Exam problems for the seminar “Representations and Probability”, Fall 2019

The highest mark (10) corresponds to the solution of any three out of the five presented problems.

Problem 1. For $t > 0$, consider the scaled semicircle law, namely the probability measure on $\mathbb{R}$:

$$d\mu_t(x) := \tfrac{1}{2\pi t}\mathbf{1}_{[-2\sqrt{t},2\sqrt{t}]}\sqrt{4t-x^2}\,dx.$$

Define for $x > 2\sqrt{t}$

$$f(t,x) := \int \tfrac{1}{x-y}d\mu_t(y) = \frac{x-\sqrt{4t-x^2}}{2t}.$$

For $x > 0$, consider the solution $X^x(t)$ to the equation

$$\dot{X} = f(t,X), \qquad X(0) = x. \tag{1}$$

This is monotone in t and x, so define $X^0(t) := \lim_{x\to 0} X^x(t)$ (which is the rigorous way of writing (1) for $x = 0$, since $f(0,x)$ is defined only for $x > 0$).
a) Why is it that $X(1) = 2$ coincides with the limit, as $N \to \infty$, of the largest eigenvalue of the GUE (with entries normalized as to have variance $1/N$)? To avoid technicalities, in the calculations, you can pretend that $1/x$ is a smooth function.
b) For $x > 0$, can you find a random $N \times N$ matrix such that the quantity $X^x(1)$ coincides with the limit as $N \to \infty$ of the largest eigenvalue of this matrix?
c) Can you write down a similar differential system for the (limit of) the second largest eigenvalue?

Problem 2. For μ being a Borel probability measure on $\mathbb{R}$ define

$$F(\mu) := \int \tfrac{x^2}{4}d\mu(x) - \tfrac{1}{2}\int \log|x-y|d\mu(y)\,d\mu(x). \tag{2}$$

Notice that this is well-defined, since $x^2/4 + y^2/4 - \log|x-y|$ is bounded from below. What is the minimizer of the F (that is, the probability measure μ that minimizes $F(\mu)$)? Give a proof that such a μ is a minimizer using random matrices, and/or a direct proof from (2).

Problem 3. The probability density function for the eigenvalues of $N \times N$-matrix from the Gaussian unitary ensemble has the form

$$p_N(x_1,\ldots,x_N) = \mathrm{const}_N \prod_{1\le i<j\le N}(x_i-x_j)^2 e^{-\sum\limits_{i=1}^{N} x_i^2/2}.$$

Prove that these eigenvalues form a determinantal process, i.e. the corresponding correlation functions have a determinantal form.
Hint. We have proven this for the N-th correlation function ρ_N, so it remains to prove this for ρ_k with $k \neq N$.

Problem 4. Prove that the variance of the sine-process has a logarithmic growth:

$$\mathrm{Var}\,\#_{[0,N]} = C\ln N + o(\ln N),$$

where $C > 0$, and $\#_{[0,N]}$ is the number of particles on the segment $[0,N]$ for this process.
Hint. Use the formula describing the variance in terms of integrals of the kernel, which has been proven on the last lecture of the corresponding part of the seminar.

Problem 5. Consider an octagon on the plane that is obtained from the square with side length equal to 3 by removing four right-angled triangles at its corners, each with both legs equal to 1. In other words, the sides of the octagon equal 1 and $\sqrt{2}$ alternatingly, and all angles equal $135°$. The flat surface M is obtained from the octagon by the factorization of four pairs of parallel sides by the corresponding translations. Finally, let the two foliations, $\mathcal{F}^-$ and $\mathcal{F}^+$, be the set of lines with angles equal to $30°$ and $120°$ with respect to the bottom side.

a) What are the singularities of these foliations? What is the genus of the surface M? Are there any closed leaves of the foliations? Are there separatrices connecting two singularities?

b) Construct two subsequent (in terms of the Rauzy induction) decomposition of the surface into rectangles. Find the intersection graph for these two partitions (i.e. one level of the infinite graded graph corresponding to the whole sequence of partitions). Find the orderings of outgoing/incoming edges for this graph.

Hint. Start with any piece I of the separatrix of $\mathcal{F}^-$ with one end at a singular point. For all separatrices in $\mathcal{F}^+$ of all singular points draw their segments till their first intersections with I. Then it remains to deal with the piece of I near its non-singular end.