DYNAMICAL SYSTEMS AND ERGODIC THEORY - SPRING 2019 - EXAM

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1. Introduction

General rules:

- Cumulative mark:
  (1) students who gave a talk and attended lectures already have 8 (out of 10);
  (2) students who participated actively in the lectures but did not give a talk have 4 (out of 10);
  (3) students who did not attend the lectures and did not give a talk have 0 (out of 10).

  If you do not know to which group do you belong please contact Alexandra Skripchenko.
- Exam: there is a list of problems below. Each block of topics is covered by 2 problems; each problem in the list costs 2 point;
- Dates of the exam:
  (1) the exam for the first and bf third category is on **May 24th, from 10:30 to 13:30 for the first category, from 13:30 to 17:00 - for the third category)**. Please contact Marina Dudina (teaching assistant) to book some more precise time slot. Please note that the morning option is only for people whose cumulative mark is equal to 8 (in case if you want to increase your mark); I also strongly recommend for the third category at least to start the examination procedure.
  (2) the exam for the second (and the continuation for the third category) is on **May 31st, from 10:30 to 13:30)**.

  If the time does not fit you, please contact Alexandra Skripchenko.

2. Problems

Problem 1. Let Γ be a Fuchsian group and let D be a Dirichlet polygon for D. We assume that D may have vertices on $\partial\mathbb{H}$ but D has no free edges; also, no side of D is paired by itself.

- prove that $\mathbb{H}/PSL(2,\mathbb{Z}$ has genus 0, one marked point of order 3, one marked point of order 2 and a cusp.

Now, suppose that $\mathbb{H}/\Gamma$ has genus g, r marked points of orders $m_1, \cdots, m_r$ and c cusps. We define the *signature* of Γ in the following way:

$$s(\Gamma) = (g; m_1, \cdots, m_r; c).$$

- use Gauss-Bonnet formula to show that
$$Area_{\mathbb{H}}(D) = 2\pi(2g - 2 + \sum_{j=1}^{r}(1 - 1/m_j) + c);$$
- show that if $c \geq 1$ then
$$Area_{\mathbb{H}}(D) \geq \frac{\pi}{3}.$$

Problem 2. Let $\Gamma = \{\gamma_n | \gamma_n(z) = 2^n z\}$. This is a Fuchsian group. Choose a suitable $p \in \mathbb{H}$ and construct a Dirichlet polygon $D(p)$.

Problem 3. We take a vector $v = (a, b, 0)$ with irrational a/b. This vector determines a flow ϕ_t on $\mathbb{T}^3$ in the following way:
$$\phi_t([x]) = [x + tv]$$
for each $x \in \mathbb{R}^3$ and $t \in \mathbb{R}$. Show that

- orbits of ϕ_t are not dense in $\mathbb{T}^3$ (and therefore are not uniformly distributed with respect to Lebesgue measure);
- each orbit is uniformly distributed in some subtorus of $\mathbb{T}^3$: given that $x = (x_1, x_2, x_3) \in \mathbb{T}^3$, let μ be a Haar measure on the horizontal 2-torus $\mathbb{T}^2 \times x_3$ that contains x. Then
$$\frac{1}{T}\int_0^T f(\phi_t(x))dt \to \int_{\mathbb{T}^2 \times x_3} f d\mu.$$

Problem 4. Let $X = \Gamma \setminus SL_2(\mathbb{R})$ be the quotient by lattice Γ. Prove that a probability measure μ on X invariant under the horocycle flow $h(s)$ that gives zero measure to the set of all periodic orbits of the horocycle flow must be the Haar measure of X.

Problem 5. Let $u \in A^{\mathbb{N}}$ be an infinite word on some finite alphabet A. Prove that the following are equivalent:

(1) u is ultimately periodic: $u = pvvv\cdots$ for some finite words p and v.
(2) $p_u(n)$ is bounded;
(3) for some n we have $p_u(n) \leq n$.

Problem 6. Compute the genus and the number of zeroes of L-shaped translation surface. Draw the holonomies of saddles connections of length less than 8 assuming that the squares have unit side.

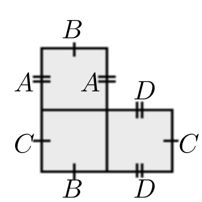

FIGURE 1. L - shaped translation surface