LECTURE 5.
COADJOINT ORBITS OF THE TRIANGULAR GROUP G_n
(PART 2)

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1. Amount of points and geometry

1.1. **General setting and examples.** We notice, that every $\mathcal{O}^*_{n,m}(K)$ is the set of K-points of an affine algebraic manifold $\mathcal{O}^*_{n,m}$ defined over $\mathbb{Z}$, i.e. a subset of $\mathbb{A}^N$, given by the algebraic equations and inequalities with integer coefficients. For such manifolds M the important information can be obtained by considering the number $M(q)$ of points of M over a finite field $\mathbb{F}_q$ of q elements. For instance, for "nice" smooth projective manifolds M the following formula holds:

$$M(q) = \sum_{k=0}^{\dim M} b_{2k} \cdot q^k, \qquad b_{2k} = \dim H^{2k}(M(\mathbb{C}), \mathbb{R}).$$

The ultimate reason for this is that M can be represented as the union of affine cells of different dimensions. Each cell of dimension k contributes q^k in $M(q)$ and 1 in b_{2k}.

For some quasi-affine algebraic manifolds the quantity $M(q)$ is still a polynomial in q with integer coefficient (which could be negative however).

Example 1. $M = G_{n,k}$, the Grassmann manifold of k-planes in n-space. Here

$$G_{n,k}(q) = \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q}{[k]_q \cdot [n-k]_q}$$

is the Gaussian q-binomial coefficient and $[n]_q := 1 + q + q^2 + ... + q^{n-1}$.

Example 2. $M = GL_n$, the general linear group of order n.

$$GL_n(q) = \prod_{k=0}^{n-1} (q^n - q^k).$$

Example 3. More generally, if $M^c_{a,b}$ is the manifold of $(a \times b)$-matrices of rank c, we have

$$M^c_{a,b}(q) = \begin{bmatrix} a \\ c \end{bmatrix}_q \cdot \begin{bmatrix} b \\ c \end{bmatrix}_q \cdot GL_c(q).$$

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Sometimes it is more convenient to deal with Laurent polynomials in $q^{1/2}$ and $q^{-1/2}$. Let us define the *balancing map*

$$k[q] \ni P \to \widetilde{P} \in k[q^{1/2}, q^{-1/2}]$$

by

$$\widetilde{P}(q) = P(q) q^{-1/2(M+m)},$$

where $M = \deg P$ and $m = \deg \left[q^M P(q^{-1}\right]$ are the maximal and minimal degrees of non-zero monomials in P.

This map has the useful properties:

$$a)\ \widetilde{P \cdot Q} = \widetilde{P}(q) \cdot \widetilde{Q}(q), \qquad b)\ \widetilde{q} = 1,$$

which allow to compute it easily. [1]

To illustrate the use of the balancing procedure we note that there are no simple expression for the sum of q-binomial coefficients, but for corresponding balanced quantities we have the nice formula

$$\sum_{k=0}^{n} \widetilde{\begin{bmatrix} n \\ k \end{bmatrix}_q} = \prod_{s=1}^{n} (1 + q^{\frac{n+1-2s}{2}}).$$

1.2. **Coadjoint orbits of low dimension. Clusters.** In our case the quantities $\mathcal{O}^*_{n,m}(q)$ and, consequently, $\mathcal{O}^*_n(q)$ turn out to be polynomials in q with integer coefficients. The explicit computation of these polynomials is a rather difficult combinatorial problem. Still more mysterious is their behavior when $n \to \infty$. Even the asymptotics of the sequence $\{\mathcal{O}^*_n(q)\}$ for given q is unknown. However, the behavior of $\mathcal{O}^*_{n,m}(q)$ for given q and m is relatively simple.

Theorem 1. *For given q and m we have*

$$\mathcal{O}^*_{n,m}(q) = q^n \cdot L_m(n),$$

where L_m is a polynomial of degree m with coefficients from $\mathbb{Z}[q, q^{-1}]$.

Proof. The proof of this theorem is based on the observation that for fixed q, m and $n \to \infty$ the set $\mathfrak{g}^*_{n,m}(K)$ has a *cluster structure*. This means that nonzero elements of matrices $F \in \mathfrak{g}^*_{n,m}(K)$ are not numerous and gather into groups called *clusters*. To be more precise, we introduce some notations and definitions.

Let $\mathfrak{b}^-_n(K)$ denote the set of non-strictly lower triangular matrices in $\mathrm{Mat}_n(K)$. To each $F \in \widehat{G}_{n+1}(K)$ we associate the matrix $\bar{F} \in \mathfrak{b}^-_n(K)$ which is obtained from F by deleting of the first row and the last column. The simple but important observation is that $\mathrm{rk}\, B_F$ depend only on the lower part of $\bar{F}$ (i.e. does not depend on the diagonal part of $\bar{F}$).

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[1] We remark that the quantity $\widetilde{\mathbb{P}^{n-1}}(q) = \frac{q^{n/2} - q^{-n/2}}{q^{1/2} - q^{-1/2}} = \widetilde{[n]}_q$ is often considered as "the quantum analog" of an integer n, and the quantity $\widetilde{\mathcal{F}^n}(q)$, where $\mathcal{F}^n$ is the (full) flag manifold, as a quantum analog of $n!$.

Now, let us call an element $\bar{F} \in \mathfrak{b}_n^-(K)$ *decomposable*, if it has a block-diagonal form:

$$\bar{F} = \bar{F}_1 \oplus \bar{F}_2 \oplus ... \oplus \bar{F}_r, \quad r > 1,$$

where $\bar{F}_i \in \mathfrak{b}_{n_i}^-(\mathbb{F})$ and $\sum_{i=1}^r n_i = n$.

Lemma 1. *Under the above conditions we have*

$$rk\, B_F = \sum_{i=1}^r rk\, B_{F_i}. \tag{1}$$

Proof. Let $\{X_{i,j}\}, 1 \le i < j \le n+1$, be the standard basis in $\mathfrak{g}_{n+1}(K)$ and $\left\{X_{i,j}^*\right\}$ be the dual basis in $\widehat{G}_{n+1}(K)$. Then

$$B_F = \sum_{i<k<j} f_{j,i} \cdot \{X_{i,k}^*\} \wedge \{X_{k,j}^*\}.$$

Now, it is easy to see that 2-forms B_{F_i} which correspond to different summands in (1) depend on disjoint sets of $\left\{X_{i,j}^*\right\}$'s. This makes the lemma evident.

Define now the **cluster** (or **elementary particle**) of the **size** n and the **mass** (or **energy**) m as an indecomposable element $\bar{F} \in \mathfrak{b}_n^-(K)$ associated to an element $F \in \mathfrak{g}_{n+1,m}^*(K)$.

It follows from this definition that for any $F \in \mathfrak{g}_{n+1,m}^*(K)$ the associated $\bar{F} \in \mathfrak{b}_n^-(K)$ splits into certain number of clusters of the sizes $n_1, ..., n_r$ and the masses $m_1, ..., m_r$, such that

$$\sum_{i=1}^r n_i = n, \qquad \sum_{i=1}^r m_i = m.$$

It is also clear that for any q there are only finitely many clusters of given mass m. Let us denote by $C_{\nu,\mu}(q)$ the number of clusters of the size ν and of the mass μ . Then the standard combinatorial argument proves

Proposition 1.

$$\sum_{n,m} \mathfrak{g}_{n+1,m}^*(q) t^m x^n = (1 - \sum_{\nu,\mu} C_{\nu,\mu}(q) t^\mu x^\nu)^{-1}.$$

Corollary.

$$\mathcal{O}_{n+1,m}(q) = q^{-2m} \cdot \left\{ \textit{coefficient of } t^m x^n \textit{ in } (1 - \sum_{\nu,\mu} C_{\nu,\mu}(q) t^\mu x^\nu)^{-1}. \right\} \tag{2}$$

We now derive the statement of the theorem from this corollary. We observe that $C_{1,0} = q$: the corresponding clusters ("photons") are just matrices of order 1 with a coefficient in $\mathbb{F}_q$.

Moreover, $C_{n,0} = 0$ for $n > 1$, because all particles of size $n > 1$ have a non-zero mass (the corresponding 2-form B_F is non-zero).

From this we conclude that the right hand side of (1.4.4) can be written in the form

$$[1-qx-t\Phi(q,x,t)]^{-1}$$

for some $\Phi\in\mathbb{Z}[q]\,[[x,t]]$. It follows that the right hand side of (1.4.5) is equal to the coefficient of x^n in

$$\frac{\Psi(q,x,t)}{[1-qx-t\Phi(q,x,t)]^{m+1}}\,|_{t=0}=\frac{\Psi(q,x,0)}{(1-qx)^{m+1}}$$

for some $\Psi\in\mathbb{Z}[q]\,[[x,t]]$. And this coefficient has evidently the form (1.4.1).

As an illustration, we give here the description of clusters for small m and the corresponding explicit formulae for $\mathcal{O}_{n,m}$.

m $=0$. It is clear that $\mathcal{O}^*_{n,0}(q)=q^{n-1},\quad n\geq 1$. One can also formally deduce it from the above relation $C_{n,0}=q\cdot\delta(n)$.

m $=1$. One can check that $C_{2,1}=q^2(q-1), C_{3,1}=q^3(q-1)^2$ and $C_{n,1}=0$ for $n>3$. The corresponding $\bar{F}$'s look like

$$\begin{pmatrix} x & 0\\ \lambda & y\end{pmatrix}\quad\text{and}\quad\begin{pmatrix} x & 0 & 0\\ \lambda & y & 0\\ 0 & \mu & z\end{pmatrix}$$

where $\lambda,\mu\in\mathbb{F}_q^\times, x,y,z\in\mathbb{F}_q$. It follows that

$\mathcal{O}^*_{n,1}(q)=q^{n-3}(q-1)(nq-3q+1),\quad n\geq 3.$

m $=2$. Clusters of mass 2 are: $\begin{pmatrix} x & 0 & 0 & 0\\ \lambda & y & 0 & 0\\ 0 & \mu & z & 0\\ 0 & 0 & \nu & u\end{pmatrix}$, $\begin{pmatrix} x & 0 & 0 & 0 & 0\\ \lambda & y & 0 & 0 & \\ 0 & \mu & z & 0 & \\ 0 & 0 & \nu & u & \\ 0 & 0 & 0 & \rho & v\end{pmatrix}$,

$\begin{pmatrix} x & 0 & 0 & 0 & 0\\ p & y & 0 & 0 & 0\\ 0 & q & z & 0 & 0\\ 0 & \lambda & r & u & 0\\ 0 & 0 & 0 & s & v\end{pmatrix}$, where all latin letters run through $\mathbb{F}_q$ and greek letters run through $\mathbb{F}_q^\times$. It follows that $C_{4,2}(q)=q^4(q-1)^3,\ C_{5,2}(q)=q^9(q-1)+q^5(q-1)^4,\ C_{n,2}=0$ for $n>5$. $\square$

1.3. **Statistics of coadjoint orbits.** The information contained in the double sequence $\left\{\mathcal{O}^*_{n,m}(q)\right\}$ can be encoded in the ordinary sequence of polynomials

$$\mathcal{O}^*_n(q,t)=\sum_{m\geq 0}\mathcal{O}^*_{n,m}(q)\cdot t^m$$

or in the generating series

$$\mathcal{O}^*(q,t,z)=\sum_{n\geq 0}\mathcal{O}^*_n(q,t)\cdot z^n.$$

At the moment I have almost nothing to say about the last series except the very plausible conjecture that it is a formal power series in z with the zero radius of convergence for all $q > 1$ and $t \geq 1$. On the contrary, the polynomials $\mathcal{O}_n^*(q,t)$ have several interesting interpretations.

Firstly, the theorem 1.2.4 implies that after the substitution $t \to \mu^2 q^2$ the values of these polynomials can be expressed in the form which reminds partition functions in the quantum statistical physics:

$$\mathcal{O}_n^*(q,\mu^2 q^2) = \sum_{F \in \widehat{G}(\mathbb{F})} \mu^{\mathrm{rk} B_F}.$$

In other words, we consider $\widehat{G}(\mathbb{F})$ as a peculiar lattice model where $\mathrm{rk}B_F$ plays the role of the energy function and $-\log\mu$ is the analog of the inverse temperature $\beta = \frac{1}{kT}$.

The approach suggested in [K3] gives the expression of this partition function for the special cases when β is a non negative integer.

Theorem 2. *Let $r > 0$ be an integer. Then*

$$\mathcal{O}_n^*(q, q^{2-2r}) = q^{-rn(n-1)} \cdot \sum_{X,Y,F} \theta(\langle F, [X,Y]\rangle),$$

where X, Y run through $\mathfrak{g}_n(\mathbb{F}_{q^r}), F$ runs through $\mathfrak{g}_n^*(\mathbb{F}_q)$ naturally embedded in $\mathfrak{g}_n^*(\mathbb{F}_{q^r})$ and θ is any non-trivial additive character of $\mathbb{F}_{q^r}$.

The proof of this theorem is based on the following simple facts. Let $V_i, i = 1,2$, be finite dimensional vector spaces over $\mathbb{F}$ and $B : V_1 \times V_2 \to \mathbb{F}_q$ be a bilinear form. For any integer $r \geq 1$ we can consider $V_i^{(r)} = V_i \otimes_{\mathbb{F}_q} \mathbb{F}_{q^r}$ and extend B by $\mathbb{F}_{q^r}$-linearity to a bilinear form $B^{(r)} : V_1^{(r)} \times V_2^{(r)} \to \mathbb{F}_{q^r}$.

Lemma 2. *For any non-trivial additive character θ of $\mathbb{F}_{q^r}$ we have*

$$\sum_{v_i \in V_i^{(r)}} \theta(B(v_1,v_2)) = q^{r(\dim V_1 + \dim V_2 - rkB)}.$$

Proof of the lemma. By a suitable choice of bases in V_1 and V_2 we can reduce B to the form

$$B(v_1,v_2) = \sum_{j=1}^{\mathrm{rk}B} v_1^j \cdot v_2^j,$$

where $v_i^j, 1 \leq j \leq \dim V_i$, are coordinates of $v_i \in V_i$. Then we use the formula

$$\sum_{x \in \mathbb{F}_{q^r}} \theta(xy) = q^r \cdot \delta(y)$$

where $\delta(y) = \begin{cases} 0 & \text{if } y \neq 0 \\ 1 & \text{if } y = 0, \end{cases}$ and get the result.

Now, applying the lemma to the form B_F, we obtain

$$\sum_{X,Y,F} \theta(\langle F, [X,Y]\rangle) = \sum_F q^{r[n(n-1) - \mathrm{rk}B_F]}.$$

The contribution of those F for which $\mathrm{rk}B_F = 2m$ is equal to $q^{2m} \cdot \emptyset_{n,m}(q)$ times $q^{r[n(n-1)-2m]}$. This gives the left hand side of the desired equality.

Another interpretation of the polynomials $\mathcal{O}_n^*(q,t)$ arises if we accept the conjecture 2.2.1 (see section 2.2 below). Namely

$$\mathcal{O}_n^*(q, q^{-s}) = \sum_{\lambda \in \widehat{G_n(\mathbb{F})}} d(\lambda)^{-s},$$

where $\widehat{g_n^*(\mathbb{F}_q)}$ is the set of (equivalence classes of) unitary irreducible representations of $G_n(\mathbb{F}_q)$ and $d(\lambda)$ is the dimension of a representation from the class λ. These ζ-like sums seem to have interesting properties and will be discussed in the section 3.

We reproduce now another result of [K4], which is the definition of a more sophisticate partition function associated with a Lie algebra $\mathfrak{g}_n$.

Namely, we put

$$\Phi_n(Z) = \sum_{X,Y,F} \theta(\langle F, [X,Y] - Z \rangle),$$

where, as before, X, Y run through $\mathfrak{g}_n(\mathbb{F}_q), F$ runs through $\mathfrak{g}_n^*(\mathbb{F}_q)$.

Let us introduce also for any Lie algebra $\mathfrak{g}$ the algebraic manifold

$$CP_{\mathfrak{g}} = \{(X,Y) \in \mathfrak{g} \times \mathfrak{g} \mid [X,Y] = 0\}$$

which consists of pairs of commuting elements in $\mathfrak{g}$.

Theorem 3.

$$\Phi_n(0) = q^{n(n-1)/2} \cdot CP_{\mathfrak{g}_n}(q)$$

The proof of this theorem is completely analogous to the proof of the theorem 1.5.1 and also based on the lemma 1.5.2.

Remark 1. This theorem is also true for any Lie algebra over $\mathbb{F}$. To the contrary, the equality $\Phi_n(0) = q^{n(n-1)} \cdot \emptyset_n^*(q)$ should be replaced for general Lie algebra $\mathfrak{g}$ by

$$\Phi_n(0) = q^{2\dim \mathfrak{g}} \cdot \sum_{\Omega \in \widehat{G}/G} \mathrm{ind}(\Omega),$$

where $\mathrm{ind}(\Omega) = q^{-\dim \Omega} \cdot \#\Omega$. If $\Omega \simeq \mathbb{A}^{2m}$, we have $\mathrm{ind}(\Omega) = 1$.

Peculiar lattice models became rather popular in the last time. We indicate to the interested reader the paper [MS] where some of these models are discussed.

1.4. **Euler-Bernoulli triangle.** The numerical computations for small n in the simplest case $q = 2$ lead to the remarkable observation: the sequence $\{\mathcal{O}_n^*(2)\}$ is very close to the sequence $\{K_n\}$ of so called *Euler-Bernoulli numbers*, defined by the generating function

$$\sum_{n \geq 0} K_n \frac{x^n}{n!} = \frac{1 + \sin x}{\cos x}$$

Namely, we have (see K2, K3).

n	0	1	2	3	4	5	6	7	8	9
$\mathcal{O}_2^*(n)$	?	1	2	5	16	61	275	1430	8506	57205
K_{n+1}	1	1	2	5	16	61	272	1385	7936	50521

Unfortunately, it is too difficult to compute directly next terms of the second row. (First 6 entries was computed "by hand" by my former student A.Mikhailov about 20 years ago, 3 following entries are taken from [G].)

On the contrary, for computing the sequence $\{K_n\}$ there exists a very simple and efficient procedure ([A1], [A2], [KB]). Consider the triangular table – the so called *Euler-Bernoulli triangle*:

$$\begin{array}{ccccccccc}
 & & & & a_{00} & & & & \\
 & & & a_{10} & \longleftarrow & a_{01} & & & \\
 & & a_{02} & \longrightarrow & a_{11} & \longrightarrow & a_{20} & & \\
 & a_{30} & \longleftarrow & a_{21} & \longleftarrow & a_{12} & \longleftarrow & a_{03} & \\
a_{04} & \longrightarrow & a_{13} & \longrightarrow & a_{22} & \longrightarrow & a_{31} & \longrightarrow & a_{40}
\end{array}$$

which is filled up shuttlewise according to the rules:

1) $a_{0,n} = \delta(n)$;

2)$a_{k,l} = a_{k-1,l+1} + a_{l,k-1},\ \ k \geq 1,\ \ l \geq 0.$

The numerical values of the first six rows are:

$$\begin{array}{ccccccccccc}
 & & & & & 1 & & & & & \\
 & & & & 1 & \longleftarrow & 0 & & & & \\
 & & & 0 & \longrightarrow & 1 & \longrightarrow & 1 & & & \\
 & & 2 & \longleftarrow & 2 & \longleftarrow & 1 & \longleftarrow & 0 & & \\
 & 0 & \longrightarrow & 2 & \longrightarrow & 4 & \longrightarrow & 5 & \longrightarrow & 5 & \\
16 & \longleftarrow & 16 & \longleftarrow & 14 & \longleftarrow & 10 & \longleftarrow & 5 & \longleftarrow & 0
\end{array}$$

In other words, the number at the end of an arrow is equal to the sum of the number at the tail of the arrow and the number above the arrow.

The provocative conjecture arises that there exists a q-analog of the Euler-Bernoulli triangle whose side elements count the numbers of coadjoint orbits in $\mathfrak{g}_n^*(\mathbb{F}_q)$.

Namely, let us keep the rule 1) and substitute the rule 2) by the two more sophisticate ones:

$2')a_{k,l} = t^{-1}a_{k-1,l+1} + (q-1)t^{l-1}a_{l,k-1}$ for $k \geq 1,\ \ l \geq 1$;

$2'')a_{n,0} = (q-1)^{-1}a_{n-1,1}$.

We obtain in this way a triangle table whose elements are polynomials in q and t. The first few rows are:

$$\begin{array}{ccccccccc}
 & & & & 1 & & & & \\
 & & & 1 & \longleftarrow & 0 & & & \\
 & & 0 & \longrightarrow & q-1 & \longrightarrow & 1 & & \\
 & q & \longleftarrow & q(q-1) & \longleftarrow & (q-1)t & \longleftarrow & 0 & \\
0 & \rightarrow & q(q-1)t^2 & \longrightarrow & q^2(q-1)t & \longrightarrow & q^2(q-1)+(q-1)^2t & \rightarrow & q^2+(q-1)t
\end{array}$$

It turns out that $a_{n,0}(q,t) = \varnothing_n^*(q,t)$ for $n \leq 5$.

Now, it is natural to conjecture that the other quantities $a_{k,l}(q,t)$ also have an interpretation in terms of orbits. One easily checks inductively that $2'$ and $2''$ imply the equality

$$a_{n+1,0}(q,t) = \sum_{k+l=n} a_{k,l}(q,t).$$

So the orbit interpretation of $a_{k,l}(q,t)$ should be connected with some stratification of the set $\O_n^*(q)$. Namely, $a_{k,l}(q,t) = \sum_m a_{k,l}^m(q)t^m$ where $a_{k,l}^m(q)$ is the number of 2m-dimensional orbits in the k-th stratum of $\widehat{G}_{n,m}(\mathbb{F})$. The appropriate stratification can be defined as follows. Let $X_{k,l} \subset \widehat{G}_{k+l}(\mathbb{F})$ is defined by the following conditions on the coefficients $\{f_{i,j}\}$ of $F \in \widehat{G}_{k+l}(\mathbb{F})$:

$$f_{n,1} = ... = f_{n,k-1} = 0, \quad f_{n,k} \neq 0; \; n = k+l.$$

Let $x_{k,l}^m(q)$ denote the number of 2m-dimensional orbits in $X_{k,l}$ and put $x_{k,l}(q,t) = \sum_m x_{k,l}^m(q)t^m$. In [K4] we have established that $x_{k,l}(q,t) = a_{k,l}(q,t)$ for $k+l \leq 6$ with the only one exception: $x_{4,2}(q,t) - a_{4,2}(q,t) = (q-1)^3t^2(t-q)(t-q^2) \neq 0$.

Remark 1. The latter formula suggests that the generalized Euler - Bernoulli triangle may have special properties when $t = q$ or $t = q^2$. Indeed, for $t = q^2$ it looks like

$$\begin{array}{c}
1 \\
1 \quad \longleftarrow \quad 0 \\
0 \quad \longrightarrow \quad q-1 \quad \longrightarrow \quad 1 \\
q \quad \longleftarrow \quad q^2-q \quad \longleftarrow \quad q^3-q^2 \quad \longleftarrow \quad 0 \\
0 \quad \longrightarrow \quad q^6-q^5 \quad \longrightarrow \quad q^5-q^4 \quad \longrightarrow \quad q^4-q^3 \quad \longrightarrow \quad q^3 \\
q^6 \quad \longleftarrow \quad q^7-q^6 \quad \longleftarrow \quad q^8-q^7 \quad \longleftarrow \quad q^9-q^8 \quad \longleftarrow \quad q^{10}-q^9 \quad \longleftarrow \quad 0
\end{array}$$

The case $t = q$ is still under investigation.

Remark 2. Still more intriguing is the following coincidence: for $q = 3,\ t = 1$, the sequence $\{x_{n,0}\}$ have the same first entries– 1, 1, 1, 3, 11, 57, 361, ... as the sequence $\{b_n\}$ of Springer numbers of type D_n (see [A2], [Sh]).

All this gives rise to many interesting but rather difficult problems. We formulate here only three of them.

1. Could one give the explicit formula for $a_{k,l}(q,t)$ or for appropriate generating function?

We observe that in the case $q = 2, t = 1$ our generalized Euler-Bernoulli triangle becomes the usual one and the following formula holds:

$$A(u,v) := \sum_{k,l} a_{k,l}(2,1)\frac{u^k}{k!}\frac{v^l}{l!} = \frac{\cos v + \sin u}{\cos(u+v)}.$$

It can be derived from the differential equation and the initial condition satisfied by $A(u,v)$:

$$(\partial_u - \partial_v)A(u,v) = A(v,u), \qquad A(0,v) = 1$$

which follow from the rules $2'), 2'')$.

2. What is the asymptotic behavior of $a_n(q) := \sum_{k+l=n} a_{k,l}(q,1)$ when $n \to \infty$?

For $q = 2$ (1.6.1) implies that $a_n(2) \approx (\frac{2}{\pi})^n \cdot n!$ but in general case we have no results.

3. Euler and Bernoulli numbers are related to the ζ-like sums:

$$(-1)^{n-1} B_{2n} \frac{2^{2n}-1}{2(2n)!} \pi^{2n} = \sum_{k=0}^{\infty} \frac{1}{(2k+1)^{2n}}, \; E_n \frac{1}{2(2n)!} \left(\frac{\pi}{2}\right)^{2n+1} = \sum_{k=0}^{\infty} \frac{(-1)^k}{(2k+1)^{2n+1}}.$$

Recently in [BCK] the very ingenious geometric interpretation of these relations has been found in terms of volumes of some polytopes. Namely,

$$\frac{K_{n-1}}{2(n-1)!} = \frac{2^{n+1}}{\pi^n} \sum_{k=0}^{\infty} \frac{(-1)^{nk}}{(2k+1)^n} = \text{vol } (\Delta_n),$$

where Δ_n is the convex polytope in $\mathbb{R}^n$ defined by the conditions:

$$x_i \geq 0, \; x_i + x_{i+1} \leq 1, \; i \in \mathbb{N} \text{ mod } n.$$

Could it possible to find the analogous interpretation for all entries of the Euler-Bernoulli triangle and its q-analogue?

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