is easy to check that the linear mapping $j : C^k(X) \to C^k(X')$ taking each k-simplex in X to the sum of the k-simplexes of its barycentric subdivision descends to a linear mapping $j_* : H_k(X) \to H_k(X')$, which is inverse to $i_* : H_k(X') \to H_k(X)$.

Definition 9.6 *Let X, Y be two simplicial complexes. Each point $y \in Y$ belongs to the interior of exactly one simplex in Y. Let $f : X \to Y$ be an arbitrary continuous mapping. We say that a simplicial mapping $\varphi : X \to Y$ is a* simplicial approximation *of f if for each point $x \in Y$ its image $\varphi(x)$ belongs to the simplex in Y corresponding to the point $f(x)$.*

Lemma 9.7 *Any simplicial approximation φ of a continuous mapping f is homotopic to f.*

Indeed, the two mappings are connected by the homotopy $F : X \times [0,1] \to Y$, $F(x,t) \equiv (1-t)f(x) + t\varphi(x)$ (the operations are well-defined inside each simplex in Y, whence totally in Y), $F(x,0) = f(x)$, $F(x,1) \equiv \varphi(x)$.

Lemma 9.8 *Let $\varphi : X \to Y$, $\psi : Y \to Z$ be simplicial approximations of continuous maps $f : X \to Y$, $g : Y \to Z$, respectively. Then $\psi \circ \varphi : X \to Z$ is a simplicial approximation of the composition $g \circ f$.*

For a simplicial complex X, denote by $X^{(m)}$ its m th barycentric subdivision.

Theorem 9.9 *Let X, Y be simplicial complexes, and let $f : X \to Y$ be a continuous mapping. Then, for some $m \geq 0$, there is a simplicial approximation $\varphi : X^{(m)} \to Y$.*

We refer the reader to [Prasolov] for a proof of this theorem.

9.3 Homotopy invariance of homology of simplicial complexes

Theorem 9.10 *If two simplicial complexes X, Y are simplicial divisions of homotopy equivalent topological spaces, then they have the same homology.*

The proof of this theorem is based on the following

Lemma 9.11 *If $f, g : X \to Y$ are two homotopic continuous mappings of two simplicial complexes, then they induce the same linear mappings $f_*, g_* : H_*(X) \to H_*(Y)$ on homology.*

Indeed, for this purpose it suffices to construct a representation of the product $X \times [0, 1]$ as a simplicial complex, which coincides with the complex X when restricted to both $X \times \{0\}$ and $X \times \{1\}$. (In order to construct such a simplicial complex it suffices to represent in this form the product $\Delta \times I$ for a simplex Δ.)

Exercise 9.12 *Construct a simplicial division of the product $\Delta^k \times \Delta^m$ of two simplices adding no vertices. How many $(n+m)$-simplexes does it have?*

Corollary 9.13 *For $m \neq n$, the vector spaces $\mathbb{R}^m$ and $\mathbb{R}^n$ are not homeomorphic to one another.*

Indeed, suppose the converse, and let $f : \mathbb{R}^m \to \mathbb{R}^n$ be a homeomorphism. Then f is a homeomorphism between $\mathbb{R}^m \setminus \{0\}$ and $\mathbb{R}^n \setminus \{f(0)\}$. The latter spaces are homotopy equivalent to the spheres S^{m-1} and S^{n-1}, respectively. For $m \neq n$ these two spheres have different homology, and we are done.

Chapter 10

Homology with coefficients in Abelian groups

Homology of chain complexes we considered in the previous sections are homology with coefficients in a field. Specifically, it is the field $\mathbb{R}$ of real numbers. If we replace the field $\mathbb{R}$ with another field of characteristic 0, say, the field $\mathbb{C}$ of complex numbers or the field $\mathbb{Q}$ of rationals, then the homology $H_k(X)$ of a simplicial complex X will be essentially the same: these will be vector spaces over the corresponding field of the same dimension as in the case of reals. The dimension of the space of homology is the only significant information.

Considering homology with coefficients in Abelian groups allows one to replace vector spaces with Abelian groups that sometimes carry more subtle information and distinguish between topological spaces that cannot be distinguished by homology with coefficients in $\mathbb{R}$.

For a finite simplicial complex X and an Abelian group G, denote by $C_k(X, G)$ the *group of chains with coefficients in* G,

$$C_k(X, G) = \left\{ \sum a_i \Delta_i^k \right\},$$

where $a_i \in G$ and the summation is carried over all basic oriented k-dimensional simplices in X. The definition of the boundary operators $\partial_k : C_k(X, G) \to C_{k-1}(X, G)$ as well as verification of the fact that we obtain in this way a chain complex remain the same as in the case of real coefficients. Hence one may define the homology groups $H_k(X; G)$ with ceofficients in G for all k.

Theorem 10.1 *If G is the additive group of a ring, then the homology $H_k(X; G)$ are topology and homotopy invariants.*

10.1 Chain complexes of Abelian groups

Let $C_0, C_1, C_2, \ldots,$ be a sequence of finitely generated Abelian groups. Recall that each such Abelian group G is isomorphic to a group of the form $\mathbb{Z}^r \oplus (\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k})$, where the positive integers $p_1, \ldots, p_k$ are powers of primes. The summand $\mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k}$ is denoted by $\mathrm{Tors}(G)$, so that the group has the form $G = \mathbb{Z}^r \oplus \mathrm{Tors}(G)$; it is well-defined and consists of elements of finite order in G. On the contrary, the summand $\mathbb{Z}^r$ is not well-defined, since the sum of an element of infinite order and an element of finite order is an element of infinite order as well. Only the degree r of the summand $\mathbb{Z}$ is well-defined; it is called the *rank* of the Abelian group G. Thus, there is an exact sequence of Abelian groups of the form

$$0 \longrightarrow \mathbb{Z}_{p_1} \oplus \cdots \oplus \mathbb{Z}_{p_k} = \mathrm{Tors}(G) \longrightarrow G \longrightarrow \mathbb{Z}^r \longrightarrow 0.$$

A sequence $\mathcal{C}$ of Abelian groups and their homomorphisms

$$0 \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \tag{10.1}$$

is called a *chain complex of Abelian groups* if $\operatorname{Im}\partial_{k+1} \subset \operatorname{Ker}\partial_k$ for all $k = 0, 1, 2, \dots$. The quotient group $\operatorname{Ker}\partial_k / \operatorname{Im}\partial_{k+1}$ is called the k th homology group of the chain complex $\mathcal{C}$ and is denoted by $H_k(\mathcal{C})$.

The following examples show that homology of chain complexes of Abelian groups could be more sophisticated than that of chain complexes of vector spaces.

Example 10.2 *Consider the chain complex*

$$0 \longrightarrow C_1 = \mathbb{Z} \stackrel{\partial_1 = \times p}{\longrightarrow} C_0 = \mathbb{Z} \longrightarrow 0,$$

where the homomorphism ∂_1 is defined as multiplication by an integer p. Then ∂_1 has zero kernel, and its image consists of the subgroup $p\mathbb{Z} \subset \mathbb{Z}$. Therefore, $H_1(\mathcal{C}) = 0$ and $H_0(\mathcal{C}) = \mathbb{Z}/p\mathbb{Z} = \mathbb{Z}_p$.

Example 10.3 *Consider the chain complex*

$$0 \longrightarrow C_1 = \mathbb{Z}_p \stackrel{\partial_1 = \times q}{\longrightarrow} C_0 = \mathbb{Z}_p \longrightarrow 0, \tag{10.2}$$

where p is a positive integer and the homomorphism ∂_1 is defined as multiplication by an integer q. Then if q is relatively prime to p, then the chain complex is acyclic, that is, $H_1(\mathcal{C}) = H_0(\mathcal{C}) = 0$. On the contrary, if $q = p$, then we have $H_1(\mathcal{C}) = H_0(\mathcal{C}) = \mathbb{Z}_p$.

Exercise 10.4 *Find the homology of chain complex (10.2) in the case where p and q are not coprime, but $q \neq p$.*

The notion of Euler characteristic can be defined for chain complexes of Abelian groups as well. Similarly to the case of chain complexes over $\mathbb{R}$, the *Euler characteristic* $\chi(\mathcal{C})$ of a chain complex $\mathcal{C}$ of Abelian groups is defined either as the alternating sum of the ranks of groups of chains $\sum_{i=0}^n (-1)^i \operatorname{rank}(C_k)$, or as the alternating sum of the ranks of homology groups $\sum_{i=0}^n (-1)^i \operatorname{rank}(H_k)$.

Exercise 10.5 *Prove that these two definitions yield the same number.*

10.2 Fundamental class and the degree of a continuous mapping

The notion of a surface can be generalized to simplicial complexes of dimensions greater than 2.

A simplicial complex X is called a connected *pseudomanifold of dimension n* if

- each simplex in X is a face of an n-dimensional simplex;
- each simplex of dimension $n-1$ in X is a face of exactly two simplices of dimension n;
- the complex X is *strongly connected* meaning that for any two n-simplices Δ_a^n, Δ_b^n in X there is a tuple of n-simplices $\Delta_1^n, \dots, \Delta_k^n$ such that any two consecutive simplices in the extended sequence $\Delta_a^n, \Delta_1^n, \dots, \Delta_k^n, \Delta_b^n$ have a common $(n-1)$-dimensional face.

For an n-dimensional pseudomanifold, we define its $\mathbb{Z}_2$-*fundamental class* as the sum of all its n-simplices. The following statement is obvious.

Lemma 10.6 *For an n-dimensional pseudomanifold X, we have $H_n(X, \mathbb{Z}_2) = \mathbb{Z}_2$ and the $\mathbb{Z}_2$-fundamental class of X spans its n th homology with coefficients in $\mathbb{Z}_2$.*

Indeed, the boundary of the fundamental class is 0, since each $(n-1)$-dimensional simplex enters this boundary exactly twice. On the other hand, any other nonzero n-dimensional $\mathbb{Z}_2$-chain has a non-zero boundary: take an n-simplex Δ_a^n that belongs to this chain, and an n-simplex Δ_b^n that does not, and consider a tuple of n-simplices as in the definition of strong connectedness. Then there are two neighboring simplices in this tuple, one belonging to the chain, and the other one not. Their common $(n-1)$-face enters the boundary of the chain with coefficient $1 \in \mathbb{Z}_2$, whence the boundary is nonzero.

The fundamental class of a pseudomanifold can also be defined over $\mathbb{Z}$, but in order to do that, we need first to introduce the notion of orientability. We say that the orientations of two n-simplices with a common $(n-1)$-dimensional face are *compatible* if this face enters the boundary of these simplices over $\mathbb{Z}$ with opposite signs. An n-dimensional pseudomanifold is said to be *orientable* if one can choose orientations of each n-simplex in X in such a way that orientations of any two simplices with a common $(n-1)$-face are compatible. We call such a choice an *orientation* of the pseudomanifold. If a pseudomanifold X of dimension $n > 0$ is orientable, then it admits two different orientations. Picking one of the two orientations, we obtain an *oriented pseudomanifold*. The *$\mathbb{Z}$-fundamental class* of an oriented n-dimensional pseudomanifold is the sum of all oriented n-simplices in it with coefficients $1 \in \mathbb{Z}$.

Lemma 10.7 *For an oriented n-dimensional pseudomanifold X, we have $H_n(X, \mathbb{Z}) = \mathbb{Z}$ and the $\mathbb{Z}$-fundamental class of X spans its n th homology with coefficients in $\mathbb{Z}$.*

The proof is exactly the same as in the $\mathbb{Z}_2$-case.

Lemma 10.8 *For a non-orientable n-dimensional pseudomanifold X, we have $H_n(X, \mathbb{Z}) = 0$.*

Indeed, nonorientability of X means that there is a tuple of n-simplices $\Delta_1, \dots, \Delta_k$ in X such that

- each pair of simplices (Δ_1, Δ_2), (Δ_2, Δ_3), $\dots(\Delta_{k-1}, \Delta_k)$, (Δ_k, Δ_1), has a common $(n-1)$-face;
- there is no way to choose compatible orientations for all the simplices in the sequence;

(otherwise we would be able to choose compatible orientations for all the simplices in S). Without loss of generality we may assume that the orientation of each of the simplices Δ_i, $i = 2, 3, \dots, k$ is chosen in such a way that it is compatible with that of Δ_{i-1}. Suppose there is a chain of dimension n in S such that all n-simplices enter it with nonzero coefficients and such that its boundary is 0 (a nontrivial n-cycle). Then it contains all Δ_i, $i = 1, 2, \dots, k$ with nonzero coefficients. These coefficients must be the same, since the common $(n-1)$-face of Δ_{i-1} and Δ_i enters the boundary with the opposite sign for all $i = 2, \dots, k$. But then the common $(n-1)$-face of Δ_k and Δ_1 enters the boundary with the same sign, which means that the coefficient must be 0, a contradiction.

Exercise 10.9 *Prove that a nonorientable pseudomanifold admits a 2-fold covering by an orientable one.*

A simplicial mapping $f : X \to Y$ of a simplicial complex defines a homomorphism $f_* : H_*(X) \to H_*(Y)$ of homology. If both X and Y are oriented pseudomanifolds of the same dimension n, then the homomorphism $f_* : H_n(X,\mathbb{Z}) \to H_n(Y,\mathbb{Z})$ takes the fundamental class of X to an element in $H_n(Y)$ that is proportional to the fundamental class of Y. The proportionality coefficient is called the *degree* of the mapping f. Since all the simplicial approximations of any continuous mapping $f : X \to Y$ define the same homomorphism $f_* : H_*(X) \to H_*(Y)$ of homology, the notion of degree is well defined not only for simplicial mappings, but for arbitrary continuous mappings as well.

Exercise 10.10 *Let $X \to Y$ be a finite covering of an oriented surface by an oriented surface. Prove that in this case the two notions of degree coincide.*

Exercise 10.11 *Let $f : X \to \mathbb{C}\mathrm{P}^1$ be a rational function on a Riemann surface X considered as a continuous mapping of a surface to S^2. Endow both X and $\mathbb{C}\mathrm{P}^1$ with the orientation induced by the complex structure. Prove that the degree of f coincides with the number of preimages of all noncritical values in $\mathbb{C}\mathrm{P}^1$.*

The degree of a continuous mapping $X \to Y$ can be defined in the case where either X, or Y, or both are nonorientable pseudomanifolds as well. In this case the degree, which is the proportionality coefficient between the image of the $\mathbb{Z}_2$-fundamental class of X and the $\mathbb{Z}_2$-fundamental class of Y, is an element of $\mathbb{Z}_2$, that is, it is either 0 or 1.

10.3 Homology of the real projective plane

In order to apply the above constructions, let us consider homology of the real projective plane with coefficients in $\mathbb{R}, \mathbb{Z}$, and $\mathbb{Z}_2$. For $\mathbb{R}\mathrm{P}^2$, we can consider the simplicial decomposition shown in Fig. **??**. Here the projective plane is represented as the 2-dimensional disk with identified opposite points of the boundary circle. The simplicial decomposition consists of 10 2-dimensional simplices, 15 segments and 6 vertices numbered from 0 to 5. The corresponding chain complexes look like

$$0 \longrightarrow C_2(\mathbb{R}\mathrm{P}^2,\mathbb{R}) = \mathbb{R}^{10} \longrightarrow C_1(\mathbb{R}\mathrm{P}^2,\mathbb{R}) = \mathbb{R}^{15} \longrightarrow C_0(\mathbb{R}\mathrm{P}^2,\mathbb{R}) = \mathbb{R}^6 \longrightarrow 0,$$

$$0 \longrightarrow C_2(\mathbb{R}\mathrm{P}^2,\mathbb{Z}) = \mathbb{Z}^{10} \longrightarrow C_1(\mathbb{R}\mathrm{P}^2,\mathbb{Z}) = \mathbb{Z}^{15} \longrightarrow C_0(\mathbb{R}\mathrm{P}^2,\mathbb{Z}) = \mathbb{Z}^6 \longrightarrow 0,$$

and

$$0 \longrightarrow C_2(\mathbb{R}\mathrm{P}^2,\mathbb{Z}_2) = \mathbb{Z}_2^{10} \longrightarrow C_1(\mathbb{R}\mathrm{P}^2,\mathbb{Z}_2) = \mathbb{Z}_2^{15} \longrightarrow C_0(\mathbb{R}\mathrm{P}^2,\mathbb{Z}_2) = \mathbb{Z}_2^6 \longrightarrow 0,$$

respectively. Writing out the boundary operators is straightforward, and computation of homology can be easily done by a computer algebra program. However, our current knowledge allow us to simplify the computation.

Theorem 10.12 *The homology of the real projective plane are*

$$H_2(\mathbb{R}\mathrm{P}^2, \mathbb{R}) = 0, \qquad H_1(\mathbb{R}\mathrm{P}^2, \mathbb{R}) = 0, \qquad H_0(\mathbb{R}\mathrm{P}^2, \mathbb{R}) = \mathbb{R};$$

$$H_2(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) = 0, \qquad H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) = \mathbb{Z}_2, \qquad H_0(\mathbb{R}\mathrm{P}^2, \mathbb{Z}) = \mathbb{Z};$$

and

$$H_2(\mathbb{R}\mathrm{P}^2, \mathbb{Z}_2) = \mathbb{Z}_2, \qquad H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}_2) = \mathbb{Z}_2, \qquad H_0(\mathbb{R}\mathrm{P}^2, \mathbb{Z}_2) = \mathbb{Z}_2.$$

Note that homology of the real projective plane with coefficients in $\mathbb{R}$ coincide with those of a point (or any contractible simplicial complex), although $\mathbb{R}\mathrm{P}^2$ is not contractible. On the contrary, computations over $\mathbb{Z}$ and $\mathbb{Z}_2$ confirm noncontractibility.

Over reals, we know that $H_0(\mathbb{R}\mathrm{P}^2, \mathbb{R})$ is $\mathbb{R}$, since the projective plane is connected, and $H_2(\mathbb{R}\mathrm{P}^2, \mathbb{R}) = 0$, because the projective plane is nonorientable. Now we can conclude that $H_1(\mathbb{R}\mathrm{P}^2, \mathbb{R}) = 0$, since the Euler characteristic of the projective plane is 1: the image of the boundary operator $C_2(\mathbb{R}\mathrm{P}^2, \mathbb{R}) = \mathbb{R}^{10} \longrightarrow C_1(\mathbb{R}\mathrm{P}^2, \mathbb{R}) = \mathbb{R}^{15}$ is 10-dimensional, because the second homology are 0. The image of the boundary operator $C_1(\mathbb{R}\mathrm{P}^2, \mathbb{R}) = \mathbb{R}^{15} \longrightarrow C_0(\mathbb{R}\mathrm{P}^2, \mathbb{R}) = \mathbb{R}^6$ is 5-dimensonal, hence its kernel is 10-dimensional and coincides thus with the subspace of 1-boundaries.

Over integer numbers, the computation of H_2 and H_0 is exactly the same as in the case of real numbers. Computation of $H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z})$ is more subtle. The answer $\mathbb{Z}_2$ comes from the remark that the boundary of the sum of all the 2-simplices in Fig. **??** is the boundary circle of the disk, which, in its own turn, is twice the sum of the edges $2([0, 2] + [2, 1] + [1, 0])$. On the other hand, the half of the latter sum, that is, $[0, 2] + [2, 1] + [1, 0]$ is not a boundary. See the next section for discussion of computation of H_1 over $\mathbb{Z}$.

The zero homology with coefficients in $\mathbb{Z}_2$ are computed as above. The second homology result from the $\mathbb{Z}_2$-fundamental class of $\mathbb{R}\mathrm{P}^2$. In order to compute the first homology we may recall that the group $\mathbb{Z}_2$ may be also treated as the field with two elements; the groups $\mathbb{Z}_2^p$ are then vector spaces over $\mathbb{Z}_2$. The alternating sum of the dimensions of these vector spaces is then a topological invariant of the simplicial complex and coincides with its Euler characteristic, which gives $H_1(\mathbb{R}\mathrm{P}^2, \mathbb{Z}_2) = \mathbb{Z}_2$.

10.4 Homology of simplicial complexes with coefficients in $\mathbb{Z}$ and Hurewicz theorem

Let Γ be a group, not necessarily commutative. Denote by $[\Gamma, \Gamma] \subset \Gamma$ the commutator of Γ, that is, the subgroup in Γ generated by the commutators $aba^{-1}b^{-1}$ of elements a, b of Γ. If Γ is Abelian, then its commutator is the trivial subgroup, but this is not the case if Γ is noncommutative. The commutator $[\Gamma, \Gamma]$ is a normal subgroup in Γ, and the quotient subgroup $\Gamma/[\Gamma, \Gamma]$ is commutative. This quotient subgroup is called the *centralizer*, or the *commutant*, of Γ.

Theorem 10.13 (Hurewicz) *Let X be a finite connected simplicial complex, and let $\pi_1(X)$ be its fundamental group. Then the group $H_1(X, \mathbb{Z})$ of first homology of X with coefficients in $\mathbb{Z}$ is the commutant of $\pi_1(X)$.*

Corollary 10.14 *The group of first homology of a graph with coefficients in $\mathbb{Z}$ is the free commutative group $\mathbb{Z}^{b_1}$ of rank b_1, the first Betti number of the graph.*

Indeed, the commutant $F_b/[F_b, F_b]$ of the free group with b generators is the free Abelian group $\mathbb{Z}^b$.

Corollary 10.15 *The group $H_1(M_g, \mathbb{Z})$ of first homology of an oriented surface of genus g with coefficients in $\mathbb{Z}$ is the free commutative group $\mathbb{Z}^{2g}$ of rank $2g$.*

Indeed, the fundamental group $\pi_1(M_g)$ admits a presentation of the form $\langle a_1, b_1, a_2, b_2, \ldots, a_g, b_g | a_1 b_1 a_1^{-1} b_1^{-1} \ldots a_g b_g a_g^{-1} b_g^{-1} \rangle$. Therefore, since the only relation belongs to the commutator subgroup of the free group F_{2g} with the generators $a_1, b_1, a_2, b_2, \ldots, a_g, b_g$, the commutant $\pi_1(M_g)/[\pi_1(M_g), \pi_1(M_g)]$ coincides with the commutant $F_{2g}/[F_{2g}, F_{2g}]$ of the free group with $2g$ generators.

Exercise 10.16 *Find the commutator and the commutant of the fundamental groups of the projective plane and the Klein bottle.*

The proof of Theorem 10.13 proceeds as follows. Both the fundamental group and the first homology group of a simplicial complex are totally determined by the 2-skeleton of the simplicial complex. In this 2-skeleton we can contract 1-dimensional simplices so that to make a single vertex (0-dimensional simplex). Contraction of 1-dimensional simplices changes neither the fundamental, nor the homology group. The resulting single vertex can be taken for the base point of the fundamental group, and it spans the zero homology group.

The only edges that survived under contraction are loops at the single vertex; these loops both form a set of generators of the fundamental group, and span the group of 1-cycles. The 2-simplices in the 2-skeleton after contraction become discs that generate all the relations among the generators in the fundamental group. The natural mapping from the fundamental group to the first homology group preserves these relations, and adds new ones, that of commutativity. Hence, the resulting homology group is the quotient of the fundamental group modulo its commutant.

Exercise 10.17 *Use Exercise 10.9 to show that the first* $\mathbb{Z}$*-homology of a nonorientable pseudomanifold can be realized as a subgroup of index* 2 *in the first* $\mathbb{Z}$*-homology of its* 2*-fold orientable covering.*

Exercise 10.18 *Compute homology of the Klein bottle with coefficients in* $\mathbb{R}$, $\mathbb{Z}, \mathbb{Z}_2$.

Exercise 10.19 *Compute homology of a nonorientable surface of genus* g *with coefficients in* $\mathbb{R}, \mathbb{Z}, \mathbb{Z}_2$.

Exercise 10.20 *Compute homology of the real projective space* $\mathbb{R}\mathrm{P}^3$ *with coefficients in* $\mathbb{R}, \mathbb{Z}, \mathbb{Z}_2$.

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Chapter 11

Cohomology of simplicial complexes

In addition to considering homology with coefficients in an arbitrary Abelian group, there is one more way to make homology into a finer invariant of topological spaces. This way consists in introducing multiplication on homology thus making them into a graded ring instead of just a vector space (or an Abelian group). Unfortunately, there is no natural way to introduce multiplication in homology themselves, and multiplication arises naturally on their dual spaces, cohomology. This multiplication, however, reflects another natural structure on homology: comultiplication.

11.1 Cohomology of chain complexes

Let

$$0 \xrightarrow{\partial_{n+1}} C_n \xrightarrow{\partial_n} \dots \xrightarrow{\partial_{i+1}} C_i \xrightarrow{\partial_i} \dots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \xrightarrow{\partial_0} 0 \tag{11.1}$$

be a sequence of vector spaces over the field $\mathbb{R}$ of real numbers and their linear maps. Denote by $C_k^\vee$ the dual vector space for C_k, $C_k^\vee = \mathrm{Hom}(C_k, \mathbb{R})$, $k = 0, 1, \dots, n$. Here $\mathrm{Hom}(V, \mathbb{R}) = V^\vee$, for a finite dimensional vector space V over $\mathbb{R}$, denotes the vector space of linear functionals from V to $\mathbb{R}$; note that $\dim\ V = \dim\ V^\vee$. For a pair of vector spaces U, V, any linear mapping $A : U \to V$ determines the dual linear mapping $A^\vee : V^\vee \to U^\vee$ according to the rule

$$A^\vee(f)(u) = f(A(u)).$$

In this way the sequence (11.1) produces a sequence of vector spaces and linear maps

$$0 \xleftarrow{\delta_{n+1}} C_n^\vee \xleftarrow{\delta_n} \dots \xleftarrow{\delta_{i+1}} C_i^\vee \xleftarrow{\delta_i} \dots \xleftarrow{\delta_2} C_1^\vee \xleftarrow{\delta_1} C_0^\vee \xleftarrow{\delta_0} 0, \tag{11.2}$$

where we set $\delta_i = \partial_i^\vee$, $i = 0, \dots, n$.

Lemma 11.1 *If the sequence* (11.1) *is a chain complex, then the sequence* (11.2) *also is a chain complex.*

Indeed, in order to prove this statement it suffices to show that if for a triple of vector spaces and operators

$$U \xrightarrow{A} V \xrightarrow{B} W$$

we have $\mathrm{Im}\, A \subset \mathrm{Ker}\, B$, then for the dual sequence

$$U^\vee \xleftarrow{A^\vee} V^\vee \xleftarrow{B^\vee} W^\vee$$

we have $\operatorname{Im} B^\vee \subset \operatorname{Ker} A^\vee$. The latter statement follows since if $B(A(u)) = 0$ for any $u \in U$, then $A^\vee \circ B^\vee = 0$ as well.

The lemma allows one to define the homology of the chain complex $\mathcal{C}^\vee$ dual to the chain complex $\mathcal{C}$. This homology are called the *cohomology* of the initial chain complex and denoted by $H^k = \operatorname{Ker}\ \delta_k / \operatorname{Im} \delta_{k-1}$. Elements of the vector space C^k are called the *k-cochains*, elements of $\operatorname{Ker}\ \delta_k$ are *k-cocycles*, and elements of $\operatorname{Im}\ \delta_{k-1}$ are *k-coboundaries*.

Any element of $\operatorname{Ker}\ \delta_k$, being a linear functional on C_k, is a linear functional on $\operatorname{Ker}\ \partial_{k-1}$ as well. The value of this linear functional on $\operatorname{Im}\ \partial_{k+1}$ is 0.

Lemma 11.2 *The cohomology space H^k is naturally dual to the homology space H_k, that is, any two k-cocycles differing by a coboundary determine the same linear functionals on the space H_k.*

11.2 Comultiplication in homology and multiplication in cohomology

Let X be a simplicial complex, and let $H_*(X), H^*(X)$ be its homology and cohomology, respectively. Our goal in this section is to define a multiplication in cohomology $H^*(X)$. Recall that a multiplication on a vector space V is a linear mapping

$$m : V \otimes V \to V,$$

where $V \otimes V$ is the tensor square of the vector space V (the vector space freely spanned by the elements $e_i \otimes e_j$ for any basis $e_1, \dots, e_{\dim\ V}$ in V, $i, j = 1, \dots, \dim\ V$).

If a vector space V is endowed with a multiplication m, then its dual vector space $V^\vee$ is endowed with a natural comultiplication $\mu : V^\vee \to V^\vee \otimes V^\vee$ defined as follows:

$$\mu(f)(v_1 \otimes v_2) = f(m(v_1, v_2))$$

for an arbitrary $f \in V^\vee$. Conversely, a comultiplication $\mu : V \otimes V \to V$ on a vector space V induces a multiplication $m : V^\vee \otimes V^\vee \to V^\vee$ on the dual space $V^\vee$ according to the rule

$$m(f_1 \otimes f_2)(v) = (f_1 \otimes f_2)(\mu(v)).$$

Pick a numbering of the vertices of X. We are going to define an operation of comultiplication on $H_*(X)$, that is, a linear operation $\mu : H_*(X) \to$

$H_*(X) \otimes H_*(X)$. Let us start with defining the operation μ on the space of chains $C_*(X)$. For a k-simplex $[v_0, v_1, \ldots, v_k]$, we set

$$\mu : [v_0, v_1, \ldots, v_k] \to \sum_{i=0}^{k} [v_0, v_1, \ldots, v_i] \otimes [v_i, \ldots, v_k]$$

and extend the operation to linear combinations of simplices by linearity. Note that

$$\mu([v_0, v_1, \ldots, v_k]) \in C_0(X) \otimes C_k(X) \oplus C_1(X) \otimes C_{k-1}(X) \oplus \cdots \oplus C_k(X) \oplus C_0(X),$$

so that the operation μ is graded.

Lemma 11.3 *The comultiplicaton $\mu : C_*(X) \to C_*(X) \otimes C_*(X)$ descends to a graded comultiplication $\mu : H_*(X) \to H_*(X) \otimes H_*(X)$ that is independent of the choice of the numbering of the vertices of the simplicial complex X.*

We denote this comultiplication by the same letter μ.

Multiplication $m : H^*(X) \otimes H^*(X)$ on the cohomology $H^*(X)$ of a simplicial complex X results from the comultiplication μ in $H_*(X)$ and duality between $H_*(X)$ and $H^*(X)$.

11.3 Homology and cohomology of Cartesian product

Another, but an equivalent way to introduce multiplication in cohomology exploits three facts:

- there is a natural mapping $i : X \to X \times X$ of any topological space X to its Cartesian square taking any point $x \in X$ to the point $(x, x) \in X \times X$ on the diagonal;
- a continuous mapping $f : X \to Y$ induces a cohomology homomorphism $f^* : H^*(Y) \to H^*(X)$ in the opposite direction;
- there is a natural isomorphism between the cohomology $H^*(X \times Y)$ of the Cartesian product $X \times Y$ of two simplicial complexes and the tensor product $H^*(X) \otimes H^*(Y)$.

These facts imply that there is a natural linear mapping $i^* : H^*(X) \otimes H^*(X) \equiv H^*(X \times X) \to H^*(X)$, hence a multiplication on $H^*(X)$.

Exercise 11.4 *The* canonical *simplicial decomposition of the product*

$$[v_0, v_1, \dots, v_k] \times [v'_0, v'_1, \dots, v'_l]$$

of two simplicies of dimensions k and l is defined as follows. The $(k+l)$-dimensional simplices are given by sequences $[(v_0, v'_0), \dots, (v_i, v'_j), \dots, (v_k, v'_l)]$ of length $k+l+1$ such that there are no coinciding pairs and the indices in both the sequence of first coordinates $v_0, \dots, v_i, \dots, v_k$ and the sequence of second coordinates $v'_0, \dots, v'_j, \dots, v'_l$ are nondecreasing. Prove that this is indeed a simplicial decomposition of the product of two simplices.

Note that the requirement in the exercise means that any two subsequent pairs of vertices (v_i, v'_j) in a sequence of vertices of a $(k+l)$-simplex differ in the index of only one of the coordinates, and the difference is exactly 1. Therefore, the $(k+l)$-simplices in the canonical simplicial decomposition are in one-no-one correspondence with rook paths in a $(k+1) \times (l+1)$-rectangle leading from the left uppermost square to the right lowermost one and going either down or right.

Exercise 11.5 ΣX — X. , ΣX .

Exercise 11.6 , , $m+n$- m- n- .

Exercise 11.7 , 4- B^4 S^2 $\partial B^4 = S^3 \to S^2$. , . , . , $S^2 \to S^3$, .

Chapter 12

CW complexes. Cellular homology. Poincare duality. Manifolds.

Simplicial complexes are a rather efficient tool in combinatorial representation of topological objects. Nevertheless, they often lead to laborious computations, since the number of simplices may be large. CW complexes provide a way to simplify computations, since the number of cells in a CW complex representing a topological space usually can be chosen much less than that of simplices. And constructing cell decompositions is not more complicated than simplicial decompositions.

12.1 CW complexes

Definition 12.1 *A* (finite) CW complex *is a topological space X endowed with a splitting into a disjoint union of finitely many topological subspaces, called* open cells*, and continuous mappings f_i, one for each cell, from closed unit balls in Euclidean spaces to X such that*

- *the restriction of the map f_i to the interior of the unit ball is a homeomorphism to the i th open cell; we say that the i th open cell has the same dimension as the unit ball;*
- *the restriction of each f_i to the boundary of the unit ball takes this boundary to a union of open cells of smaller dimension.*

The image of the closed ball under the mapping f_i is called the i th *closed cell* of the CW-complex. The maximal dimension of open cells in a CW-complex is called the *dimension* of the CW-complex.

Example 12.2 *Each n-simplex is naturally a CW-complex. Moreover, each simplicial complex is naturally a CW-complex, with open cells of dimension k being the interiors of the k-simplices.*

Example 12.3 *One can make a CW-complex of the sphere S^2 by choosing two distinct points on the equator for 0-dimensional cells, the two half-equators connecting them for the 1-dimensional cells. Then there are two open 2-dimensional cells represented by the upper and the lower hemispheres, two open 1-cells represented by the two halves of the equator, and two open 0-cells, which are the chosen points. The mappings f_i are obvious.*

Even simpler CW-complex representing S^2 is obtained if we take for the only 0-cell an arbitrary point in S^2, and its complement for the 2-cell. The mapping $f : \overline{D}^2 \to S^2$ then contracts the boundary $\partial\overline{D}^2 = S^1$ of the closed unit disk to the chosen point.

Exercise 12.4 *Construct a CW-decomposition of the* 3*-sphere* S^3.

Exercise 12.5 *Construct a CW-decomposition of the projective plane* $\mathbb{R}\mathrm{P}^2$.

Exercise 12.6 *Construct a CW-decomposition of an orientable surface of genus g containing a single* 0*-cell and a single* 2*-cell. How many* 1*-cells does this decomposition have?*

12.2 Homology of CW-complexes

To each finite CW-complex X, we associate a chain complex of Abelian groups. Let $C_k(X)$ denote the group of k-chains spanned over $\mathbb{Z}$ or $\mathbb{Z}_2$ by open k-cells in X. The union of the closures of open k-cells in X forms the *k-skeleton* of X.

In order to define the differential $\partial_k : C_k(X) \to C_{k-1}(X)$, we must define its action on each open k-cell. Let $f : \overline{D}^k \to X$ be the corresponding map of the closed k-disk. The differential ∂_k takes an open k-cell to a linear combination of open $(k-1)$-cells:

$$\sum d_i(f) c_i^{k-1},$$

where the sum is taken over all $(k-1)$-cells c_i^{k-1} in X and the coefficients $d_i(f)$ are defined as follows.

By contracting the complement to the open cell c_i^{k-1} in the $(k-1)$-skeleton of X, we make the closure $\bar{c}_i^{k-1}$ of this open cell into the $(k-1)$-sphere S^{k-1}. The restriction of the mapping f to the boundary sphere $\partial\overline{D}^k = S^{k-1}$, after this contraction, becomes a mapping $S^{k-1} \to S^{k-1}$. The degree of this mapping is well defined, since both the source and the target sphere have well-defined fundamental class. This is obvious over $\mathbb{Z}_2$, but is also true over $\mathbb{Z}$. Indeed, each open cell in X is an image of an open oriented ball of the same dimension, whence is endowed with the induced orientation. Therefore, the integer fundamental class of the $(k-1)$-sphere obtained by contracting the complement to a $(k-1)$-cell in the $(k-1)$-skeleton of X is well-defined. In addition, the boundary $\partial\overline{D}^k = S^{k-1}$ of a closed k-disc is also endowed with a natural orientation. We set the number $d_i(f)$ equal to the degree of the mapping $S^{k-1} \to S^{k-1}$ of the two oriented spheres.

Theorem 12.7 *The homomorphisms ∂_k thus defined make the sequence of Abelian groups $C_k(X)$ into a chain complex.*

Homology of this complex are called the *cellular homology* of X.

Exercise 12.8 *Check that for a simplicial decomposition of a topological space X its cellular homology with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_2$ coincide with the corresponding simplicial homology.*

Example 12.9 *Consider the CW-complex splitting the sphere S^n, for $n > 0$, into two open cells: one of dimension 0, and its complement, which is an open set of dimension n. Then the cellular homology of this CW complex are $\mathbb{Z}$ (or $\mathbb{Z}_2$) in dimensions n and 0, and are 0 in all other dimensions,*

$$H_k(S^n) = \begin{cases} \mathbb{Z} & k = 0, \\ \mathbb{Z} & k = n, \\ 0 & k \neq 0, n. \end{cases}$$

We see that the cellular homology of the sphere coincide with the simplicial one, but the computation in this case is much easier.

Theorem 12.10 *The cellular homology of a given topological space X are independent of the chosen CW-complex homeomorphic to X.*

Corollary 12.11 *If a topological space X admits a simplicial decomposition, then its simplicial homology, with coefficients in $\mathbb{Z}$ or $\mathbb{Z}_2$, coincide with cellular homology. In particular, the Euler characteristic of a CW complex defined as the alternating sum of the numbers of open cells of each dimension is well-defined.*

Exercise 12.12 *Construct a CW-decomposition of the real projective space $\mathbb{R}\mathrm{P}^n$ containing one open cell of each dimension from 0 to n. Using this decomposition compute cellular homology of $\mathbb{R}\mathrm{P}^n$ with coefficients in $\mathbb{Z}, \mathbb{Z}_2$.*

Exercise 12.13 *Construct a CW-decomposition of the complex projective space $\mathbb{C}\mathrm{P}^n$ containing one open cell of each even dimension from 0 to $2n$. Using this decomposition compute cellular homology of $\mathbb{C}\mathrm{P}^n$ with coefficients in $\mathbb{Z}, \mathbb{Z}_2$.*

Exercise 12.14 *Construct a CW-decomposition of the orientable surface of genus g. Using this decomposition compute its cellular homology with coefficients in $\mathbb{Z}, \mathbb{Z}_2$.*

Exercise 12.15 *Construct a CW-decomposition of the nonorientable surface of genus g. Using this decomposition compute its cellular homology with coefficients in $\mathbb{Z}, \mathbb{Z}_2$.*

12.3 Manifolds

Pseudomanifolds are simplicial analogs of manifolds. In particular, each compact manifold admits a representation as a finite pseudomanifold. Before introducing the notion of manifold, we define a smooth submanifold in an Euclidean space.

Definition 12.16 *A subset $M \subset \mathbb{R}^n$ is called an m-*dimensional smooth submanifold *if each point $A \in M$ possesses a neighborhood $U_A \subset \mathbb{R}^n$ such that the intersection $M \cap U_A$ is the set of common zeroes of an $(n-m)$-tuple of smooth functions $f_1, \dots, f_{n-m} : U_A \to \mathbb{R}$ such that their differentials $df_1, \dots, df_{n-m}$ are linearly independent at A (or, which is the same, the rank of the matrix of derivatives*

$$\begin{pmatrix} \frac{\partial f_1}{\partial x_1} & \dots & \frac{\partial f_1}{\partial x_n} \\ \dots & \dots & \dots \\ \frac{\partial f_{n-m}}{\partial x_1} & \dots & \frac{\partial f_{n-m}}{\partial x_n} \end{pmatrix}$$

is $n - m$).

Note that since the differentials depend on the point continuously, their linear independency at A causes their linear independency at some neighborhood of A (we do not require that the latter neighborhood coincides with U_A). The implicit function theorem implies that in some neighborhood of each point $A \in M$ local coordinates $x'_1, \dots, x'_n$ can be chosen in such a way that the intersection of M with this neighborhood is given by the equations $x'_{m+1} = \dots = x'_n = 0$. Then the coordinates $x'_1, \dots, x'_m$ can serve as local coordinates in M in some neighborhood of A.

A function $f : M \to \mathbb{R}$ is said to be *smooth* if, written in any such system of local coordinates, it is infinitely differentiable. A mapping $F : M \to N$ of an m-dimensional submanifold M to an n-dimensional submanifold N is said to be *smooth* if it is coordinatewise smooth when written in any system of local coordinates at any point $A \in M$ and the point $F(A) \in N$. A smooth mapping $F : M \to N$ is called a *diffeomorphism* if it is one-to-one and its inverse $F^{-1} : N \to M$ also is smooth. Diffeomorphisms preserve dimensions of the submanifolds.

If there exists a diffeomorphism $F : M \to N$, then the submanifolds M and N are said to be diffeomorphic. Being diffeomorphic is an equivalence relations on submanifolds. An *m-dimensional manifold* is an equivalence class of m-dimensional submanifolds in Euclidean spaces (of arbitrary dimension).

Theorem 12.17 *Each compact manifold admits a finite simplicial decomposition.*

12.4 Morse functions

Let M be an m-dimensional manifold, and let $f : M \to \mathbb{R}$ be a smooth function on M. If df is nonzero at a point $A \in M$, then there is a local coordinate system $(x_1, \ldots, x_m)$ at a neighborhood of A such that

$$f(x) = f(A) + x_1$$

at this neighborhood (any function can be linearized at its noncritical point).

A point $A \in M$ is a *critical point of f* if the differential df is zero at A, $df|_A = 0$. At a critical point, a function cannot be linearized; its any coordinate presentation has zero linear part. A critical point is said to be *nondegenerate* if there is a system of local coordinates $x_1, \ldots, x_m$ around A such that the function f has the form

$$f(x_1, \ldots, x_m) = f(A) \pm x_1^2 \pm x_2^2 \pm \cdots \pm x_m^2;$$

in these coordinates. Here the number of squares coincides with the dimension m of M. If a critical point is nondegenerate, then the number of positive squares and the number of negative squares do not depend on the chosen coordinate system. The number of negative squares is called the *index* of the critical point. A critical point of index 0 is a local minimum of the function f, while a critical point of index m is a local maximum. Critical points of index between 0 and m are called *saddles*.

The value of f at a critical point of f is called a *critical value* of f. A function $M \to \mathbb{R}$ with finitely many critical points, all of whom are nondegenerate, is called a *Morse function*. If, in addition, the critical values of a Morse function are pairwise distinct, then the function is said to be *strongly Morse*.

Example 12.18 *Let $S^1 \subset \mathbb{R}^2$ be the unit circle in the Euclidean plane with the coordinates (x, y). Obviously, the function $y|_{S^1}$ is a Morse function* *on S^1. This Morse function has two critical points, one being a minimum, of index 0, the other one the maximum, of index 1. Functions of this type (restrictions of one of the coordinates of the ambient Euclidean space to the submanifold) are called* height functions*. It often happens that there is a Morse function among height functions.*

Other useful type of functions is represented by distance functions*. These are functions of the form $\rho_A(x) = ||x - A||^2$, where $x \in M$ and A is an arbitrary point in the ambient Euclidean space of M. For the unit circle, for example, the distance function $\rho_A(x)$ is a Morse function for all points $A \in \mathbb{R}^2$ with the exception of the origin. Indeed, the function $\rho_0(x) \equiv 1$ is a constant function on the circle, so that all its points are degenerate critical ones.*

Height functions can be considered as limit versions of (square roots of) distance functions, as the base point A of the distance function $\rho_A(\cdot)$ goes to infinity in the direction prescribed by the chosen coordinate.

Example 12.19 *The torus $S^1 \times S^1 \subset \mathbb{R}^3$ given, for example, by the parametric equation*

$$(u, v) \mapsto (8 + (3 + \cos v)\cos u, 3 + \sin v, 4 + (3 + \cos v)\sin u)$$

together with the coordinate z provide a standard example of a Morse height function. This Morse function has a single critical point of index 0, two critical points of index 1, and a single critical point of index 1.

Exercise 12.20 *Construct Morse functions on*

- *an orientable surface of genus 2;*
- *real projective plane $\mathbb{R}\mathrm{P}^2$;*
- *complex projective plane $\mathbb{C}\mathrm{P}^2$;*
- *Klein bottle.*

Specify the critical points and their indices for each function.

12.5 Constructing cell decomposition of a manifold

A Morse function on a manifold provides a tool for constructing its cell decomposition. Let M be an m-dimensional manifold, and let $f : M \to \mathbb{R}$ be a strongly Morse function on M.

Theorem 12.21 *The manifold M is homotopy equivalent to a cell complex, with the number of cells of dimension k equal to the number of singular points of f of index k, for $k = 0, 1, \ldots, m$.*

Corollary 12.22 (Weak Morse inequalities) *The dimension of the vector space $H_k(M, \mathbb{R})$ is at most the number of critical points of index k of M, for $k = 0, 1, \ldots, m$. The same is true for the Betti numbers $b_k(M)$ (which are the ranks of the homology groups $H_k(M, \mathbb{Z})$).*

We construct a CW-structure on M by considering how the part M_t of M defined by the inequality

$$M_t = \{x \in M, f(x) \leq t\}$$

grows as t varies from $-\infty$ to $+\infty$. Note that since any manifold can be represented as a submanifold in an Euclidean space, each manifold can be endowed with a smooth Riemanniann structure that is induced from the ambient Euclidean space. We suppose that one of these structures is picked for M.

The process of growing splits into several kinds of steps.

Step 0. For t sufficiently small, the submanifold M_t is empty, since f, being a continuous function on a compact topological space, achieves its minimal value.

Step 1. Let $t_{\min}$ be the minimal value of the function f on M, $t_{\min} = \mathrm{Inf}_{x \in M}\{f(x)\}$. And let $\varepsilon > 0$ be such that f has no critical values on the halfsegment $(t_{\min}, t_{\min} + \varepsilon]$. Then the topological space $M_{t_{\min}+\varepsilon}$ is homeomorphic to the closed m-disk D_m. Indeed, this statement follows from the fact that a critical point that is a local minimum (which, in our case, is even a global one) of the function has index 0 and the subspace $M_{t_{\min}+\varepsilon}$ is homeomorphic to the subset $\{(x_1, \ldots, x_m) | x_1^2 + \cdots + x_m^2 \leq \varepsilon^2\}$, which is the m-disk.

Step 2. Let t_0 be a noncritical value of the function f, and suppose there are no critical values of f on the segment $[t_0, t_0 + \varepsilon]$, $\varepsilon > 0$. Then the two subspaces M_{t_0} and $M_{t_0+\varepsilon}$ are homeomorphic to one another.

The proof uses the gradient vector field of the function f on the layer $M_{t_0+\varepsilon} \setminus M_{t_0}$. Recall that, for a given Riemanniann structure $\langle\cdot,\cdot\rangle$ on M, the gradient vector field $\operatorname{grad} f$ of a smooth function $f : M \to \mathbb{R}$ is the vector field such that $\langle\operatorname{grad} f, \cdot\rangle = df$. To the gradient vector field $\operatorname{grad} f$, we associate a deformation retraction $M_{t_0+\varepsilon} \to M_{t_0}$ in the following way: each point of M_{t_0} remains fixed, while points in the layer $M_{t_0+\varepsilon} \setminus M_{t_0}$ move along the vector field $-\operatorname{grad} f$ with the velocity ε, until they reach the level set $f = t_0$ of f.

Exercise 12.23 *Prove that the mapping $df : TM \to TR$ takes the vector field* $\operatorname{grad} f$ *to the vector field $\partial/\partial t$ on $\mathbb{R}$.*

Step 3. Suppose there is a single critical value on the interval $(t_0, t_0+\varepsilon)$. Let k denote the index of the corresponding critical point. Then the space $M_{t_0+\varepsilon}$ is homotopy equivalent to the space M_{t_0} with a k-cell attached. Indeed, in order to prove this statement, it suffices to consider how the subspace M_t changes in a neighborhood of a nondegenerate singular point. A function with a singular point of index k admits a coordinate presentation

$$f(x_1, \ldots, x_m) = -x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2$$

(we assume that f vanishes at the singular point). In particular, for $m = 2$, $k = 1$, we pass from the subset

$$-x_1^2 + x_2^2 < -\varepsilon$$

to the subset

$$-x_1^2 + x_2^2 < \varepsilon,$$

for $\varepsilon > 0$. The result is homotopy equivalent to attaching a segment to the subset

$$-x_1^2 + x_2^2 < -\varepsilon,$$

the ends of the segment being attached to different connected components of the subset.

For $m = 3$, $k = 1$, the domain

$$-x_1^2 + x_2^2 + x_3^2 < -\varepsilon$$

is bounded by the hyperboloid

$$-x_1^2 + x_2^2 + x_3^2 = -\varepsilon$$

of two sheet and consists of two connected components. The domain

$$-x_1^2 + x_2^2 + x_3^2 < \varepsilon,$$

bounded by a hyperboloid of one sheet is homotopy equivalent to the initial one with a segment (a cell of dimension 1) attached, the two ends being attached to different connected components of the initial domain.

On the other hand, for $m = 3$, $k = 2$, the domain

$$-x_1^2 - x_2^2 + x_3^2 < -\varepsilon$$

is bounded by the hyperboloid

$$-x_1^2 - x_2^2 + x_3^2 = -\varepsilon$$

of one sheet and consists of a single connected components. The domain

$$-x_1^2 - x_2^2 + x_3^2 < \varepsilon,$$

bounded by a hyperboloid of two sheet is homotopy equivalent to the initial one with a 2-disc (a cell of dimension 2) attached, the boundary circle of the disc attached to the circle $x_3^2 = 0$, $-x_1^2 - x_2^2 = -\varepsilon$, in the initial domain.

For arbitrary n and k the situation is exactly the same: the domain

$$-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2 < \varepsilon$$

is homotopy equivalent to the domain

$$-x_1^2 - \cdots - x_k^2 + x_{k+1}^2 + \cdots + x_m^2 < -\varepsilon$$

with the boundary sphere of the k-disc attached along the sphere $-x_1^2 - \cdots - x_k^2 = -\varepsilon$, $x_{k+1} = \cdots = x_m = 0$.

12.6 Existence of Morse functions

Morse functions exist on each manifold. Moreover, Morse functions are dense in the space of all smooth functions (endowed with any reasonable topology). The same is true for strong Morse functions.

Lemma 12.24 *If M is a compact smooth manifold, then there is a function $f : M \to \mathbb{R}$ having finitely many critical points. Moreover, there is a Morse function. Moreover, there is a strong Morse function.*

Indeed, let M be represented by a smooth m-dimensional submanifold in $\mathbb{R}^N$. For a point $A \in \mathbb{R}^N$, denote by $\rho_A : \mathbb{R}^N \to \mathbb{R}$ the function defined as

$$\rho_A(x) = ||x - A||^2,$$

that is, the square of the distance between a point x and the point A. This function obviously is smooth, hence its restriction $\rho_A : M \to \mathbb{R}$ also is smooth. We claim that there is a Morse function (and even a strong Morse function) among the functions ρ_A, for different A.

Consider the mapping $E : \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}^N$ taking a point (x, v) to the point $x + v$, $E : (x, v) \mapsto x + v$. Consider the restriction of this mapping to the smooth submanifold $T^\perp M$ of $\mathbb{R}^N \times \mathbb{R}^N$, which consists of points (x, v) such that $x \in M$, and v is orthogonal to the tangent plane $T_x M$ to M at x. The dimension of the submanifold $T^\perp M$ is N. The restriction of E to $T^\perp M$ is a smooth mapping of an N-dimensional manifold to $\mathbb{R}^N$. The critical points of $E : T^\perp M \to \mathbb{R}^N$ are those points where the rank of its differential dE is less than N. According to Sard's lemma, the set of critical values of E (the values of E at the critical points) has measure zero in the range $\mathbb{R}^N$.

Now, take for A an arbitrary point in $\mathbb{R}^N$ that is not a critical value of E. Then the function $\rho_A : M \to \mathbb{R}$ is a Morse function. Indeed, a point $x \in M$ is a critical point of ρ_A iff the vector $A - x$ is orthogonal to M at x. For a compact manifold M, each line orthogonal to a given tangent plane $T_x M$ contains only finitely many points such that x is a degenerate singular point of the function ρ_A. These points are exactly the critical values of E (the *focal points* of the submanifold M).

The following exercise completes the proof of Lemma 12.24.

Exercise 12.25 *Prove that in any neighborhood of a point $A \in \mathbb{R}^N$ such that the function ρ_A is Morse there is a point A' such that the function $\rho_{A'}$ is strongly Morse on M.*

12.7 Poincaré duality

Theorem 12.26 *For an orientable manifold M of dimension m, we have $H^i(M, \mathbb{R}) = H_{m-i}(M, \mathbb{R})$, for $k = 0, 1, \ldots, m$.*

Corollary 12.27 (Poincaré duality) *There is a nondegenrate pairing $H_i(M, \mathbb{R}) \times H_{m-i}(M, \mathbb{R}) \to \mathbb{R}$.*

Indeed, the corollary follows from the theorem and the natural identification between $H_i(M,\mathbb{R})$ and the dual vector space to $H^i(M,\mathbb{R})$.

We give two proofs of this theorem.

The first proof uses the fact that a manifold admits a simplicial subdivision. Let δ be a simplex of dimension k in M. We associate to this simplex a cell $D\delta$ of dimension $m-k$ in M in the following way. For each simplex Δ of dimension m, we define the intersection of $D\delta$ with Δ as the convex hull of the barycentres of all subsets of the vertices of Δ that contain δ. The cell $D\delta$ is, by definition, the union of all these intersections over all simplices Δ of maximal dimension in M. It is clear that the dimension of the cell $D\delta$ is $m-k$, and that the cells $D\delta$, for all δ, form a CW-decomposition of M. This CW-decomposition is said to be *dual* to the initial simplicial complex structure. For $m=2$ this construction coincides with that of the dual graph for a given triangulation of a surface.

Now, each k-dimensional cycle in the simplicial decomposition of M defines a linear functional on $(m-k)$-chains of M constructed from the dual CW-decomposition: the value of such a cycle on a chain of cells is the sum of intersection indices of the simplices of the cycle with the cells of the chain. Here the *intersection index* of a simplex δ with the cell $D\delta$ is ± 1, the sign depending on the chosen orientations, and it is 0 with all other cells. The constructed pairing depends on the homology class of the chain rather than on the chain itself, whence descends to the homology level.

The second proof exploits Morse functions and the duality between the CW-complex structure based on a Morse function f and its opposite $-f$, which also is a Morse function. The k-cell we attach to the subspace $M_{t_0-\varepsilon}$, for a critical point of index k of f, can be treated as the lower separatrix disk. By definition, the *lower separatrix manifold* of a critical point of a function $f : M \to \mathbb{R}$ consists of those points $y \in M$ that approach this critical point along the gradient vector field $\operatorname{grad} f$. Respectively, the *upper separatrix manifold* of a critical point consists of those points $y \in M$ that approach this critical point along the gradient vector field $-\operatorname{grad} f$. For a Morse critical point of index k, $f(x_1,\dots,x_m) = -x_1^2 - \dots - x_k^2 + x_{k+1}^2 + \dots + x_m^2$, the lower separatrix manifold is the coordinate plane $x_{k+1} = \dots = x_m = 0$, and the function f is a negatively determined quadratic form on it, while the upper separatrix manifold is the coordinate plane $x_1 = \dots = x_k = 0$, and the function f is a positively determined quadratic form on it. All points in $\mathbb{R}^m$ not belonging to the two submanifolds demonstrate hyperbolic behavior: they first approach the origin under the vector field $\operatorname{grad} f$ (or $-\operatorname{grad} f$), and then go away from it. The two submanifolds are transversal to one another and, in addition, orthogonal with respect to the metric in the ambient Euclidean space.

The dimension of the lower separatrix disc is k, while that of the upper separatrix disc is $m-k$. The intersection index of a k-cycle and an $(m-k)$-cycle consisting of these cells is equal to the algebraic number of intersection points of these complementary discs, with the signs defined by the chosen orientations, see details in the next Chapter.

Chapter 13

Morse complexes. Morse inequalities. Witten–Morse complexes.