PROBLEMS

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In the Spring semester of 2019 I am teaching at HSE the course

"Topics in the modern representation theory. The orbit method for the triangular group over a finite field."

Here I collect some problems for students of this course. They can be used as compulsory exercise (homeworks), or as subjects for independent study, or as possible themes for research. Problems of the first kind are marked by circles, of the third kind by stars.

0.1. **Notations.** .

$\mathrm{Fun}(X)$ - the space of functions (usually, complex-valued) on the set X.

$\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, $\mathbb{R}$, $\mathbb{C}$, $\mathbb{H}$ - sets of natural numbers, integers, rationals, reals, complex numbers and quaternions respectively.

K - a field, $K^+ = (K, +)$ - the additive group of K, $K^\times = (K\backslash\{0\}, \cdot)$ - the multiplicative group of K.

F_q - a finite field with q elements, where $q = p^k$, p is a prime number.

$f_1 * f_2$ - a convolution product in $\mathrm{Fun}(G)$ defined by

$$(f_1 * f_2)(g) = \sum_{h\in G} f_1(h) f_2(h^{-1}g) = \sum_{g_1 g_2 = g} f_1(g_1) f_2(g_2).$$

(π, V) - linear representation of a group G in the vector space V

(π^*, V^*) - the dual representation of G in the dual vector space V^*, defined by $\langle \pi^*(g) f, v\rangle = \langle f, \pi(g^{-1}v\rangle$.

$I(\pi_1, \pi_2) := \mathrm{Hom}_G(V_1, V_2)$ - the space of intertwining operators (or **intertwiners**) between two representations (π_1, V_1), (π_2, V_2) of G, i.e., the operators $A : V_1 \to V_2$ for which the following diagram is commutative:

$$\begin{array}{ccc} V_1 & \xrightarrow{\pi_1(g)} & V_1 \\ {\scriptstyle A}\downarrow & & \downarrow{\scriptstyle A} \\ V_2 & \xrightarrow{\pi_2(g)} & V_2 \end{array}$$

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$i(\pi_1, \pi_2) := \dim I(\pi_1, \pi_2)$ - the **intertwining number** between π_1, π_2.

$\chi_\pi(g) := \operatorname{tr} \pi(g)$ - the character of π.

unirrep - a short name for a unitary irreducible representation.

$\widehat{G}$ - the set of (equivalence classes of) unirreps of G; (π_λ, V_λ) - a unirrep of G, which is a representative of the class $\lambda \in \widehat{G}$; $d(\lambda) = \dim V_\lambda$. For an abelian group G the set $\widehat{G}$ is also a group, called the **Ponriagin dual** of G.

$L^2(G)$ - the space of complex-valued functions on G with the inner product

$$(f_1, f_2) = \frac{1}{|G|} \sum_{g \in G} f_1(g) \overline{f_2(g)}.$$

$L^2(\widehat{G})$ - the space of matrix-valued functions on $\widehat{G}$ $\Big($where $f(\lambda) \in \mathrm{Mat}_{d(\lambda)}(\mathbb{C})\Big)$

with the inner product $\quad (f_1, f_2) = \sum_{\lambda \in \widehat{G}} \operatorname{tr} \Big(f_1(\lambda) f_2^*(\lambda) \Big).$

Fourier transform on G - the unitary bijection between $L^2(G)$ and $L^2(\widehat{G})$, which sends a function $f \in L^2(G)$ to the function $\widehat{f} \in L^2(\widehat{G})$, given by

$$\widehat{f}(\lambda) = \sum_{g \in G} f(g) \pi_\lambda^*(g).$$

Inverse Fourier transform - the relation $f(g) = \sum_{\lambda \in \widehat{G}} \operatorname{tr} \Big(\widehat{f}(\lambda) \pi_\lambda(g) \Big)$.

1. Finite fields

1.1. General facts. .

1.° Show that in a finite field K the multiplicative group $K^\times := (K \backslash \{0\}, \cdot)$ is cyclic.

2.° Compile the addition and multiplication tables for the fields $\mathbb{F}_4$, $\mathbb{F}_8$, $\mathbb{F}_9$.

3.° Describe all homomorphisms from $\mathbb{F}_q$ to $\mathbb{F}_{q'}$, where $q = p^k$, $q' = (p')^{k'}$, p, p' are prime numbers.

4. Compute the inverse map for $Y = \exp_3(X) = X + \frac{X^2}{2}$ as a formal power series in Y with coefficients in $\mathbb{F}_3$. In other words, find the coefficients $a_k \in \mathbb{F}_3$, $k \geq 0$, so that $X = \sum_{k \geq 0} \alpha_k Y^k$ satisfies $X + \frac{X^2}{2} = Y$. (Note that in $\mathbb{F}_3$ we have $\frac{1}{2} = -1$.)

1.2. **Relations between finite fields.** .

1.° If p, p' are different primes, then there is no homomorphisms between $\mathbb{F}_q$ and $\mathbb{F}_{q'}$ for $q = p^k$, $q' = (p')^{k'}$.

2.° Show that a homomorphism $\alpha : \mathbb{F}_{p^m} \to \mathbb{F}_{p^n}$ exists iff $m|n$ (m is a divisor of n). In this case α is unique, is an embedding and its image consists of all $x \in \mathbb{F}_{p^n}$, satisfying $x^{p^m} = x$.

3.° Let p be a prime and $q = p^k$. a) Show that the **Frobenius map** $Fr\colon x \mapsto x^p$ is an automorphism of $\mathbb{F}_q$.

b) Show that the group $\mathrm{Aut}(\mathbb{F}_q)$ of all automorphisms of $\mathbb{F}_q$ is cyclic of order k with Fr as generator.

4.° Write the addition and multiplication tables for the fields $\mathbb{F}_q$ for $q = 3, 8, 9$.

5.*(Optional) Try to prove the uniqueness of $\mathbb{F}_q$, using the cyclic nature of $\mathbb{F}_q^{\times}$.

6.* a) Show that there exists a countable field $\overline{\mathbb{F}}_2$ which for any $k \in \mathbb{N}$ contains a unique subfield with 2^k elements. It is called the algebraic closure of $\mathbb{F}_2$.

b) Show that all such fields are isomorphic.

c) Prove that the multiplicative group of $\overline{\mathbb{F}}_2$ is isomorphic to the group of complex numbers of the form $z = e^{\frac{2k\pi i}{n}}$, n odd.

2. Complex linear representations of finite groups

1°. Show that $\widehat{(f_1 * f_2)} = \widehat{f_1}\widehat{f_2}$.

2°. Prove that the following are equivalent:
a) All characters χ of a group G are real-valued.
b) Every element $g \in G$ is conjugate to its inverse g^{-1}.

3. Formulate and prove the Schur's Criterion for a unirrep (π, V) of G.

4. Let $Inv(G)$ denote the set of all involutions in G (i.e., elements G satisfying $g^2 = e$). Assume that all unirreps of G are of a real type. Then

$$\sum_{\lambda \in \widehat{G}} d(\lambda) = |Inv(G)|.$$

5. Möbius inversion formula. For any function $f : \mathbb{N} \to \mathbb{R}$ define the function $F : \mathbb{N} \to \mathbb{R}$ by

$$F(m) = \sum_{d|m} f(d), \quad \text{where } d|m \text{ means that } d \text{ is a divisor of } m. \tag{1}$$

Show that f can be expressed in terms of F by the formula

$$f(m) = \sum_{d|m} \mu(d) F(\frac{m}{d}), \tag{2}$$

where the Möbius function μ is defined by

$$\mu(m) = \begin{cases} (-1)^k & \text{if } m \text{ is a product of } k \text{ distinct primes} \\ 0 & \text{otherwise.} \end{cases} \tag{3}$$

Hint. Consider first the case when f and F vanish outside the geometric progression $\{p^k\}_{k \in \mathbb{Z}_+} \subset \mathbb{N}$.

6. Fix a non-trivial additive character of $\mathbb{F}_q$, i.e., a character of the group $(F, +)$. Let ρ be a non-trivial multiplicative character of $\mathbb{F}_q$, i.e., a character of the group $(F^\times, \cdot)$. Extend it to the function $\widetilde{\rho}$ on $\mathbb{F}_q$ by the zero value at 0. Show that the Fourier transform of $\widetilde{\rho}$ is proportional to the extended inverse character $\widetilde{\rho^{-1}}$. The proportionality coefficient c depends on the multiplicative character ρ and is given by so-called **Gauss sum**:

$$c(\rho) = \frac{1}{q} \sum_{x \in \mathbb{F}_q^\times} \rho(x) \overline{\chi(x)}.$$

a)$^\circ$ Compute $c(\rho)$ For the only non-trivial multiplicative characters of $\mathbb{F}_3$.

b)* The same for two non-trivial multiplicative characters of $\mathbb{F}_4$.

7.* Compare the computation of Gauss sum with its classical analogues for real field. Recall that the the classical additive and multiplicative characters for $\mathbb{R}$ are:

$$\chi_\lambda(x) = e^{i\lambda x},\ \lambda \in \mathbb{R}, \quad \rho_{\alpha,\varepsilon}(x) = |x|^{i\alpha} (\operatorname{sgn} x)^\varepsilon,\ \alpha \in \mathbb{R},\ \varepsilon = 0, 1.$$

a) Show that the Fourier transform of a multiplicative character $\pi_{\alpha,\varepsilon}$ on $\mathbb{R}$, defined by

$$\widehat{\pi_{\alpha,\varepsilon}}(y) = \int_{\mathbb{R}^\times} |x|^{i\alpha} (\operatorname{sgn} x)^\varepsilon e^{-ixy} dx, \tag{4}$$

is a distribution on $\mathbb{R}$, satisfying $\widehat{\pi_{\alpha,\varepsilon}}(\lambda y) = |\lambda|^{-1} (\operatorname{sgn} \lambda)^\varepsilon \pi_{-\alpha,\varepsilon}(y)$.

b) Derive from a), that this distribution is expressed in terms of Euler Γ-function. Namely, on $\mathbb{R}^\times$ it has the form $c \cdot |y|^{-1} \pi_{-\alpha,\varepsilon}(y)$, where the constant c is

$$c = \int_{\mathbb{R}^\times} |x|^{i\alpha} (\operatorname{sgn} x)^\varepsilon e^{-ix} dx = \Gamma(1 + i\alpha) + (-1)^\varepsilon \Gamma(1 - i\alpha).$$

Recall, that

$$\Gamma(1 - i\alpha) = \overline{\Gamma(1 + i\alpha)} \quad \text{and} \quad \Gamma(1 + z)\Gamma(1 - z) = \frac{\pi z}{\sin \pi z}.$$

3. Symplectic and Poisson G-manifolds

Exercise 1. Let M be the unit sphere in $\mathbb{R}^3$ with the ordinary metric (inherited from $\mathbb{R}^3$). Define the symplectic structure on M by the restriction to M of the 2-form $\omega = xdy \wedge dz + ydz \wedge dx + zdx \wedge dy$ on $\mathbb{R}^3$.

Show that for any smooth function f on M the skew gradient s-grad$f$ is obtained from the ordinary gradient grad$f$ by a rotation on $90°$.

Exercise 2. Express the symplectic form ω in terms of local coordinates $(x,\, y)$ on the upper and lower part of M.

Exercise 3. Compute the volume of the unit ball in $\mathbb{R}^3$, using the integral $\int_M \omega$.

Exercise 4. Find the image of ω under the stereographic projection $s : M \ni (x, y, z) \mapsto (\frac{x}{}) \in \mathbb{R}^2$.

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