LECTION 6.
A MODEL FOR THE GROUP S_n

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1. General setting

1.1. **Representations with a simple spectrum.** In this section we consider unitary representations π of a finite group G in a Hilbert space V.[1]

We say that a representation π has a **simple spectrum**, if it is a direct sum of pairwise non-equivalent unirreps. Such representation are characterized by a simple algebraic condition.

Proposition 1. *The representation (π, V) has a simple spectrum, iff the algebra of intertwining operators $I(\pi, \pi) = End_G(V)$ is commutative.*

Proof. Let $V = \oplus_k V_k$ be the decomposition of V into irreducible subspaces. Choose an orthonormal basis B_k in every V_k and let $B = \bigcup B_k$. Then B will be an orthonormal basis in V. We can assume that for every pair of equivalent subrepresentations $(\pi|_{V_i}, \pi|_{V_j})$ the bases B_i, B_j are chosen so that matrices of $\pi(g)|_{V_i}$ and $\pi(g)|_{V_i}$ coincide. Then every intertwining operator A in V has the matrix of block form with blocks $A_{k,l} \in \mathrm{Hom}_G(V_k, V_l)$, which are zero when $\pi|_{V_k}$ and $\pi|_{V_l}$ are not equivalent and can be arbitrary scalar matrices $c_{k,l} \cdot 1$, when $\pi|_{V_k} \simeq \pi|_{V_l}$.

We see that the algebra $I(\pi, \pi)$ is isomorphic to $\oplus_k \mathrm{Mat}(\mu_k, \mathbb{C})$, where μ_k are multiplicities of irreducible components. $\square$

1.2. **Big subgroups.** A subgroup H of a finite group G is called **big**, if the following equivalent conditions are satisfied.

1. For every unirrep π of G, it restriction $\mathrm{Res}_H^G \pi$ has a simple spectrum.

2. For every unirrep ρ of H, the induced representation $\mathrm{Ind}_H^G \rho$ has a simple spectrum.

The equivalence follows from the Frobenius Formula

$$i(\mathrm{Res}_H^G \pi, \rho) = i(\pi, \mathrm{Ind}_H^G \rho). \quad (1)$$

Exercise 1. Let H, K are subgroups of G. If $H \subset K \subset G$ and H is a big subgroup of G, then so is K.

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[1] Actually, the notions we introduce below make sense in more general setting, namely, for continuous unitary representations of compact topological groups. Most of the statements also remain true in this situation.

Exercise 2. * Let $\mathbb{C}[G]^H$ be the subalgebra in the group algebra $\mathbb{C}[G]$, which consists of functions f, satsfying

$$f(hgh^{-1}) = f(g) \quad \text{for all} \quad h \in H,\, g \in G.$$

The subgroup H is big, iff the subalgebra $\mathbb{C}[G]^H$ is commutative.

Known examples. In the following pairs every group is a big subgroup of the following one for $n \geq 1$:

$$S_n \subset S_{n+1}, \qquad S_2 \times S_n \subset S_{n+2}, \qquad U_n \subset U_{n+1},$$
$$U(1) \subset SU(2) \qquad SU_{n+1} \subset SU_{n+2}, \qquad Spin_{n+2} \subset Spin_{n+3}.$$

1.3. **Definition of model representation.** A representation π of a finite group G, is called a **model** for G, if its decomposition into unirreps contains every type with the multiplicity 1. Note, that this definition make sense also for continuous representations of compact groups. The first example, which was the source of the notion, is the representation of $G = \mathrm{SO}(3, \mathbb{R})$ in $L^2(S^2)$. Here the irreducible components have dimensions $2k + 1$ and consist of homogeneous polynomials of degree k in variables x, y, z. Later many other examples of model representations were discovered (see [**??**][],[]).

Exercise 3. Show that the regular representation of G in $\mathbb{C}[G]$ has a simple spectrum iff the group G is commutative.

2. Basic facts about S_n

2.1. **Definitions and notations.** The group S_n of permutations of n objects is the most important and most studied family of finite groups. By definition, it is a group of automorphisms of X_n, a finite set with n elements. Usually, X_n is realized as $\{1, 2, \ldots, n\}$, so that there are natural inclusions:

$$X_1 \subset X_2 \subset \cdots \subset X_n \subset \ldots \qquad \text{and} \qquad S_1 \subset S_2 \subset \cdots \subset S_n \subset \ldots.$$

There are several ways do describe elements $s \in S_n$:

a) As a bijective function $k \mapsto s(k)$, $1 \leq k \leq n$.

b) As a row vector $\Big(s(1), s(2), \ldots, s(n)\Big) \in \mathbb{N}^n$.

c) As a graph $\Gamma(s)$ of the form $\begin{matrix} 1 & 2 & \ldots & n-1 & n \\ \\ 1 & 2 & \ldots & n-1 & n \end{matrix}$, where the set V of vertices is the union of two copies (upper and lower) of X_n and the set A of arrows consists of a_k, $1 \leq k \leq n$, which joins the upper copy of k with the lower copy of $s(k)$.

A pair $(i, j) \subset X_n \times X_n$ is called bf inversion for $s \in S_n$, if $i < j$ and $s(i) > s(j)$. The number of inversions is called the **length** of s and is denoted $l(s)$. It is equal to the number of the intersection points for the arrows in $\Gamma(s)$.

Proposition 2. *The function* $sgn\, s = (-1)^{l(s)}$ *is multiplicative:*

$$sgn\,(s_1 s_2) = sgn\,(s_1) \cdot sgn\,(s_2)$$

and is the only non-trivial character of S_n*.*

2.2. **Subgroups and conjugacy classes.** For any partition of the set X_n into disjoint parts $X_1, \ldots, X_k$ we denote by $Y_{X_1, \ldots, X_k}$ the subgroup of S_n, consisting of permutations, which preserve the partition. The subgroups of this kind are called **Young subgroups**. As abstract groups, they are determined up to isomorphism by the numbers $\lambda_i = |X_i|$, $1 \leq i \leq k$, and are isomorphic to one of the groups $Y_\lambda := S_{\lambda_1} \times \cdots \times S_{\lambda_k}$.

To any standard Young tableau T we associate two Young subgroups: $Y_{row}(T)$ (resp. $Y_{col}(T)$) which correspond to the partitions of X_n into rows (resp. columns) of T. Up to isomorphism, they are determined by the corresponding Young diagram D (or by the partitions $\lambda(D)$ and $\lambda^*(D)$).

Introduce also the notation $s_{row}(T)$ (resp. $s_{col}(T)$) for the element $s \in S_n$ which permutes cyclically the elements of every row (resp. every column) of T. Sometimes, these elements are called **horizontal** (resp. **vertical**) permutations.

It is well-known that every conjugacy class $C \subset S_n$ contains a horizontal (resp. vertical) permutation for an appropriate tableau T. Moreover,

Proposition 3. *There is a bijection between conjugacy classes in* S_n *and partitions* $\lambda \in \mathcal{P}_n$ *such that the class* C_λ *contains a horizontal permutation for a tableau* T *of type* λ *(resp. a vertical permutation for a tableau* T *of the dual type* λ^**).*

2.3. **Representation of S_n, induced from Young subgroups.** Let λ be a partition of n. Denote by $\mathcal{X}_\lambda$ the set of all partitions of the set $X_n = \{1, 2, \ldots, n\}$ into disjoint parts of cardinalities $\lambda_1, \ldots, \lambda_k$. It is clear that the group S_n acts transitively on $\mathcal{X}_\lambda$ and the stabilizer of the point $(X_1, \ldots, X_k)$ is the Young subgroup we denoted above by $Y(X_1, \ldots, X_k)$, or Y_λ.

Consider two kinds of induced representation of S_n:

$$\Pi_\lambda = \mathrm{Ind}_{Y_\lambda}^{S_n} 1 \quad \text{and} \quad \Pi'_\mu = \mathrm{Ind}_{Y_\mu}^{S_n} \mathrm{sgn} \simeq \Pi_\mu \otimes \mathrm{sgn}\,.$$

The computation of the intertwining numbers between these representation is a beautiful and non-trivial group-theoretic (and combinatorial) problem. To describe the result, we have to introduce a partial order in the set $\mathcal{P}_n$ of partitions. We say that λ dominates μ and write $\lambda \succeq \mu$ if

$$\lambda_1 + \cdots + \lambda_k \geq \mu_1 + \cdots + \mu_k \quad \text{for all} \quad k \geq 1. \tag{2}$$

Let $\lambda, \mu \in \mathcal{P}_n$ be partitions.

Proposition 4. *The table of intertwining numbers for representaions* Π_λ *and* Π'_μ *have a unitriangular form:*

$$i(\Pi_\lambda, \Pi'_\lambda) = 1 \quad \textit{and} \quad i(\Pi_\lambda, \Pi'_\mu) = 0 \textit{ unless } \lambda \succeq \mu. \tag{3}$$

The proof is based on a following combinatorial fact.

Lemma 1. *Let λ does not dominate μ. Then for any standard tableau T of shape λ and any standard tableau T' of shape μ, there are two numbers $i, j \in X_n$ such that they are situated in one row of T and in one column of T'.*

As a corollary, we obtain the bijection between the set $\widehat{S_n}$ of (equivalence classes of) unirreps of S_n and the set $\mathcal{P}_n$ of partitions of n.

Indeed, the first relation (3) means that the (reducible) representations Π_λ and Π'_λ have a unique unirrep in common. We denote this common unirrep by π_λ. The space V_λ of this representation contains a unique (up to scalar factor) vector v_λ^+, which is invariant under all operators $\pi_\lambda(g)$, $g \in Y_\lambda$. It also contains a unique (up to scalar factor) vector v_λ^-, which is invariant under all operators $\pi'_\lambda(g)$, $g \in Y_\lambda$.

3. Vector bundles and induced representations

Let G be a group and $H \subset G$ be a subgroup. Then there is the natural **restriction functor** Res_H^G from the category of all representations of G, denoted $\mathcal{R}ep(G)$, to the category $\mathcal{R}ep(H)$.

The notion of an induced representation was defined by G.Frobenius as a dual functor Ind_H^G from $\mathcal{R}ep(H)$ to $\mathcal{R}ep(G)$, satisfying the **duality formula**:

$$(4) \qquad i(\mathrm{Res}_H^G \pi,\, \rho) = i(\pi,\, \mathrm{Ind}_H^G)\rho \quad \text{for all} \quad \pi \in \mathcal{R}ep(G),\, \rho \in \mathcal{R}ep(H).$$

The geometric version of this construction is based on the notion of vector bundle, introduced below.

3.1. **Vector bundles.** An n-dimensional complex **vector bundle** L over a topological space X is a topological space L, endowed with a continuous map $p : L \to X$ such that for any $x \in X$ the **fiber** $F_x = p^{-1}(x)$ has a structure of n-dimensional complex vector space.

Such bundles form a category, where a **morphism** from $(L_1 \overset{p_1}{\to} X_1)$ to $(L_2 \overset{p_2}{\to} X_2)$ is a commutative diagram of the form $\begin{array}{ccc} L_1 & \xrightarrow{\phi} & L_2 \\ \downarrow{\scriptstyle p_1} & & \downarrow{\scriptstyle p_2} \\ X_1 & \xrightarrow{\psi} & X_2 \end{array}$, where ϕ and ψ are continuous maps with the additional condition: the restriction of ϕ to every fiber F_x is a linear operator from F_x to $F_{\psi(x)}$.

Remark. In most applications the sets L and X are assumed to be a smooth manifolds, the projection p is smooth and the bundle structure is locally trivial.

The last property means that any point $x \in X$ has a nejghborhood $U \ni x$ such that $p^{-1}(U)$ is isomorphic to the trivial bundle $L_0 = U \times \mathbb{C}^n$ with the natural projection to U. I.e., we have a commutative diagram

$$\begin{array}{ccc} L_0 & \xrightarrow{\phi} & U \times \mathbb{C}^n \\ \downarrow{\scriptstyle p} & & \downarrow \\ U & \xrightarrow{\iota} & U \end{array}$$, where ι is the inclusion and for every $x \in U$ the restriction of ϕ to the fiber F_x is an isomorphism of F_x onto $\{x\} \times \mathbb{C}^n$.

Let $U \subset X$ be a subset. A map $s : U \to L$ is called a section of L over U if the composition $p \circ s = \mathrm{Id}_U$. In other words, for any point $x \in U$ the value $s(x)$ belongs to the fiber F_x. The set of all sections of L over U is denoted by $\Gamma(L, U)$. It is a natural generalization of the space of vector-functions $\mathrm{Fun}(U, \mathbb{C}^n)$. (Note, that for trivial bundles these notions coincide.)

3.2. **G-bundles over G-sets.** A vector bundle L over a right G-set is called a G-**bundle**, if G acts also on L (also from the right) and

1. This action commutes with the projection $p : L \to X$.
2. The restriction of G-action on every fiber is a linear map.

In the space $\Gamma(L, X)$ the linear representation Π of the group G arises. It acts by the shifts of argument:

$$\Big(\Pi(g)\gamma\Big)(x) = \gamma(x \cdot g). \tag{5}$$

When the bundle L is trivial and 1-dimensional, the above formula gives just the geometric representation of G in the complex function space $\mathrm{Fun}(X, \mathbb{C})$.

The most interesting case is when the G-set X is homogeneous. The notations are slightly shorter if we assume that X is a right G-set, hence isomorphic to the right coset space $H\backslash G$. Below we consider this situation in more details.

To study the sections of L, we have to write these sections in a convenient form. Denote by $x_0 \in X$ the initial point (the coset $H\backslash H$ in $H\backslash G$) and by W the fiber F_{x_0}. Choose a basis $B = (e_1, \ldots, e_n)$ in W.

Using the action of the element $g \in G$, we can translate this basis to the basis $B \cdot g = (e_1 \cdot g, \ldots, e_n \cdot g)$ in the fiber $F_{x \cdot g} = W \cdot g$. A section $\gamma \in \Gamma(L, X)$ is determined by the vector-function $w_\gamma = (w_\gamma^1, \ldots, w_\gamma^n)$ on G, given by

$$w_\gamma^i(g) = (\gamma(x_0 \cdot g), e_i \cdot g). \tag{6}$$

But not every vector-function is associated to a section.

Proposition 5. *Let (ρ, W) be the representation of $H = Stab(x_0)$ on the fiber W over x_0. A vector-function w on G correspond to a section of the bundle L over X iff it satisfies the relation*

$$w(hg) = \rho(h)w(g). \tag{7}$$

Proof. By definition of ρ, the bases $B \cdot h$ and B are connected by the relation

$$B \cdot h = (e_1 \cdot h, \ldots, e_n \cdot h) = (\rho(h)e_1, \ldots, \rho(h)e_n) = \rho(h)B. \tag{8}$$

Applying the action of g to this relation, we get $B \cdot hg = \rho(h)B \cdot g$, which implies (7) and the necessity of the condition. The sufficiency follows from the explicit formula $\gamma(x_0 \cdot g) = w^i(g)e_i \cdot g$.

Proposition 6 (Corollary)**.** *There is natural bijection between the (equivalence classes of) n-dimensional G-bundles over $X = H\backslash G$ and (equivalence classes of) n-dimensional complex representations of H.*

We denote L_ρ the bundle, corresponding to $\rho \in \mathcal{R}ep(H)$.

Theorem 1. *The representation Π of G in the space $\Gamma(L_\rho, X)$, defined by* (5)*, is equivalent to the induced representation $Ind_H^G \rho$.*

Proof. We show that Π possesses the characteristic property of induced representations, i.e. satisfies the Frobenius formula (1) with Π instead of $Ind_H^G \rho$:

$$i(\mathrm{Res}_H^G \pi,\, \rho) = i(\pi,\, \Pi).$$

To prove it, we establish the explicit correspondence between the two spaces of intertwining operators: $I(\mathrm{Res}_H^G \pi,\, \rho)$ and $I(\pi,\, \Pi)$.

Let V be the space of the representation π. By definition, an operator $A \in I(\pi,\, \Pi)$ sends a vector $v \in V$ to some section of the vector bundle L_ρ, which we denote γ_v. The intertwining property is: $\gamma_{\pi(g)v} = \Pi(g)\gamma_v$, or

$$\gamma_{\pi(g)v}(x) = \gamma_v(xg) \quad \text{for all} \quad x \in X,\, g \in G.$$

The section γ_v determines the vector-function on G which we denote by $w_v(g)$. The G-action on L_ρ in terms of these vector-functions looks like $w_{\pi(g)v} = \Pi(g)w$.

Finally, we associate to A the operator $\widetilde{A} : V \to W$, which sends $v \in V$ to the value $w_v(e) \in W$.

We have to check that $\widetilde{A}$ belongs to $i(\mathrm{Res}_H^G \pi,\, \rho)$ and the correspondence $A \to \widetilde{A}$ is a bijection. The intertwining property of $\widetilde{A}$ follows from

$$\widetilde{A}\pi(h)v = w_{\pi(h)v}(e) = w_v(h) = \rho(h)w_v(e) = \pi(h)\widetilde{A}v.$$

Further, every operator B from $i(\mathrm{Res}_H^G \pi,\, \rho)$ has the form $\widetilde{A}$, where A is the operator, which sends $v \in V$ to the section $\gamma(x_0 \cdot g) = \pi(g)Bv$. $\square$

4. Involutions in S_n and the model representation

4.1. **Involutions.** In this section we call **involution** every permutation $s \in S_n$, satisfying $s^2 = e$ (including e itself). It is clear that the cycle type of such permutation is $1^k 2^l$, $k + 2l = n$. The conjugacy class $C[s]$ is an homogeneous S_n-set, isomorphic to the coset space $S_n/Z(s)$, where $Z(s)$ is the centralizer of the element $s \in S_n$.

To describe the centralizer $Z(s)$, consider first the extremal cases $n = k$ and $n = 2l$. In the first case $s = e$ and $Z(s) = S_n$. In the second case we realize X_{2l} as the set $X'_{2l} = \{\pm 1, \pm 2, \dots, \pm l\}$ and put $s(i) = -i$. The centralizer $Z(s)$ contains permutations of two types:

$$a)\, s(i) = \mathrm{sgn}\,(i) \cdot \sigma(|i|), \text{ where } \sigma \in S_l, \qquad b)\, s(i) = \varepsilon(i) \cdot i,\, \varepsilon(i) = \pm 1.$$

The permutations of type a) form a group, isomorphic to S_k, those of second type form a group, isomorphic to $\mathbb{Z}/2\mathbb{Z})^l$. The whole centralizer is a semi-direct product of these two groups: $C_l \simeq S_l \rtimes (\mathbb{Z}/2\mathbb{Z})^l$. (The group C_l is isomorphic to the full symmetry group of an l-dimensional cube).

In general case, when $n = k + 2l$, the centralizer $\mathbb{Z}(s)$ is the direct product $S_k \rtimes C_l$. Its cardinality is $\frac{|S_n|}{|S_k| \cdot |C_l|} = \frac{(k+2l)!}{k!\,(2l)!!}$.

5. The universal line bundle over $Inv(S_n)$

The whole set $Inv(S_n)$ of involutions in S_n is the disjoint union of subsets $Inv_l(S_n)$, $0 \le l \le \frac{n}{2}$, which are just conjugacy classes $C_{1^k 2^l}$, $k + 2l = n$. We define an S_n-line bundle L over $Inv(S_n)$ as follows.

As we showed above, a complex G-bundle L over an homogeneous right G-set $X = H \backslash G$ is determined by the action of the subgroup H in the fiber $F_{x_0} \simeq \mathbb{C}$ over the initial point $x_0 \in X$.

In our case this action is given by the family χ of characters $\{\chi_l\}$, $0 \le l \le n/2$. The space $\Gamma(Inv_l(S_n),\, L\chi_l)$ of sections of L over $Inv_l(S_n)$ is identified with the space of complex-valued functions φ on S_n, satisfying

$$\varphi(gh) = \chi_l(h)\varphi(g), h \in S_k \rtimes C_l, \tag{9}$$

which is a particular case of (7), where representation ρ is one dimensional.

Theorem 2 (A.A.Klyachko)**.** *Define the character χ_l of $S_k \rtimes C_l$ as trivial on S_k and as sgn on C_l. Then the space*

$$\Gamma(L\chi,\, Inv(S_n)) = \oplus_{l=0}^{[n/2]} \Gamma(L\chi_l,\, Inv_l(S_n))$$

will be a model for S_n.

□

Example 1. For $n = 3$ we have $Inv_0 = \{e\}$, $Inv_1 = \{(1,2),\, (2,3),\, (1,3)\}$.

The line bundle L_χ over Inv_0 is trivial and in the space $\Gamma(Inv_0,\, \chi_0$ the trivial representation π_0 is realized (as a geometric representation) .

The line bundle L_χ over Inv_1 corresponds to the character *sgn* and is the tensor product of geometric representation in Fun(Inv_1) and 1-dimensional representation $\pi_1 = sgn$. So, in the space $\Gamma(Inv_0,\, \chi_0$ the reducible representation $\pi_1 \oplus \pi_2$ is realized.

Example 2. $n = 4$ we have $Inv_0 = \{e\}$, $Inv_1 = \{(1,2),\, (1,3),\, (1,4),\, (2,3),\, (2,4),\, (3,4)\}$, $Inv_2 = (12)(34),\, (13)(24),\, (14)(23)$.

Here again, in sections of L_χ over Inv_0 the trivial representation π_0 acts.

The 3-dimensional representation in the space of sections of L_χ over Inv_2 is again the tensor product of geometric representation in $\mathrm{Fun}(Inv_2)$ and 1-dimensional representation $\pi_1 = sgn$. So, splits as $\pi_1 \oplus \pi_2$.

Finally, the 6-dimensional representation in the space of sections of L_χ over Inv_1 is the sum of two 3-dimensional unirreps of S_4.

It would be interesting to describe explicitly the distribution of unitrreps of S_n between the $[n/2]$ spaces $\Gamma(L_\chi, Inv_l(S_n))$.

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