LECTURE 4.
COADJOINT ORBITS OF THE TRIANGULAR GROUP G_n

A.A.KIRILLOV

Recall that $G_n(K)$ is the group of K-points of the algebraic group G_n of upper unitriangular $n \times n$ matrices, $\mathfrak{g}_n(K)$ is its Lie algebra and $\mathfrak{g}^*(K)$ is the dual K-vector space. We omit the mention of K when it is clear from the context, or is irrelevant.

When $K = \mathbb{R}$ or $\mathbb{C}$, the groups G_n are connected and simply connected nilpotent Lie groups. For such groups the method of orbits establishes the bijection between the set $\widehat{G}_n$ of equivalence classes of unirreps of G_n and the set $\mathcal{O}_n^* = \mathfrak{g}_n^*/G_n$ of coadjoint orbits (i.e. G_n-orbits in $\mathfrak{g}_n^*$).[1]

Unfortunately, the structure of the sets $\mathcal{O}_n^*$ of coadjoint orbits for triangular groups is much more complicated, than for classical groups.

Here, as in the case of adjoint orbits, the result is practically independent of the basic field K. So, we try to understand first the structure of $\mathcal{O}_n^*(K)$ for finite fields $K = \mathbb{F}_q$.

1. Basic facts and definitions

Recall first, that for a connected and simply connected nilpotent Lie group G the exponential map $\exp : \mathfrak{g} \to G$ is a diffeomorphism, i.e. a smooth bijection.

For algebraic groups over finite fields the standard formula for exp does not make sense. But for $K = \mathbb{F}_q$, $q = p^k$, we can replace it by the "fake" exponential

$$\exp_p(X) = \sum_{k=0}^{p-1} \frac{X^k}{k!}, \tag{1}$$

or, even by more simple map $\exp_2(X) = 1 + X$. It is clear that these maps establish the bijection from $\mathfrak{g}_n(K)$ to $G_n(K)$ and are compatible with the action of $G_n(K)$:

$$\exp_p(\mathrm{Ad}h\, X) = h \exp_p(X) h^{-1} \quad \text{for} \quad h \in G_n(K),\ X \in \mathfrak{g}_n(K). \tag{2}$$

They also send subalgebras of $\mathfrak{g}_n(K)$ to subgroups of $G_n(K)$.

Date: Spring 2019.

[1] The general references for this and other facts related to the orbit method are [?K0], [?K1] and [?K4].

For any $g \in G_n(K)$ the operator of coadjoint action of g on $\mathfrak{g}_n^*(K)$ is $(\operatorname{Ad} g^{-1})^*$. Let us denote it, for short, by $K(g)$. Then

$$\langle K(g)F,\, X\rangle = \langle F,\, \operatorname{Ad} g^{-1} X\rangle. \tag{3}$$

The corresponding action of the Lie algebra g_n has the form:

$$\langle K_*(X)F,\, Y\rangle = -\langle F,\, [X,\, Y]\rangle. \tag{4}$$

Now we formulate the simple but very important property of coadjoint orbits.

Theorem 1. *a) The dimension of the orbit $\Omega \subset \mathfrak{g}_n^*(K)$, passing through the point $F \in \mathfrak{g}_n(K)$, is equal to the rank of the bilinear form B_F on $g_n(K)$, given by $B_F(X,\, Y) = \langle F,\, [X,\, Y]\rangle$.*

b) On every coadjoint orbit $\Omega \subset \mathfrak{g}_n^*(K)$ there is a canonically defined differential 2-form σ_Ω. Namely, let $X,\, Y \in \mathfrak{g}_n,\ F \in \Omega$; define the vectors $\xi,\, \eta \in T_F\Omega$ by $\xi = K_*(X)F,\ \eta = K_*(Y)F$. Then

$$\sigma_\Omega(F)(\xi,\, \eta) = B_F(X,\, Y). \tag{5}$$

c) The form σ_Ω is closed, non-degenerate and $G_n(K)$-invariant.

Proof. a) Let $Stab(F)$ be the stabilizer of F in $G_n(k)$. The direct computation (see (4)) shows that the Lie algebra $stab(F)$ of $Stab(F)$ coincides with the kernel of B_F. Hence, $\dim \Omega_F = \dim T_F\Omega_F = \dim \mathfrak{g} - \dim\ker B_F = \operatorname{rk} B_F$.

The proof of b) and c) needs some (elementary) technique from differential geometry of homogeneous manifolds. So, we readdress the reader to books [?6, ?8]. (See also the section below)

Note, that the structure of the coadjoint orbits as algebraic homogeneous manifolds is very simple. They are smooth algebraic submanifolds in $\mathfrak{g}_n^*$, isomorphic to even-dimensional affine spaces $\mathbb{A}^{2m}$. The element $g \in G_n(K)$ acts on $\mathfrak{g}^*(K)$ by a linear transformation $K(g)$ and the matrix elements of this transformation are polynomial functions on $G_n(K)$ (see examples below).

Since the form B_F is antisymmetric, its rank is an even number. Thus, the space $\mathfrak{g}^*(K)$ splits into parts $\mathfrak{g}_{n,m}(K)$, where this rank equals $2m$. Each part $\mathfrak{g}_{n,m}(K)$ splits into coadjoint orbits of dimension $2m$. We denote the collection of all coadjoint orbits of dimension $2m$ for $G_n(K)$ by $\mathcal{O}_{n,m}(K)$. We consider them in more details below (see section 3.).

1.1. **Examples.** Before going farther, it is instructive to look in details on two examples in small dimensions.

Example 1. The Lie algebra $\mathfrak{g}_3$ consists of matrices $X = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & 0 & \beta \\ 0 & 0 & 0 \end{pmatrix}$.

The elements of $\mathfrak{g}_3^*$ we write in the form $F = \begin{pmatrix} * & * & * \\ x & * & * \\ z & y & * \end{pmatrix}$ (guess, why?).

For $X = \begin{pmatrix} 0 & \alpha & \gamma \\ 0 & \beta & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $X' = \begin{pmatrix} 0 & \alpha' & \gamma' \\ 0 & 0 & \beta' \\ 0 & 0 & 0 \end{pmatrix}$ we have $[X,\, X'] =$ $\begin{pmatrix} 0 & 0 & \alpha\beta' - \alpha'\beta \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$ and $B_F(X,\, X') := \langle F,\, [X,\, X'] \rangle = z(\alpha\beta' - \alpha'\beta)$.

The matrix of the form B_F on $\mathfrak{g}_3$ in a natural basis is $\begin{pmatrix} 0 & z & 0 \\ -z & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$.

So, the subset $\mathfrak{g}_{3,0}^* \subset \mathfrak{g}_3^*$ is given by the equation $z = 0$. It has dimension 2 and splits into zero-dimensional orbits, the points $(x, y, 0)$.

The set $\mathfrak{g}_{3,1}^*$ is given by the inequality $z \neq 0$, has dimension 3 and splits into 2-dimensional orbits, the planes $z = \mathit{const}$.

Finally, we get: $\mathcal{O}_{3,0}^* \simeq \mathbb{A}^2_{x,y}$, $\mathcal{O}_{3,1}^* \simeq \mathbb{A}^1_z \backslash \mathbb{A}^0 \simeq GL_1$.

Example 2. The space $\mathfrak{g}_4^*$ consists of matrices $F = \begin{pmatrix} * & * & * & * \\ x_1 & * & * & * \\ y_1 & x_2 & * & * \\ z & y_2 & x_3 & * \end{pmatrix}$.

For $X = \begin{pmatrix} 0 & \alpha_1 & \beta_1 & \gamma \\ 0 & 0 & \alpha_2 & \beta_2 \\ 0 & 0 & 0 & \alpha_3 \\ 0 & 0 & 0 & 0 \end{pmatrix}$, $X' = \begin{pmatrix} 0 & \alpha_1' & \beta_1' & \gamma' \\ 0 & 0 & \alpha_2' & \beta_2' \\ 0 & 0 & 0 & \alpha_3' \\ 0 & 0 & 0 & 0 \end{pmatrix}$ we have

$$[X,\, X'] = \begin{pmatrix} 0 & 0 & \alpha_1\alpha_2' - \alpha_1'\alpha_2 & \alpha_1\beta_2' + \beta_1\alpha_3' - \alpha_1'\beta_2 - \beta_1'\alpha_3 \\ 0 & 0 & 0 & \alpha_2\alpha_3' - \alpha_3'\alpha_2 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

Therefore, $B_F(X,\, X') = y_1(\alpha_1\alpha_2' - \alpha_1'\alpha_2) + y_2(\alpha_2\alpha_3' - \alpha_3'\alpha_2) + z(\alpha_1\beta_2' + \beta_1\alpha_3' - \alpha_1'\beta_2 - \beta_1'\alpha_3)$.

The matrix of B_F is $\begin{pmatrix} 0 & y_1 & 0 & 0 & z & 0 \\ -y_1 & 0 & y_1 & 0 & 0 & 0 \\ 0 & -y_2 & 0 & z & 0 & 0 \\ 0 & 0 & -z & 0 & 0 & 0 \\ -z & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}$

The subset $\mathfrak{g}_{4,0}^*$ is given by $z = y_1 = y_2 = 0$, has dimension 3 and splits into zero-dimensional orbits, distinguished by values of x_1, x_2, x_3.

The set $\mathfrak{g}^*_{4,1}$ is given by $z = 0,\ (y_1, y_2) \neq (0, 0)$, has dimension 5 and splits into 2-dimensional orbits, distinguished by values of y_1, y_2 and the quantity $z' = x_1y_2 + x_3y_2$.[2]

The set $\mathfrak{g}^*_{4,2}$ is given by $z \neq 0$, has dimension 6 and splits into 4-dimensional orbits, distinguished by values of two invariants: z and $\Delta := zx_2 - y_1y_2$.

Finally, we get: $\mathcal{O}^*_{4,0} \simeq \mathbb{A}^3_{x_1,x_2,x_3},\ \mathcal{O}^*_{4,1} \simeq \mathbb{A}^3_{y_1,y_2,z'} \backslash \mathbb{A}^1_{z'},\ \mathcal{O}^*_{4,2} \simeq \mathbb{A}^2_{z,\Delta} \backslash \mathbb{A}^1_{z}$.

2. Symplectic and Poisson manifolds

2.1. **Symplectic manifolds.** The notion of a symplectic manifold $(M,\, \sigma)$ is, in a sense, an odd analogue of a Riemannian manifold $(M,\, g)$. Instead of a **metric** g, which is determined by a positively defined symmetric bilinear form $g_{i,j}dx^idx^j$ in every tangent space T_mM, we consider the **symplectic structure** σ, determined by a non-degenerate antisymmetric bilinear form[3] $\sigma_{i,j}dx^i \wedge dx^j$.

More precisely, we consider the odd analogue of a **flat metric** on M. The point is that the coefficients $g_{i,j}$ of the metric tensor are in general smooth functions of the local coordinates, which can not be made constant, if the so-called **curvature tensor** $R^l_{i,j,k}$ does not vanish. For a differential 2-form σ the analogue of the curvature tensor is the **exterior differential** $d\sigma = d\sigma_{i,j}(x) \wedge dx^i \wedge dx^j = \partial_k\sigma_{i,j}(x) \wedge dx^i \wedge dx^j \wedge dx^k$.

The definition of symplectic manifold includes also the condition that the form σ is **closed**, i.e., $d\sigma = 0$. It is well-known that such a form has no local invariants. In appropriate local coordinates $(q^1,\, \dots,\, q^m;\, p_1,\, \dots,\, p_m)$, called **Darboux coordinates**, it takes the form

$$\sigma = \sum_i dq^i \wedge dp_i.$$

On every symplectic manifold $(M,\, \sigma)$ we can define the **skew-gradient** of a smooth function f. By definition, it is a vector field s-grad f on M, such that

$$\sigma(\text{s-grad}\, f,\, v) = vf \quad \text{for any vector field } v \text{ on } M. \tag{6}$$

In the Darboux coordinates we have $\text{s-grad}\, f = \sum_i \left(\frac{\partial f}{\partial p_i}\partial_{q^i} - \frac{\partial f}{\partial q^i}\partial_{p_i}\right)$.

Now we can define the so-called **Poisson bracket** of two smooth functions on M by

$$\{f_1,\, f_2\} = (\text{s-grad}\, f_1)\, f_2 = -(\text{s-grad}\, f_2)\, f_1 = \sum_i \frac{D(f_1,\, f_2)}{D(p_i,\, q^i)}. \tag{7}$$

[2]Actually, when $z = 0$, the entry $(F^2)_{41} = x_1y_2 + x_3y_2$ does not depend on values of stars and is a **secondary G-invariant**.

[3]Since such bilinear forms do exist only in even-dimensional vector spaces, every symplectic manifold M has even dimension $n = 2m$.

Exercise 1. a) Show that the Poisson bracket define in the space $C^\infty(M)$ the structure of a Lie algebra, i.e., the following Jacobi identity holds:

$$\{\{f_1,\, f_2\},\, f_3\} + \{\{f_2,\, f_3\},\, f_1\} + \{\{f_3,\, f_1\},\, f_2\} = 0.$$

b) Derive from this result that the fields s-grad f are infinitesimal symmetries of $(M,\, \sigma)$. (It means that the Lie derivative of the form σ along s-grad f vanishes.)

Hint. Use the **Cartan identity** $L_v = \iota_v \circ d + d \circ \iota_v$, where L_v is the Lie derivative, d is the exterior differential and ι_v is the interior product.

2.2. **Poisson manifolds.** Let M be a smooth manifold. Call it a **Poisson manifold** if in the space $C^\infty(M)$ a structure of a Lie algebra $[\cdot,\, \cdot]$ is defined with the following properties.

1. The map $C^\infty(M) \times C^\infty(M) \to C^\infty(M):\ (f_1,\, f_2) \mapsto [f_1,\, f_2]$ is continuos in C^∞ topology.

2. It is related to the multiplication in $C^\infty(M)$ by the relation

$$[f_1,\, f_2 f_3] = [f_1,\, f_2] f_3 + f_2 [f_1,\, f_3]. \tag{8}$$

Exercise 2. Show that the condition above implies that there is a smooth bivector field $c = c^{i,j}\partial_i \wedge \partial_j$ on M such that

$$[f_1,\, f_2] = c^{i,j}\partial_i f_1 \partial_j f_2.$$

Exercise 3 (Optional)**.** Show that every Poisson manifold $(M,\, [\cdot\,,\, \cdot])$ can be split into submanifolds M_α, $\alpha \in A$, (the **symplectic leaves**) so, that

a) Every M_α has a symplectic structure σ_α; we denote by $[\cdot\,,\, \cdot]_\alpha$ the corresponding Poisson bracket.

b) For any point $m \in M_\alpha \subset M$ we have

$$[f_1,\, f_2](m) \ =\ [f_1|_{M_\alpha},\, f_2|_{M_\alpha}]_\alpha. \tag{9}$$

Exercise 4. Let $\mathfrak{g}$ be a Lie algebra with a basis $(x_1, \ldots, x_n)$ and the structure constants c_{ij}^k, so that $[x_i,\, x_j] = \sum_k c_{ij}^k x_k$. Let $\mathfrak{g}^*$ be the dual vector space. We consider $(x_1, \ldots, x_n)$ as coordinates on $\mathfrak{g}^*$. Show that

a) the bivector $c = c_{ij}^k x_k \partial^i \wedge \partial^j$ defines on $\mathfrak{g}^*$ the structure of a Poisson manifold

b) the role of symplectic leaves on $\mathfrak{g}^*$ is played by the coadjoint orbits.

2.3. **Universality of $\mathfrak{g}^*$.** Let G be a connected simply connected Lie group and $\mathfrak{g} = \mathrm{Lie}(G)$ be its Lie algebra. Consider the category *mathcalPG* of Poisson G-manifolds, where objects are Poisson manifolds $(M,\, [\cdot\,,\, \cdot])$, on which the group G acts in such a way that the following condition is satisfied.

Let L_X be the vector field on M, corresponding to $X \in \mathfrak{g}$. There is a homomorphism $X \mapsto f_X$ of the Lie algebra $\mathfrak{g}$ to the Lie algebra $(\mathbb{C}^\infty(M), [\cdot, \cdot])$ such that for any $X \in \mathfrak{g}$, we have $L_X f = [f_X, f]$.

Morphism in $\mathcal{PG}$ from $(M', [\cdot, \cdot]')$ to $(M, [\cdot, \cdot])$ is a smooth G-equivariant map $\varphi : M' \to M$ such that

$$[f_1 \circ \varphi, f_2 \circ \varphi] = [f_1, f_2]' \circ \varphi,$$

It turns out that the Poisson G-manifold $(\mathfrak{g}^*, c)$ is a **universal final object** in $\mathcal{PG}$ (i.e., for any object $(M, [\cdot, \cdot])$ of $\mathcal{PG}$ there is a unique morphism from $(M, [\cdot, \cdot])$ to $(\mathfrak{g}^*, c)$.

In down-to-earth words it means that for any Poisson G-manifold $(M, [\cdot, \cdot])$ there is a canonically defined smooth map μ from M to $\mathfrak{g}^*$, called the **moment map**. In the notations above the functional $F = \mu(m)$ is defined by

$$\langle F, X \rangle = f_X(m) \quad \text{for any} \quad m \in M, X \in \mathfrak{g}. \tag{10}$$

Historical Note. The notion of the moment map is a far-going generalization of the notions of the impulse and the moment of impulse in mechanics. For a 1-parametric Lie groups it reduces to the famous Emmy Noether theorem about link between conservation laws and symmetry. In full generality it was introduced by J.-M.Souriau, [?S], B.Kostant [?Ko] and myself [?K0, ?K1, ?K8].

2.4. **From symplectic to Poisson G-manifolds.** The mathematical model of the classical mechanics is based on the notion of a symplectic manifold. But the quantization procedure needs the additional structure of a Poisson manifold. For homogeneous G-manifolds it leads to the beautiful theory of projective unitary representations of Lie groups and to the notion of central group extensions.

2.4.1. *Central extensions.* Recall that a a sequence of groups and homomorphisms is called **exact** if the image of every homomorphism coincides with the kernel of the next one.

A group G is an **extension** of a group G_1 by a group G_2, if we have an exact sequence

$$\{e\} \;\to\; G_2 \;\stackrel{i}{\to}\; G \;\stackrel{p}{\to}\; G_1 \;\to\; \{e\}. \tag{11}$$

In other words, the map i is an injection and the map p is a surjection. Therefore, we can identify G_2 with a normal subgroup $i(G_2) \subset G$ and G_1 with the corresponding quotient group $G/i(G_2)$.

The extension is called **central**, if $i(G_2)$ is contained in the center of G.

We say that the extension (11) **splits**, or is **trivial**, if there exists a homomorphism $j : G_1 \to G$ such that $p \circ j = \mathrm{Id}$. In this case we call G a **semi-direct product** of G_1 and G_2 and write $G = G_1 \ltimes G_2$.

The map $(g_1, g_2) \;\to\; g = i(g_1)j(g_2)$ establishes a bijection between the sets $G_1 \times G_2$ and G. But in general it is not a group isomorphism. The group law in the semi-direct product contains the additional information: the action of G_1 by automorphisms on G_2. Namely, to an element $g_1 \in G_1$ we associate an automorphism $\alpha_{g_1} \in \mathrm{Aut}(G_2)$ which sends $g_2 \in G_2$ to $j(g_1)g_2 j(g_1^{-1}) \in G_2$.

On the other hand, knowing the group laws in G_1 and G_2 and the homomorphism α, we can reconstruct the group law in the semi-direct product $G_1 \ltimes G_2$ by the formula

$$(g_1,\, g_2)\cdot(g_1',\, g_2') = \Big(g_1 g_1',\, \alpha_{g_1'}^{-1}(g_2)g_2'\Big).$$

2.4.2. *Projective unitary representations.* The phase space in quantum mechanics is a projectivization PV of a Hilbert space V, i.e., the collection of unit vectors $\psi \in V$, up to proportionality: ψ and $e^{i\alpha\psi}$ define the same state $[\psi]$ of the system. It is a metric space with the distance

$$d([\psi_1],\, [\psi_2]) = \sqrt{1 - |(\psi_1,\, \psi_2)|}.$$

It is known that every isometry of this space can be realized by the action of a unitary or anti-unitary operator on V. Hence, every connected group G can act as a symmetry group of the system only by so-called projective representation π, i.e., a unitary operator-valued function on G, satsfying

$$\pi(g_1 g_2) = c(g_1,\, g_2)\pi(g_1)\pi(g_2), \quad \text{where } |c(g_1,\, g_2)| = 1. \tag{12}$$

Exercise 5. a) Show that the function c above is a **cocycle**, i.e., satisfies the following **cocycle equation**:

$$c(g_1,\, g_2)c(g_1g_2,\, g_3) = c(g_1,\, g_2g_3)c(g_2,\, g_3).$$

b) Show that if c is a cocycle and b is any map from G to $\mathbb{T}^1$, then the function $c'(g_1,\, g_2) = \frac{b(g_1)b(g_2)}{b(g_1g_2)}c(g_1,\, g_2)$ is also a cocycle; such cocycles c, c' are called **equivalent**. A cocycle is called **trivial**, if it is equivalent to the cocycle $c(g_1g_2) \equiv 1$. (In this case the projective representation π is equivalent to an ordinary linear representation.)

Exercise 6. Let $\widetilde{G}$ be a group and $C \subset G$ is a subgroup of the center of $\widetilde{G}$. Show that every unirrep $\widetilde{\pi}$ of $\widetilde{G}$ defines the projective representation π of the factor-group $H = G/C$ as follows. For any $h \in H$ choose a representative $s(h)$ of the class h and put $\pi = \widetilde{\pi}(s(h))$.

Exercise 7. * Let π be a unitary projective representation of a group G. Show that there is a central extension $\{e\} \to C \xrightarrow{i} \widetilde{G} \xrightarrow{p} G \to \{e\}$ and a linear representation $\widetilde{\pi}$ of $\widetilde{G}$ such that π is obtained from $\widetilde{\pi}$ as in the previous exercise.

2.4.3. *Homogeneous symplectic and Poisson manifolds.* Let (M,ω) be a symplectic manifold, where a connected Lie group G acts transitively, preserving the form ω. To every element $X \in \mathfrak{g} = \mathrm{Lie}(G)$ there corresponds the 1-parametric subgroup $g(t) = \exp(tX)$ and the flow of diffeomorphisms

$\Phi(t) : m \mapsto g(t)m$. The velocity of this flow is a tangent vector field L_X on M, called the **Lie field**. Since $\Phi(t)$ preserves ω, we have $L_X\omega = 0$. Using the Cartan formula $L_X = \imath_X \circ d + d \circ \imath_X$, we get $d(\imath_X\omega) = 0$. Therefore, in some neighborhood of any point $m \in M$ there exist a smooth function f_X such that $\imath_X\omega = df_X$, hence, $L_X = \text{s-grad} f_X$.

Actually, the 1-form df_X is defined globally on the whole M, so f_X can be extended to a globally defined function, which, however, can be multivalued.

Let $\widetilde{M}$, $\widetilde{G}$ be simply-connected coverings of M, G respectfully. Then $\widetilde{M}$ is a symplectic manifold (with "the same" form ω), where the group $\widetilde{G}$ act transitively and preserving the form ω. The functions f_X, $X \in \mathfrak{g}$ are now defined globally on $\widetilde{M}$.

Moreover, we can assume that the correspondence $X \mapsto f_X$ is linear. Indeed, if $(X_1, \ldots, X_N)$ is a basis in $\mathfrak{g}$, then we can choose the globally defined functions f_{X_k}, $1 \leq k \leq N$ and for any $X = c^k X_k$ put $f_X = c^k f_{X_k}$. But in general this correspondence will be not a homomorphism of Lie algebra $\mathfrak{g}$ to the Lie algebra $(\mathrm{Fun}(\widetilde{M}), [\cdot\,, \cdot])$. Instead, we have a relation

$$f_{[X,\,Y]} = \{f_X,\, f_Y\} + c(X,\, Y), \tag{13}$$

where the constants $c(X, Y)$ form a "cocycle" in the sense that they satisfy the **cocycle equation**

$$c([X,\, Y],\, Z) + c([Y,\, Z],\, X) + c([Z,\, X],\, Y) = 0. \tag{14}$$

Every such cocycle defines a new Lie algebra $\widetilde{\mathfrak{g}}$ as follows. As a vector space, $\widetilde{\mathfrak{g}} = \mathfrak{g} \oplus \mathbb{R}$ and the commutator looks like

$$[(X,\, a),\, (Y,\, b)] = \Big([X,\, Y],\, c(X,\, Y)\Big). \tag{15}$$

Define the map of $\widetilde{\mathfrak{g}}$ to $\mathrm{Fun}(\widetilde{M})$ by the simple rule:

$$f_{(X,\,a)} = f_X + a.$$

then it will be a Lie algebra homomorphism of $\widetilde{\mathfrak{g}}$ to $\Big(\mathrm{Fun}(\widetilde{M}),\, [\cdot\,, \cdot]\Big)$.

We reached our goal: from a symplectic G-manifold M we constructed a Poisson G'-manifold M' where the pair (G', M') differs from the original pair (G, M) by two corrections.

The topological correction is a passage to universal covering and the algebraic correction is a passage to a central extension.

E-mail address: kirillov@math.upenn.edu