HANDOUT 3.
ANATOMY OF TRIANGULAR GROUP G_n

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1. Preface

The set of triangular matrices is one of the most fundamental objects in mathematics (as well as the set of natural numbers or the group of permutations). It can be endowed by several structures and the interaction between these structures is very interesting and leads to many remarkable discoveries.

Our main goal is to understand the structure of the group G_n of upper unitriangular $n \times n$-matrices.[1] The experimental data suggests that the answer depends rather weakly on the basic field K. Thus, the tempting idea is to try to understand the main cases $K = \mathbb{R}$, $\mathbb{C}$ by studying first the case of a finite field $\mathbb{F}_q$, when our group is finite. Triangular group and its Lie algebra

The description of conjugacy classes for G_n is rather complicated (and still unsolved) problem. But there are at least two coarser equivalence relation between group elements.

One of them replace the adjoint action of the group G on its Lie algebra

$$\mathfrak{g}: \operatorname{Ad}(g): \; X \mapsto gXg^{-1}$$

by the two-side action of $G \times G$

$$A(g_1,\, g_2): \; X \mapsto g_1 X g_2^{-1}.$$

So, instead of adjoin orbits in $\mathfrak{g}$, corresponding to conjugacy classes in G, we get so-called **superclasses**, which are much easier do describe and classify. This approach was invented independently by my former graduate student Ning Yang [?a]nd by Carlos Andre [?]. See the survey paper of 28 authors [?].

Second is related with Jordan theorem about conjugacy classes in $GL(n)$ and for the triangular group was developed by T.Springer and M. Gerstenhaber. Interesting new facts were discovered recently in papers by Anna Melnikov [?].

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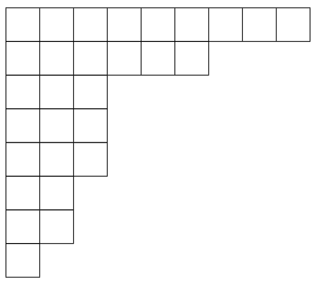
[1]This group is also denoted by TU_n or N_n. In this handout we use the notation G_n for our group and $\mathfrak{g}_n$ for its Lie algebra, omitting the index n when possible.

2. Springer theory

2.1. **Partitions.** Consider a presentation of a natural number n as an (unordered) sum of natural summands. It is called **partition** and denoted by

$$(1) \qquad \lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_s) \quad \text{where} \quad \sum_{i=1}^{s} \lambda_i = n \qquad (\lambda-\textbf{notation}).$$

Let $\mathcal{P}_n$ denote the set of all partitions of n. A convenient graphic image of a partition λ is the so-called **Young diagram** $D = D(\lambda)$ of n boxes arranged in s rows of lengths λ_i, $1 \leq i \leq s$. E.g., $D(\lambda)$ looks like



for $\lambda = (9, 6, 3, 3, 3, 2, 2, 1)$.

The lengths of columns in $D(\lambda)$ form the so-called **dual partition**

$$\lambda^* = (\lambda_1^* \geq \lambda_2^* \geq \dots \lambda_t^*)$$

of the same number n. For an example above $\lambda^* = (8, 7, 5, 2, 2, 2, 1, 1, 1)$. (A control question: is it clear that $s = \lambda_1^*, \quad t = \lambda_1$?)

Another way to write a partition λ is to use two symbolic notations.

$$\alpha\textbf{-notation}: \qquad \lambda = 1^{\alpha_1} 2^{\alpha_2} \cdots n^{\alpha_n}, \text{ where } \sum_{k=1}^{n} k\alpha_k = n.$$

Here $\alpha_k = \alpha_k(\lambda)$ is the number of parts λ_i of size k. E.g., for our example

$$\lambda = 1^1 2^2 3^3 4^0 5^0 6^1 7^0 8^0 9^1, \quad \lambda^* = 1^3 2^3 4^0 5^1 6^0 7^1 8^1, \quad n = 29.$$

$$\mu\beta\textbf{-notation}: \qquad \lambda = \mu_1^{\beta_1} \mu_2^{\beta_2} \cdots \mu_m^{\beta_m}, \text{ where } \sum_{i=1}^{m} \beta_i \mu_i = n.$$

Here we assume $\mu_1 > \cdots > \mu_m > 0$ (and omit powers $\beta_i = 1$); for the example above we have: $\quad \lambda = 963^3 2^2 1, \quad \lambda^* = 8752^3 1^3$.

Warning. The numbers μ_i in $\mu\beta$-notation are taken from numbers λ_i in (1): we get rid of repetitions and mention every number only once. The powers β's are just non-zero α's.

The following relations are evident (follow immediately from definitions):

$$(2) \qquad \begin{array}{c} a)\ \lambda_k^* = \#\{i \,|\, \lambda_i \geq k\}, \qquad b)\ \lambda_k^* - \lambda_{k+1}^* = \alpha_k(\lambda), \\ c)\ \lambda_i = \mu_k \text{ and } \alpha_{\lambda_i} = \beta_k \quad \text{for} \quad i \in (\beta_1 + \cdots + \beta_{k-1},\ \beta_1 + \cdots + \beta_k]. \end{array}$$

2.2. **The Jordan types of triangular matrices.** Every matrix $g \in G_n(F)$ has the form $g = 1 + A$, where $A \in \mathfrak{g}_n$ is an upper triangular nilpotent $n \times n$-matrix with $A^n = 0$. It is well-known that in an appropriate basis A takes the block-digonal form with diagonal blocks looking like

$$J_{\lambda_i} = \begin{pmatrix} 0 & 1 & 0 & \dots & 0 & 0 \\ 0 & 0 & 1 & \dots & 0 & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & \dots & 0 & 1 \\ 0 & 0 & 0 & \dots & 0 & 0 \end{pmatrix}.$$

The sizes λ_i, $1 \le i \le s$, of these **Jordan blocks** are defined uniquely up to order. Thus, they form a **partition** $\lambda \in \mathcal{P}_n$. We call this partition the **Jordan type** of a nilpotent matrix A and denote it by $J(A)$. The set of all matrices of type λ in $\mathfrak{g}$ we denote by $\mathfrak{g}_\lambda$.

2.3. **How to compute $J(A)$ for a given $A \in \mathfrak{g}$.** The obvious invariants of a matrix A under conjugation are the ranks of powers of A. It is known, that for a nilpotent $A \in \mathrm{Mat}(n, K)$ it is a full system of invariants: two nilpotent matrices are conjugate, iff their powers have the same ranks. Actually, for $n \times n$-matrices it is enough to check this for powers $1, 2, \dots, n-1$.

Let us denote $\mathrm{rk}\,(A^k)$ by $r_k(A)$. It is convenient to put $A^0 = 1$. Then for any $A \in \mathfrak{g}_n$ we have

$$n = r_0(A) > r_1(A) \ge \dots \ge r_{n-1}(A) \ge r_n(A) = 0.$$

The relations between the sequence $\{r_k(A)\}$ and the Jordan type of A are described by

Lemma 1. *Let $J(A) = \lambda$. Then for $k \ge 1$ we have the relations:*

$$(3) \qquad r_{k-1}(A) - r_k(A) = \lambda_k^*, \qquad r_{k-1}(A) - 2r_k(A) + r_{k+1}(A) = \alpha_k(\lambda).$$

Proof. Denote by J_n the nilpotent Jordan block of size n. To write the formula for ranks of powers of $(J_n)^m$, we introduce for $l \in \mathbb{Z}$ the notations

$$(l)_+ = \begin{cases} l, & \text{if } \quad l > 0 \\ 0 & \text{otherwise,} \end{cases} \qquad \theta(l) := (l)_+ - (l-1)_+ = \begin{cases} 1 & \text{if } \quad l > 0 \\ 0 & \text{otherwise.} \end{cases}$$

Using these notations, we can write: $r_k(J_n) = (n-k)_+$. Also, from the decomposition $A = \oplus_i J_{\lambda_i}$ we obtain: $r_k(A) = \sum_i (\lambda_i - k)_+$. Hence,

$$\begin{aligned} r_{k-1}(A) - r_k(A) &= \textstyle\sum_i \big((\lambda_i - k + 1)_+ - (\lambda_i - k)_+\big) \\ &= \textstyle\sum_i \theta(\lambda_i - k + 1) = \#\{i \,|\, \lambda_i \ge k\} = \lambda_k^*. \end{aligned}$$

We proved the first of relations (3). The second follows now from (2).

□

Thus, the type $J(A)$ is determined by the sequence $\{r_k(A)\}$. Moreover, from $\lambda_k^* \ge \lambda_{k+1}^*$ it follows that the sequence $\{r_k(A)\}$ is not only weakly decreasing from n to zero, but is convex: $r_k(A) \le \frac{r_{k-1}(A) + r_{k+1}(A)}{2}$.

Exercise 1. Show that a convex weakly decreasing sequence starting with $r_0 = n$ and finishing with $r_n = 0$, must look like

$$n = r_0 > r_1 > \cdots > r_l = r_{l+1} = \cdots = r_n = 0 \tag{4}$$

for some number l between 0 and n.

Hint. For $k \in [1, \ldots, n]$ the "slope" $s_k := r_{k-1} - r_k$ is non-negative and weakly decreasing. Moreover, $\sum_{k=1}^{n} s_k = n$. Show that (4) is true for $l = \min\{k \mid r_k = 0\}$.

2.4. **Neighbor partitions.** We say that partitions $\lambda \in \mathcal{P}_n$ and $\Lambda \in \mathcal{P}_{n+1}$ are **neighbor**, if the diagram $D(\Lambda)$ is obtained from $D(\lambda)$ by adding one box. In this case we write $\lambda \to \Lambda$, or, in more details, $\lambda \overset{(i,j)}{\to} \Lambda$, when the extra box is in i-th row and j-th column. Such pairs of partitions naturally arise when we consider a matrix $\mathcal{A} \in \mathfrak{g}_{n+1}$ of the form

$$\mathcal{A} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}, \quad A \in \mathfrak{g}_n, \quad a \in K^n. \tag{5}$$

Theorem 1. *Let $A \in \mathfrak{g}_n$ be of Jordan type $\lambda \in \mathcal{P}_n$ and $\mathcal{A} \in \mathfrak{g}_{n+1}$ is defined as in (5) for some $a \in K^n$. Then*

a) The partition $\Lambda = J(\mathcal{A}) \in \mathcal{P}_{n+1}$ is a neighbor of λ.

b) If Λ is a neighbor of λ, then the set of all $a \in K^n$ for which $J(\mathcal{A}) = \Lambda$, is a non-empty quasi-affine algebraic manifold of the form K^t, or $K^t \backslash K^s$.

Proof. For given pair (A, a) and $k \geq 0$ put $\delta_k = r_k(\mathcal{A}) - r_k(A)$.

Since $\mathcal{A}^k = \begin{pmatrix} A^k & A^{k-1}a \\ 0 & 0 \end{pmatrix}$, the numbers δ_k can take only two values:

$$\delta_k = \begin{cases} 0, & \text{if the last column of } \mathcal{A}^k \text{ is dependent on other columns} \\ 1, & \text{if it is independent.} \end{cases}$$

More precisely, define the subspaces $W_k(A) \subset K^n$ by $W_0 := \emptyset$ and

$$W_k(A) = \{a \in K^n \mid A^{k-1}a \in \mathrm{im}\,(A^k)\} \text{ for } k \geq 1. \tag{6}$$

Denote by $w_k(A)$ the dimension of $W_k(A)$.[2] Since this dimension depends only on the type $\lambda = J(A)$, we sometimes write $w_k(\lambda)$ instead of $w_k(A)$.

The dependence of numbers δ_k on $a \in K^n$ looks as follows:

$$\delta_k = \begin{cases} 0, & \text{if } a \in W_k, \\ 1, & \text{if } a \notin W_k. \end{cases} \tag{7}$$

We observe that $A^{k-1}a \in \mathrm{im}(A^k)$ implies $A^k a \in \mathrm{im}(A^{k+1})$. Hence, the sequence $\{W_k\}$ is weakly increasing:

$$\emptyset = W_0(A) \subset W_1(A) \subset \cdots \subset W_n(A) \subset W_{n+1}(A) = K^n.$$

[2] Here it is convenient to consider the empty set as a vector space of dimension $-\infty$. Then the formula $\#W_k(\mathbb{F}_q) = q^{w_k}$ holds true also for $k = 0$.

Therefore, $\delta_k = 0$ implies $\delta_{k+1} = 0$. Since $\delta_0 = 1$ and $\delta_{n+1} = 0$, there must be a number $s \in \{1, 2, \ldots, n\}$ such that

$$\delta_k = \begin{cases} 1, & \text{for } k < s \\ \\ 0, & \text{for } k \geq s. \end{cases}$$

From (3) we obtain

$$\delta_{k-1} - \delta_k = \Lambda^*_k - \lambda^*_k, \quad \delta_{k-1} - 2\delta_k + \delta_{k+1} = \alpha_k(\Lambda) - \alpha_k(\lambda), \tag{8}$$

which means

$$\Lambda^*_i - \lambda^*_i = \begin{cases} 1 & \text{for } i = s \\ 0 & \text{otherwise,} \end{cases} \qquad \alpha_i(\Lambda) - \alpha_i(\lambda) = \begin{cases} -1, & \text{for } i = s-1 \\ 1 & \text{for } i = s \\ 0 & \text{otherwise.} \end{cases} \tag{9}$$

The first relation shows that the columns of $D(\Lambda)$ have the same lengths as the columns of $D(\lambda)$, with one exception: in the s-th column an extra box appears.

The second relation means that one of rows[3] of length $s-1$ is replaced by a row of length s. Let t be the number of the new row. Then $\lambda \overset{(t,s)}{\to} \Lambda$ and we have proved the part a). To prove the part b) we need the explicit formula for $w_k(A)$. It is given by

Lemma 2.

$$w_k(A) = n - \lambda^*_k. \tag{10}$$

Proof. The operator A^{k-1} sends K^n onto $\operatorname{im}(A^{k-1})$ and annihilate the subspace $\ker(A^{k-1})$ of dimension $n - r_{k-1}$. Thus, it established a bijection of the factor space $K^n/\ker(A^{k-1})$ and $\operatorname{im}(A^{k-1})$. Therefore, the preimage of $\operatorname{im}(A^k)$ in $K^n/\ker(A^{k-1})$ has the same dimension as $\operatorname{im}(A^k)$, which is $r_k(A)$. Hence, $W_k/\ker(A^{k-1})$ has dimension $r_k(A)$ and we get the desired relation $\dim W_k = n - r_{k-1}(A) + r_k(A) = n - \lambda^*_k$. □

We use the formula (10) to compute recursively the number of points in the set $\mathfrak{g}_\lambda(\mathbb{F}_q)$ of all matrices of type λ in $\mathfrak{g}(\mathbb{F}_q)$.

Let $p_n : \mathfrak{g}_{n+1} \to \mathfrak{g}_n$ be the natural projection, acting by deleting of the last column from every matrix. Lemma 3 shows that $p_n(\mathfrak{g}_\Lambda) = \bigcup_{\lambda \to \Lambda} \mathfrak{g}_\lambda$.

For any pair of neighbor partition $\lambda \overset{(t,s)}{\to} \Lambda$ we choose a matrix $A \in \mathfrak{g}_\lambda$ and denote by $C(A, \Lambda)$ the set of all $a \in \mathbb{F}_q^n$ such that the matrix $\mathcal{A} = \begin{pmatrix} A & a \\ 0 & 0 \end{pmatrix}$ has type Λ. The cardinality of $C(A, \Lambda)$ is the same for all A of the Jordan type $J(A) = \lambda$; we denote it by $c(\lambda, \Lambda)$.

[3]Actually, the first (upper) of such rows.

Theorem 2. *Use the $\mu\beta$-notation and write λ in the form $\mu_1^{\beta_1}\mu_2^{\beta_2}\cdots\mu_m^{\beta_m}$ with $\mu_1 > \cdots > \mu_m > 0$ and $\beta_i > 0$. If $\lambda \overset{(t,s)}{\to} \Lambda$, then*

$$c(\lambda,\,\Lambda) = \#(W_s\backslash W_{s-1}) = q^{n-\beta_1-\cdots-\beta_{s-1}}(1-q^{-\beta_s}). \tag{11}$$

The proof follows from (9) and (10).

Proposition 1 (The recurrent formula)**.** *The cardinalities $P_\lambda(q) := \#\mathfrak{g}_\lambda$ are polynomials in q, satisfying recurrent relations*

$$P_\Lambda(q) = \sum_{\lambda\to\Lambda} c(\lambda,\,\Lambda)P_\lambda(q) \quad \textit{for all} \quad \Lambda\in\mathcal{P}. \tag{12}$$

2.4.1. *A worked example.* Let $\lambda = 9,6,3,3,3,2,2,1$. Suppose we take a matrix $A \in \mathfrak{g}_{29}$ of type λ and add to it randomly one extra column (in q^{29} equiprobable ways). Then we obtain a matrix of type $\Lambda^{(k)}$ with an extra box in k-th column with some probability p_k, depending on k, q and λ. In our example only six types have the non-zero probabilities:

$\Lambda^{(1)} = (10,6,3,3,3,2,2,1)$, $\Lambda^{(2)} = (9,7,3,3,3,2,2,1)$,
$\Lambda^{(3)} = (9,6,4,3,3,2,2,1)$, $\Lambda^{(6)} = (9,6,3,3,3,3,2,1)$,
$\Lambda^{(7)} = (9,6,3,3,3,2,2,2)$, $\Lambda^{(8)} = (9,6,3,3,3,2,2,1,1)$.

The numerical data are collected in the table, where $\varepsilon := q^{-1}$.

k	0	1	2	3	4	5	6	7	8	9	10
λ_k		9	6	3	3	3	2	2	1	0	0
r_k	29	21	14	9	7	5	3	2	1	0	0
λ_k^*		8	7	5	2	2	2	1	1	1	0
α_k		1	2	3	0	0	1	0	0	1	0
w_k	$-\infty$	21	22	24	27	27	27	28	28	28	29
p_k	0	ε^8	$\varepsilon^7-\varepsilon^8$	$\varepsilon^5-\varepsilon^7$	$\varepsilon^2-\varepsilon^5$	0	0	$\varepsilon-\varepsilon^2$	0	0	$1-\varepsilon$

2.5. **Young graph Y and the polynomials $P_\lambda(q)$.** Let Y be a graph, whose vertices are partitions λ (or their young diagrams $D(\lambda)$), and the arrows $\lambda\to\Lambda$ correspond to neighbor partitions. For every matrix $A\in\mathfrak{g}_n$ denote by $A\rfloor_k$ the matrix, formed by k first columns of A. The sequence $\{J(A\rfloor_k)\}$, $0\le k\le n$, defines a path in Y which starts at empty diagram at time 0 and finished at $D(\lambda)$ at the time n. To determine this path one has to fill up the boxes of the diagram $D(\lambda)$ by numbers $1,2,\ldots,n$, so that the boxes with numbers $\le k$ form the diagram $J(A\rfloor_k)$. The resulting table is called *standard tableau* and is denoted by $T(A)$.

It is known that the unirreps of the group S_n are naturally labeled by partitions $\lambda\in\mathcal{P}_n$. Moreover, the dimension d_λ of the unirrep π_λ is equal to the number of standard tableaux of shape λ.

We denote by $P_T(q)$ the number of matrices $A\in\mathfrak{g}_n(\mathbb{F}_q)$ with $T(A)=T$. Clearly,

$$\#\mathfrak{g}_\lambda = \sum_{T\mapsto\lambda} P_T(q), \quad \text{where } T\mapsto\lambda \text{ means that } T \text{ is of shape } \lambda. \tag{13}$$

From (11) and (12) we get a recurrent relations for P_T:

Proposition 2. *If* $\lambda \overset{t,s}{\to} \Lambda$, $T_n \mapsto \lambda$ *and* $T_{n+1} \mapsto \Lambda$, *then*

$$P_{T_{n+1}}(q) = q^n(q^{-\lambda^*_s} - q^{-\lambda^*_{s-1}})P_{T_n}(q). \tag{14}$$

In the section 2.2 we introduced the notation $\mathfrak{g}_\lambda$ for the subset of $\mathfrak{g}(\mathbb{F}_q)$ consisting of matrices of Jordan type λ and denoted by $P_\lambda(q)$ the number of points in $\mathfrak{g}(\mathbb{F}_q)$. The proposition above implies that $P_\lambda(q)$ is a polynomial in q and allows to compute it recursively. To write the result, we need some notations.

In combinatorics the quantity $\frac{q^k-1}{k-1}$ is denoted $[k]_q$ and considered as a "q-analogue" of a natural number k . Analogously, one define the q-analogues of the factorial and binomial coefficients. It is convenient also to have a special notation Q for the number $q-1=|\mathbb{F}_q^\times|$.

One can check that in this notations at least some polynomials P_λ have a simple form. In particular,

$$P_{1^k}=1, \quad P_{k+1}=q^{\binom{k}{2}}Q^k, \quad P_{21^{k-1}}=kq^k-[k]_q,$$
$$P_{k\,1}=q^{\binom{k}{2}-1}Q^{k-1}(kq+1),\dots$$

Exercise 2 (Optional)**.** Find the explicit formula for $P_{k\,1^m}$ (hook diagram).

Exercise 3 (Optional)**.** Same for $P_{2^k\,1^m}$ (two-columns diagram).

Exercise 4 (Optional)**.** Find the pattern in the very remarkable family of polynomials $A_n=\sum_{2k+m=n}P_{2^k\,1^m}$ (see the table below).

$A_0 = 1,$
$A_1 = 1,$
$A_2 = q,$
$A_3 = 2q^2 - q,$
$A_4 = 2q^4 - q^2,$
$A_5 = 5q^6 - 4q^5,$
$A_6 = 5q^9 - 5q^7 + q^5,$
$A_7 = 14q^{12} - 14q^{11} + q^7,$
$A_8 = 14q^{16} - 20q^{14} + 7q^{12},$
$A_9 = 42q^{20} - 48q^{19} + 8q^{15} - q^{12},$
$A_{10} = 42q^{25} - 75q^{23} + 35q^{21} - q^{15},$
$A_{11} = 132q^{30} - 165q^{29} + 44q^{25} - 10q^{22}$
$A_{12} = 132q^{36} - 275q^{34} + 154q^{32} - 11q^{26} + q^{22}$
$A_{13} = 429q^{42} - 572q^{41} + 208q^{37} - 65q^{34} + q^{26}$
$A_{14} = 429q^{49} - 1001q^{47} + 637q^{45} - 77q^{39} + 13q^{35}$
$A_{15} = 1430q^{56} - 2002q^{55} + 910q^{51} - 350q^{48} + 14q^{40} - q^{35}$
$A_{16} = 1430q^{64} - 3640q^{62} + 2548q^{60} - 440q^{54} + 104q^{50} - q^{40}$
$P_{17} = 4862q^{72} - 7072q^{29} + 3808q^{67} - 1700q^{64} + 119q^{56} - 16q^{51}.$

3. Statistical mechanics approach

3.1. **Setting.** The set $\Phi := \mathfrak{g}_n(\mathbb{F}_q) \times \mathfrak{g}_n(\mathbb{F}_q) \times \mathfrak{g}_n^*(\mathbb{F}_q)$ can be considered as a phase space of some statistical mechanical system. The natural candidate to the partition function is the expression

$$\Sigma_n(q) = \sum_{X,Y\in\mathfrak{g}_n, F\in\mathfrak{g}_n^*} \chi(\langle F,\, [X,\, Y]\rangle). \tag{15}$$

Here χ is any non-trivial additive character[4] of the field $\mathbb{F}_q$. To compute this expression, we use the identity

$$\sum_{f\in V^*} \chi(\langle f,\, v\rangle) = q^N \delta(v) \quad \text{for any } f \in V^*,$$

where V is N-dimensional vector space over $\mathbb{F}_q$, V^* is the dual vector space and $\delta(v) = \begin{cases} 1 & \text{if} \quad v = 0, \\ 0 & \text{otherwise.} \end{cases}$

Applying this identity for the case $V = \mathfrak{g}$, $v = [X,\, Y]$, $N = \binom{n}{2}$, we get

$$\Sigma_n(q) = q^{\binom{n}{2}} \sum_{X,\, Y\in\mathfrak{g}} \delta([X,\, Y]) = |G| \cdot |CP(\mathfrak{g})|, \tag{16}$$

where $CP(\mathfrak{g}) \subset \mathfrak{g} \times \mathfrak{g}$ denotes the set of pairs of commuting elements in the Lie algebra $\mathfrak{g}$.

On the other hand, rewriting $\langle F,\, [X,\, Y]\rangle$ as $\langle [Y,\, F],\, X\rangle$, we get

$$\Sigma_n(q) = q^{\binom{n}{2}} \sum_{Y\in\mathfrak{g}, F\in\mathfrak{g}^*} \delta(K(Y)F) = \#G \cdot \sum_{F\in\mathfrak{g}^*} |stab_{\mathfrak{g}}(F)|.$$

The last sum over $F \in \mathfrak{g}^*$ we split into sums over $F \in \Omega$ for a given coadjoint orbit Ω and the sum over all coadjoint orbits $\Omega \in \mathfrak{g}^*/G$. Since

$$|\Omega_F| = \frac{|G|}{|Stab_G(F)|} = \frac{|\mathfrak{g}|}{|stab_{\mathfrak{g}}(F)|},$$

we finally get

$$\Sigma_n(q) = q^{n(n-1)} |\mathfrak{g}^*/G| = q^{n(n-1)} |\widehat{G}|. \tag{17}$$

It implies the useful relation (which can be derived in many other ways):

$$|CP(G)| = |G| \cdot |\widehat{G}|. \tag{18}$$

3.2. **Additional parameter: degree of an extension.** Let $\mathfrak{g}_{n,m}^*$ denote the union of all $2m$-dimensional orbits in $\mathfrak{g}^*$. Recall that the main information of the structure of $\widehat{G}$ in terms of coadjoint orbits is encoded in the family of polynomials $P_n(q,\, t) = \sum_{m\geq 0} t^m \cdot |\mathfrak{g}_{n,m}^*(\mathbb{F}_q)|$.

It is interesting that these polynomials also can be expressed as partition function, similar to Σ_n above. We come to this in the next chapter.

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[4]The sum in question does not depend on the choice of χ.