HANDOUT (ΛICTOK) 2.
CRASH COURSE IN REPRESENTATION THEORY

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The text below is the second handout to my course "Topics in the modern representation theory."

1. Basic definitions

Consider first the case of a finite group G. In a wide sense, **representation** of this group means any realization of an abstract group G as a group of transformation of some mathematical object.

The simplest object is a finite set X with cardinality $|X| = n$. In this way we get a so-called **permutational** representation of G, i.e. a homomorphism $\alpha: \ G \to S_n$, the group of permutations of n symbols.

The next simple object is a n-dimensional vector space V over some field F. We get a so-called **linear** representation of G, i.e. a homomorphism $\pi\colon G \to \mathrm{Aut}(V)$, the group of invertible linear operators in V.

Any choice of a basis in V establishes an isomorphism of $\mathrm{Aut}(V)$ with the group $GL(n,\, F)$ of $n \times n$ invertible matrices with elements from F. So, we can consider π as a matrix-valued function on G, satisfying

$$\pi(g_1 g_2) = \pi(g_1)\pi(g_2) \quad \text{ and } \quad \pi(g^{-1}) = (\pi(g))^{-1}.$$

We shall consider mainly the cases $F = \mathbb{C}$ or $\mathbb{R}$ and call them complex or real representation.

A linear representation $(\pi,\, V)$ is called

a) **reducible**, if the space V has a subspace $V_1 \subset V$, which is stable under the action of all operators $\pi(g),\ g \in G$; otherwise, it is called **irreducible**.

Algebraically, reducibility means that by an appropriate choice of a basis in V, all representation operators acquire the block-triagular form:

$$\pi(g) = \begin{pmatrix} \pi_1(g) & \pi_{12}(g) \\ 0 & \pi_2(g) \end{pmatrix};$$

b) **decomposable** if the space V is a direct sum: $V = V_1 \oplus V_2$, where V_i are stable under the action of all operators $\pi(g),\ g \in G$; otherwise, it is called **indecomposable**.

Date: Jan 2019.

Algebraically, decomposability means that by an appropriate choice of a basis, all operators $\pi(g)$ become block-diagonal:

$$\pi(g) = \begin{pmatrix} \pi_1(g) & 0 \\ 0 & \pi_2(g) \end{pmatrix};$$

c) **orthogonal** (resp. **unitary**) if V admits a G-invariant inner product. Show that for orthogonal and unitary representations the property a) implies b): any reducible representation is decomposable.

Exercise 2. Show that for any complex or real representation (π, V) of a finite group G there exists a G-invariant inner product in V.

Let (π_1, V_1) and (π_2, V_2) be two representations of the same group G over the same field F. An operator $A : V_1 \to V_2$ is called **intertwining** operator (or, simply **intertwiner**), if the following diagram is commutative:

$$\begin{array}{ccc} V_1 & \xrightarrow{\pi_1(g)} & V_1 \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ V_2 & \xrightarrow{\pi_2(g)} & V_2 \end{array} \qquad \text{for all } g \in G.$$

Let $\operatorname{Hom}_G(V_1, V_2)$ be the set of all intertwiners between V_1, V_2. It is clear that it is a vector space over F. The dimension of this space is denoted $i(\pi_1, \pi_2)$ and is called **intertwining number** for π_1, π_2. Representations π_1, π_2 are called **equivalent**, if they admit an invertible intertwiner. In appropriate bases equivalent representations are given by the same matrix-valued function. Therefore, there is no reason to distinguish equivalent representations and consider the **equivalence classes** as main object.

Physicists inwented a convenient short name **unirrep** for unitary irreducible representations. The set of equivalence classes of unirreps of a group G is called the **dual object** to G and is denoted $\widehat{G}$. For an abelian G the set $\widehat{G}$ has itself a group structure and is called a **dual** group to G.

Exercise 3. Show, that for the cyclic group $G = \mathbb{Z}/n\mathbb{Z}$ the dual group $\widehat{G}$ is isomorphic to G. (But there is no canonical isomorphism.)

2. First results

Theorem 1 (Schur's Lemma). *If π_1 and π_2 are irreducible, then*

$$i(\pi_1, \pi_2) = \begin{cases} 1 & \text{if } \pi_1, \pi_2 \text{ are equivalent,} \\ 0 & \text{otherwise.} \end{cases} \tag{1}$$

Hint. 1. Assume that there exist a non-zero intertwiner $A : V_1 \to V_2$. Then $i(\pi_1, \pi_2) \geq 1$. The subspaces $\ker A \subset V_1$ and $\operatorname{im} A \subset V_2$ are G-stable (check it!). For an irreducible representation every G-stable subspace is either $\{0\}$, or the whole space. Since $A \neq 0$, it can be only if $\ker A = \{0\}$ and $\operatorname{im} A = V_2$. Hence, A establishes an equivalence of π_1 and π_2. So, we can assume $\pi_1 = \pi_2$ and $A = 1$. For any other intertwiner B and any number $\lambda \in \mathbb{C}$ the linear combination $B - \lambda \cdot 1$ an intertwiner, hence, either zero, or invertible. Hence, any intertwiner has the form $\lambda \cdot 1$ and $i(\pi_1, \pi_2) = 0$.But the equation $\det(B - \lambda \cdot 1) = 0$ has a solution.

2. If $i(\pi_1, \pi_2) = 0$, there is no non-zero intertwiners. □

In the space $\mathrm{Fun}(G)$ of real or complex functions on G there is a natural G-invariant inner product

$$(f_1, f_2)_G = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}. \tag{2}$$

Let (π, V) be a unirrep of a finite group G. Choose an orthonormal basis $B = \{v_1, v_2, \ldots, v_n\}$ in V. The functions $\pi_{ij}(g) = (\pi(g)v_i, v_j)$, $1 \leq i, j \leq n$, are called **matrix elements** of π.

Theorem 2 (Orthogonality relations)**.** *a) If π is irreducible, then*

$$(\pi_{ij}, \pi_{kl})_G = \begin{cases} \frac{1}{\dim V} & \text{if } i = k,\ j = l, \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

b) If π and π' are not equivalent, then

$$(\pi_{ij}, \pi'_{kl})_G = 0 \quad \text{for all} \quad i, j, k, l. \tag{4}$$

Hint. a) For any operator $A \in \mathrm{Hom}(V, V')$ define the "averaged" operator

$$\overline{A} = \frac{1}{|G|} \sum_{g \in G} \pi'(g^{-1}) A \pi(g). \tag{5}$$

Check that $\overline{A}$ is an intertwiner between V, V'. Since π, π' are irreducible, it must be zero or invertible. The part b) follows.

If $\pi = \pi'$, then $\overline{A}$ must be a scalar operator $c \cdot 1$. Clearly, $\mathrm{tr}\, \overline{A} = \mathrm{tr}\, A$. Therefore, $c = \frac{\mathrm{tr}\, A}{\dim V}$. Taking $A = E_{ij}$, we get the part a). □

As a corollary we get:

Proposition 1. *For a finite group G the set $\widehat{G}$ is also finite.*

3. Characters and conjugacy classes

Let (π, V) be a representation of G. The matrix-valued function π determines the scalar function on G: $g \mapsto \mathrm{tr}\, \pi(g)$, which is called **character** of π and is denoted $\chi_\pi(g)$ (or, simply $\chi(g)$, if π is understood). Note, that this function does not depend on the choice of a basis in V.

The remarkable fact is that the character determines the representation up to equivalence.

Proposition 2. *Two representations of a finite group are equivalent, iff they have the same character.*

Recall that two elements g_1, g_2 from G are called **conjugate**, if

$$g_2 = x^{-1} g_1 x \quad \text{for some} \quad x \in G. \tag{6}$$

The set of all elements, conjugated to g is called a **conjugacy class** and is denoted by $C(g)$, or $[g]$. The collection of all conjugacy classes in G we denote by $Cl(G)$.

Exercise 4. a) Show that $[g_1] = [g_2]$ iff there exist $x, y \in G$ such that

$$g_1 = xy, \; g_2 = yx.$$

b) Show that any character χ is a constant function on any conjugacy class $[g]$.

Every $x \in G$ defines an **inner automorphism** $A(x)$ of G, acting by the formula

$$A(x) \colon g \mapsto g^x := x^{-1} g x.$$

The orbits of this action are exactly conjugacy classes. Let $\mathrm{Fun}^G(G)$ denote the subspace of functions on G, invariant under inner automorphisms.

Proposition 3. *The characters of unirreps of G form an orthonormal basis in the subspace* $Fun^G(G)$.

It follows from orthogonality relations for matrix elements. As a corollary we get

Proposition 4.

$$a) \quad |Cl(G)| = |\widehat{G}|, \qquad b) \quad \sum_{\lambda \in \widehat{G}} (\dim\, \pi_\lambda)^2 = |G|. \tag{7}$$

4. Functors Res and Ind

Let G be a finite group and $H \subset G$ be a subgroup. For any representation (π, V) of G we can restrict the map $\pi: \; G \to \mathrm{Aut}\, V$ to the subgroup H and obtain a representation $(\pi|_H, V)$ of H. It is denoted $\mathrm{Res}_H^G \pi$. The correspondence $\pi \rightsquigarrow \mathrm{Res}_H^G \pi$ defines a **functor** from the category $\mathcal{R}ep(G)$ of representations of G to the category $\mathcal{R}ep(H)$ of representations of H. It admits a remarkable dual functor Ind_H^G from $\mathcal{R}ep(H)$ to $\mathcal{R}ep(G)$, defined as follows.

For a representation (ρ, W) of the group H consider the space $L(G, H, \rho)$ of W-valued functions φ on G, satisfying the condition

$$\varphi(hg) = \rho(h)\varphi(g) \quad \text{for all} \quad h \in H, \, g \in G. \tag{8}$$

Clearly, this space is stable under the right shifts on G. We define the **induced** representation $\pi = \mathrm{Ind}_H^G$ in the space $V = L(G, H, \rho)$ by the formula

$$\Big(\pi(g)\varphi\Big)(g') = \varphi(g'g). \tag{9}$$

Exercise 5. Show that dim $\mathrm{Ind}\,_H^G\rho = \frac{|G|}{|H|}$ dim ρ.

There is another, often more convenient, definition of this representation. Let $X = H\backslash G$ be the set of right H-cosets in G. For every coset $x \in X$ choose a representative $s(x) \in x \subset G$. The map $x \mapsto s(x)$ is a **section** of the natural projection $p\colon G \to X$, i.e. has the property $p \circ s = \mathrm{Id}$.

Every element $g \in G$ can be uniquely written in the form

$$g = hs(x) \quad \text{for some } h \in H,\, x \in X, \tag{10}$$

thus providing the identification

$$G \simeq H \times X, g \simeq (h,\, x), \quad \text{where} \quad x = p(g),\, h = g(s(x))^{-1}.$$

We can now identify the space $L(G,\, H,\, \rho)$ with the space W-valued functions on X. Namely, a function $\varphi \in L(G,\, H,\, \rho)$ corresponds to the function $f \in \mathrm{Fun}(X,\, W)$ given by $f(x) = \varphi(s(x))$. In this notation the induced representation takes the form

$$\Big(\pi(g)f\Big)(x) = A(x,\, g)f(xg), \tag{11}$$

where the operator-valued function A is defined as follows. Take the element $s(x)g \in G$ and write it, using (**??**), in the form

$$\text{Master equation :} \qquad s(x)g = h(x,\, g)s(xg). \tag{12}$$

Then put $A(x,\, g) = \rho(h(x,\, g))$.

In particular, when ρ is the trivial 1-dimensional representation $\rho_0 \equiv 1$, the induced representation $\mathrm{Ind}_H^G\rho_0$ is the so-called **geometric representation** of G by shifts in the space $\mathrm{Fun}(X)$.

Thus, induced representations are the natural generalization of geometric representations: instead of numerical functions on an homogeneous space $X = H\backslash G$ we consider the sections of a G-vector bundle over X, defined by a representation ρ of H.

Exercise 6. Show, that the formula (**??**) defines a representation[1] of G iff there exists a representation $(\rho,\, W)$ of H and an invertible operator-valued function B on X, such that

$$A(x,\, g) = B(xg)\rho(h(x,\, g))B(x)^{-1},$$

where $h(x,\, g)$ is defined by the Master equation (**??**) above. In this case the representation π is equivalent to $\mathrm{Ind}_H^G\rho$.

Now, I explain in what sense the functors Res and Ind are dual. Recall, that for any pair of representation of a finite group we have defined above the intertwining number which plays, in a sense, the role of inner product.

[1] I.e., $\pi(g_1g_2) = \pi(g_1)\pi(g_2)$ and $\pi(e) = 1$.

Theorem 3 (Frobenius duality)**.** *Let* $H \subset G$*,* $\pi \in \mathcal{R}ep(G)$*,* $\rho \in \mathcal{R}ep(H)$*. Then*

$$i(Res_H^G \pi,\, \rho) = i(\pi,\, Ind_H^G \rho) = (\chi_\pi|_H, \chi_\rho)_H. \tag{13}$$

The so-called **left regular** (resp. **right regular**) representation of G acts in the space $\mathrm{Fun}(G)$ by the formula

$$\Big(R_{left}(g)f\Big)(x) = f(g^{-1}x), \qquad \Big(R_{right}(g)f\Big)(x) = f(xg).$$

They are equivalent and have dimension $|G|$.

Corollary. *The regular representation of* G *contains any unirrep* π *with multiplicity* $\dim \pi$*.*

Indeed, the regular representation is equivalent to $\mathrm{Ind}_{\{e\}}^G 1$.

5. Schur index

Here we consider relations between real, complex and quaternionic representations.

5.1. Real and complex representations.

Any real representation $(\pi,\, W)$ of a group G in a real vector space W defines a complex representation $\pi^{\mathbb{C}}$. Indeed, since $\mathrm{Mat}(n,\, \mathbb{R}) \subset \mathrm{Mat}(n,\, \mathbb{C})$, the real matrix $\pi(g)$ can be considered as a complex matrix, which defines a linear operator in the complex space $W^{\mathbb{C}} = W \otimes_{\mathbb{R}} \mathbb{C}$.[2] Note, that the character of π^C is the same as the character of π.

On the other hand, any complex vector space V can be considered as a real vector space $V_{\mathbb{R}}$ of double dimension. Namely, we replace a complex number $a+bi$ by a real 2-vector $(a,\, b)^t$.[3] So, a complex representation $(\pi,\, V)$ determines the real representation $(\pi_{\mathbb{R}},\, V_{\mathbb{R}})$.

Warninig. The operations $W \rightsquigarrow W^{\mathbb{C}}$ and $V \rightsquigarrow V_{\mathbb{R}}$ are not reciproque! What are $(W^{\mathbb{C}})_{\mathbb{R}}$ and $(V_{\mathbb{R}})^{\mathbb{C}}$?

Not every real vector space W has the form $V_{\mathbb{R}}$. It must have an additional structure: the operator $J \in \mathrm{End}\, W$ with $J^2 = -1$.

Further, for any complex representation $(\pi,\, V)$ we can define the **dual** or complex conjugate representation $(\overline{\pi},\, \overline{V})$.[4] In the appropriate bases in V and $\overline{V}$ we have

$$\overline{\pi}(g) = \overline{\pi(g)} \quad \text{for all } g \in G.$$

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[2]If B is any basis in W, the elements of $W^{\mathbb{C}}$ are linear combinations of vectors from B with complex coefficients.

[3]We prefer to deal with column vectors $\begin{pmatrix} a \\ b \end{pmatrix}$ which by typographic reason are written as a transposed row vector $(a,\, b)^t$.

[4]There is another definition of a dual space and dual representation, for which the notation $(\pi^*,\, V^*)$ is used. For finite dimensional representations of finite groups the two definitions are equivalent and both notations can be used.

It is clear that $\chi_{\overline{\pi}}(g) = \overline{\chi_\pi(g)}$. So, the character χ_π takes real values, iff $\overline{\pi}$ is equivalent to π.

5.2. **Complex and quaternionic representations.** Recall, that the skew field $\mathbb{H}$ of quaternions has a realization as a subalgebra of $\mathrm{Mat}(2, \mathbb{C})$. Namely, to a quaternion $q = a+b\mathbf{i}+c\mathbf{j}+d\mathbf{k}$ there corresponds the matrix $\begin{pmatrix} a+bi & -c+di \\ c+di & a-bi \end{pmatrix}$.

Since quaternions in general do not commute, there are two types of quaternionic vector spaces: left and right. I prefer to realize $W = \mathbb{H}^n$ as a set of column n-vectors w with quaternionic entries. The "number" $q \in \mathbb{H}$ acts from the right: $w \mapsto wq$. And an element $A \in \operatorname{End} V$ is a quaternionic matrix from $\mathrm{Mat}(n, \mathbb{H})$ acting on a column vector by multiplication from the left: $w \mapsto Aw$.

Every quaternionic space W gives rise a complex space $W_{\mathbb{C}}$ of double dimension just by replacing a quaternionic entry $q = a + b\mathbf{i} + c\mathbf{j} + d\mathbf{k}$ by the complex column 2-vector $(a+bi,\, c+di)^t$. Therefore, any real representation (π, W) of a group G in a quaternionic vector space W defines a complex representation $\pi_{\mathbb{C}}$ in the space $W_{\mathbb{C}}$ (since $\mathrm{Mat}(n, \mathbb{H}) \subset \mathrm{Mat}(2n, \mathbb{C})$). The character of this representation takes real values, because for any quaternionic matrix $A \in \mathrm{Mat}(n, \mathbb{H})$ its image $A_{\mathbb{C}} \in \mathrm{Mat}(2n, \mathbb{C}))$ has real trace.

Not every complex vector space V has the form $W_{\mathbb{C}}$. It must have an additional structure: the antilinear[5] operator J on V with $J^2 = -1$.

5.3. **Definition of the Schur index.** It is interesting an important to know when a given unirrep (π, V) comes from a real or from a quaternionic representation, i.e., has the form $(\pi^{\mathbb{C}}, W^{\mathbb{C}})$, or $(\pi_{\mathbb{C}}, W_{\mathbb{C}})$, for some real or quaternionic representation (π, W). The evident necessary condition is: the character χ_π must take real values. It turns out that this condition is also sufficient.

Theorem 4. *The unirrep (π, V) has a real character iff it comes from a real or from quaternionic representation.*

The reason is that a unirrep π with a real character is equivalent to its dual $\overline{\pi}$. Let $J \in \mathrm{Hom}(V, \overline{V})$ be the invertible intertwiner. Then $\overline{J} \in \mathrm{Hom}(\overline{V}, V)$ will be also an intertwiner and so will be $J\overline{J} \in \mathrm{End}(\overline{V})$ and $\overline{J}J \in \mathrm{End}(V)$. Since π and $\overline{\pi}$ are irreducible, we get $J\overline{J} = c \cdot 1_{\overline{V}}$, $\overline{J}J = c' \cdot 1_V$. If we multiply J by a constant λ, the numbers c, c' will be multiplied by $|\lambda|^2$. So, we can assume $c = \pm 1$, which implies $c' = c = \pm 1$.

Exercise 7. Show that in the case $c = c' = 1$ we have $(\pi, V) = (\pi^{\mathbb{C}}, W^{\mathbb{C}})$ for some real (π, W) and in the case $c = c' = -1$ we have $(\pi, V) = (\pi_{\mathbb{C}}, W_{\mathbb{C}})$ for some quaternionic (π, W).

The number c is called the **Schur index** of the unirrep π with a real character. If χ_π takes a non-real value, we put $\operatorname{ind} \pi = 0$.

[5]An operator A on a complex vector space is antilinear if $A(cv_1 + v_2) = \bar{c}Av_1 + Av_2$.

5.4. **Computation of the Schur index.**

Theorem 5 (Schur formula)**.**

$$ind\,\pi = \frac{1}{|G|}\sum_{g\in G}\chi_\pi(g^2). \tag{14}$$

The proof needs some digression in the representation theory.

There are several useful functors in the category $\mathcal{V}ect_F$ of vector spaces over a given field F. We are interested in three of them: $V \rightsquigarrow V\otimes V = V^{\otimes 2}$, $V \rightsquigarrow S^2(V)$, $V \rightsquigarrow \wedge^2(V)$, called tensor square, symmetric square and exterior square. They are related by the identity

$$V\otimes V = S^2(V)\oplus\wedge^2(V).$$

Correspondingly, for any representation $(\pi,\, V)$ we can define representations $\pi^{\otimes 2},\ S^2(\pi),\ \wedge^2(\pi)$. We also can consider **virtual** functors, which are formal linear combinations of genuine functors. E.g., for our purpose we shall use the virtual functor $\Psi_2 := S^2 - \wedge^2$. If A is an operator in n-dimensional space V with eigenvalues $(\lambda_1,\dots,\lambda_n)$, then we have

$$\operatorname{tr} S^2(A) = \sum_{i\leq j}\lambda_i\lambda_j,\quad \operatorname{tr}\,\wedge^2(A) = \sum_{i\leq j}\lambda_i\lambda_j,\quad \operatorname{tr}\Psi^2(A) = \sum_i\lambda_i^2.$$

Denote by $\mathcal{X}$ the character of the virtual representation $\Psi^2(\pi)$. Then, by putting $A = \pi(g)$ in the last formula above, we get $\mathcal{X}(g) = \chi_\pi(g^2)$ and

$$\frac{1}{|G|}\sum_{g\in G}\chi_\pi(g^2) = (\mathcal{X},\,1)_G = (S^2(\pi),\,1)_G - (\wedge^2(\Pi),\,1)_G \tag{15}$$

When χ_π is a non-real character, then $i(\pi,\,\overline{\pi}) = 0$ and the space $V^{\otimes 2}\simeq \operatorname{Hom}(V,\,V^*)$ contains no G-invariant elements. Therefore, both summands in the RHS of (**??**) vanish and RHS is zero.

When χ_π is real, then $i(\pi,\,\pi^*) = 1$ and $V\otimes V$ contains a non-zero G-invariant element J, which is unique, up to a scalar factor.

If $\operatorname{ind}\pi = 1$, then J belongs to $S^2(V)\subset V\otimes V$ and defines on V the G-invariant real structure, i.e., an isomorphism of V with some $W^{\mathbb{C}}$. In this case the RHS is equal 1.

If $\operatorname{ind}\pi = -1$, then J belongs to $\wedge^2(V)\subset V\otimes V$ and defines on V the G-invariant quaternionic structure, i.e., an isomorphism of V with some $W_{\mathbb{C}}$. In this case the RHS is equal -1.

We see, that in all cases LHS is equal $\operatorname{ind}\pi$. □

5.5. **Groups of $\mathbb{R}$-$\mathbb{H}$ type.** There is a wide class of groups G for which $C(g) = C(g^{-1})$ for all $g\in G$. E.g., all symmetric groups S_n and the group $SU(2)$ are of this type. I propose for them the name $\mathbb{R}$-$\mathbb{H}$-groups, because of the following

Proposition 5. *Every unirrep π of a $\mathbb{R}$-$\mathbb{H}$-group G has a real character, hence comes from a real or quaternionic representation. Conversely, every group with real characters of unirreps is a $\mathbb{R}$-$\mathbb{H}$-group.*

An interesting feature of $\mathbb{R}$-$\mathbb{H}$-group is

Theorem 6. *Let G be a $\mathbb{R}$-$\mathbb{H}$-group. Denote by $Inv(G)$ the set of all involutions in G, i.e., elements, satisfying $g^2 = e$. Then*

$$\sum_{\lambda \in \widehat{G}} \dim \pi_\lambda = |Inv(G)|. \tag{16}$$

The proof is based on the study of the operator σ in $\mathrm{Fun}(G)$, acting by $\big(\sigma f\big)(g) = f(g^{-1})$. We compute the trace of σ in two different way. First, choose in the space $\mathrm{Fun}(G)$ the basis of functions $\delta_x(g) = \begin{cases} 1 & \text{if } g = x \\ 0 & \text{otherwise.} \end{cases}$

It is clear that σ sends δ_x to $\delta_{x^{-1}}$ and $\operatorname{tr} \sigma = |Inv(G)|$.

Now, consider in $\mathrm{Fun}(G)$ the basis of matrix elements $(\pi_\lambda)_{ij}$ of unirreps. Decomposition of a function f with respect to this basis associates to f its Fourier transform, which is a matrix-valued function $\widehat{f}$ on $\widehat{G}$. The explicit formula is

$$\widehat{f}(\lambda) = \frac{1}{|G|} \sum_{g \in G} f(g) \pi_\lambda^*(g). \tag{17}$$

The original function f is reconstructed from $\widehat{f}$ by the formula

$$f(g) = \sum_{\lambda \in \widehat{G}} \operatorname{tr}\big(\widehat{f}(\lambda) \pi_\lambda(g)\big). \tag{18}$$

Let us describe the action of σ in terms of Fourier transform. Let $\lambda^* \in \widehat{G}$ denote the class of unirreps which contains $(\pi_\lambda)^*$ and choose $(\pi_\lambda)^*$ as π_{λ^*}. Then, clearly, $\widehat{f}(\lambda^*) = \big(\widehat{f}(\lambda)\big)^*$. So, unirreps, for which $\lambda^* \neq \lambda$ do not contribute to $\operatorname{tr} \sigma$.

Consider now the case $\lambda^* = \lambda$. We can write the operator $\pi_\lambda(g)$ as a complex form of real or a quaternionic matrix.

Exercise 8. Show that

$$\widehat{\sigma(f)}(\lambda) = \begin{cases} \widehat{f}(\lambda)^t & \text{if } \lambda \text{ is of real type} \\ -\widehat{f}(\lambda)^t & \text{if } \lambda \text{ is of quaternionic type.} \end{cases} \tag{19}$$

Corollary. *The contribution of $\lambda = \lambda^* \in \widehat{G}$ in trace of σ is $\dim \pi_\lambda$ for real type and $-\dim \pi_\lambda$ for quaternionic type.*

□

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