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Discrete Integrable Equations and Their Reductions

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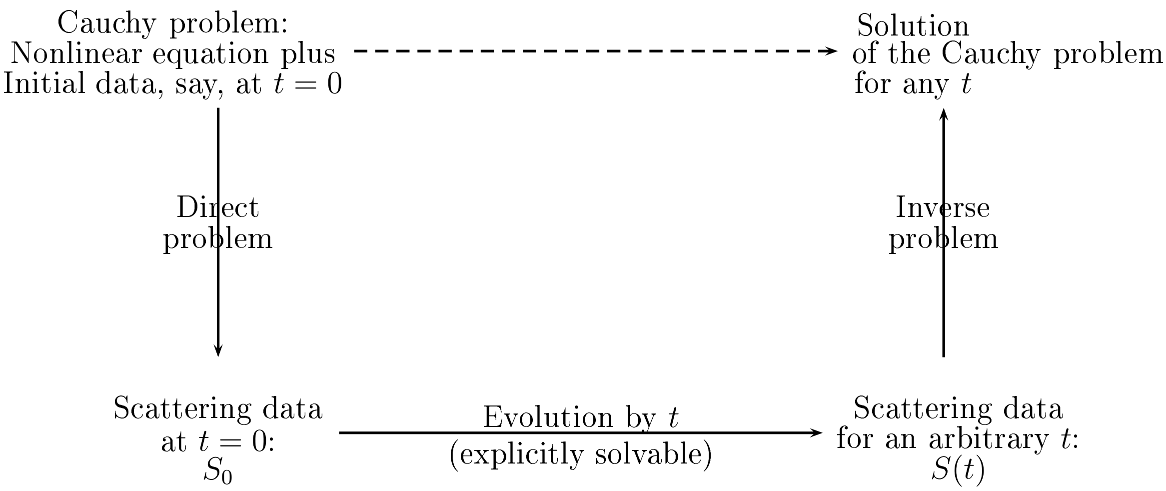
Contents

1 Lecture

1.1 Introduction

General scheme of the Inverse Scattering Transform (IST) in application to partial differential, difference-differential and difference equation can be demonstrated by the following diagram:



By means of initial data of the Cauchy problem one constructs corresponding Scattering data S_0 of the Lax operator L. Existence of the Lax representation, $L_t = [A, L]$ for the original nonlinear equation enables explicit solution of the evolution equations for the Scattering data. This gives $S(t)$. By means of the equations of the Inverse problem one reconstruct solution of the Cauchy problem for an arbitrary moment of time. In this course we start from the bottom horizontal line of this diagram: we start with construction of **linear** equations that can be uplifted to integrable nonlinear equations. Our main object of investigation will be **difference integrable equations**.

1.2 Commutator identity and linear equation on an associative algebra

We start with the following simple observation. Let we have an associative algebra with unit I and let for some element A in this algebra there exist inverse elements $(A - a_i I)^{-1}$ for some constants $a_1, a_2, a_3 \in \mathbb{C}$ ($a_1 \neq a_2 \neq a_3 \neq a_1$). In what follows we omit unity operator and write $A - a_i$ in such cases. Let B be any other element of this algebra. It is easy to check that there exists commutator identity

$$\begin{aligned}
&a_{12}\big\{(A-a_1)(A-a_2)B(A-a_1)^{-1}(A-a_2)^{-1} + (A-a_3)B(A-a_3)^{-1}\big\}+ \\
&+a_{23}\big\{(A-a_2)(A-a_3)B(A-a_2)^{-1}(A-a_3)^{-1} + (A-a_1)B(A-a_1)^{-1}\big\}+ \\
&+a_{31}\big\{(A-a_3)(A-a_1)B(A-a_3)^{-1}(A-a_1)^{-1} + (A-a_2)B(A-a_2)^{-1}\big\} = 0,
\end{aligned} \tag{1.1}$$

where we denoted differences

$$a_{ij} = a_i - a_j \neq 0 \text{ for } i \neq j. \tag{1.2}$$

Eq. (1.1) also can be written as

$$\begin{aligned} &a_{12}\bigl\{(A-a_1)(A-a_2)B(A-a_1)^{-1}(A-a_2)^{-1}+ \\ &+(A-a_3)B(A-a_3)^{-1}\bigr\} + \text{cycle}(1,2,3) = 0, \end{aligned} \tag{1.3}$$

Notice that if some difference equals to zero, say a_{12}, we do not need to make calculations based on properties of the algebra to prove that (1.1) is identity of the kind $0 = 0$. So in what follows we set that all differences in (1.2) are nonzero.

In this case in order to prove (1.1) we have to use associativity of the algebra that enables to open parenthesis. But as a simplified approach, also in search for such identities we can consider some special realization of elements A and B of the algebra as operators in some space. Say, let we have L^2 on the real axis, $f(x) \in L^2$. Let A be multiplication operator: $(Af)(x) = xf(x)$, and $(Bf)(x) = intdyB(x,y)f(y)$ be some integral operator. Applying then the l.h.s. of (1.1) to f we get that in the integral with respect to y integrand $B(x,y)f(y)$ got factor

$$\begin{aligned} a_{12}\left\{\frac{(x-a_1)(x-a_2)}{(y-a_1)(y-a_2)} + \frac{x-a_3}{y-a_3}\right\} + a_{23}\left\{\frac{(x-a_2)(x-a_3)}{(y-a_2)(y-a_3)} + \frac{x-a_1}{y-a_1}\right\}+ \\ + a_{31}\left\{\frac{(x-a_3)(x-a_1)}{(y-a_3)(y-a_1)} + \frac{x-a_2}{x-a_2}\right\} = 0, \end{aligned} \tag{1.4}$$

where now equality follow by simple calculations and takes place for any x and y.

In this course we consider discrete equations, i.e., equations depending on discrete variables 1,2, etc. In other words we consider functions and operators (matrices) $u(m)$, $F(m)$, etc., depending on $m = \{m_1, M_2, M_3, \ldots\}$, where $m_i \in \mathbb{Z}$. Throughout this text we use the following notation

$$\begin{gathered} F^{(1)}(m) = F(m_1+1, m_2, m_3), \qquad F^{(2)}(m) = F(m_1, m_2+1, m_3), \\ F^{(2,3)}(m) = F(m_1, m_2+1, m_3+1), \quad \text{etc.} \end{gathered} \tag{1.5}$$

so that upper indexes 1, 2, 3 and so on in parenthesis denote unit shifts of the variable with the same number. It is clear that such shifts commute:

$$F^{(1,2)} \equiv (F^{(1)})^{(2)} \equiv (F^{(2)})^{(1)} \equiv F^{(2,1)}. \tag{1.6}$$

Existence of identity (1.3) suggests to introduce dependence of B on three discrete variables m_1, m_2, m_3 belonging to $\mathbb{Z}$ by means of equalities

$$\begin{aligned} B^{(1)} &= (A-a_1I)B(A-a_1I)^{-1}, \\ B^{(2)} &= (A-a_2I)B(A-a_2I)^{-1}, \\ B^{(3)} &= (A-a_3I)B(A-a_3I)^{-1}, \end{aligned} \tag{1.7}$$

Then by (1.1), or (1.3) $B(m)$ obeys linear difference equation

$$a_{12}\bigl\{B^{(12)} + B^{(3)}\bigr\} + \text{cycle}\{1,2,3\} = 0. \tag{1.8}$$

While this equation, valid on an arbitrary associative algebra, will be the main subject of our consideration here, it is not the only identity of such kind. Let us consider "derivation" of the above equality. Following (1.7) and above realization of elements of associative algebra we denote $\alpha_i = \dfrac{x-a_i}{y-a_i}$. Next, we use, say, equations for $i=1,2$ in order to write x and y as functions of α_1 and α_2. Then we insert these values in α_3 that gives polynomial relation

$$a_{12}\bigl\{\alpha_1\alpha_2+\alpha_3\bigr\}+\text{cycle}\{1,2,3\}=0, \tag{1.9}$$

that is the way to write (1.8) in terms of that special realization. Of course, finally one has to check that equalities (1.1), or (1.8) are valid for any elements of associative algebra.

In what follows we show that linear difference equation (1.8) can be lifted to nonlinear integrable difference equation—the famous Hirota difference equation, that has a lot of literature.

Problem 1 *A, a_1, a_2 and a_3 are mutually commuting elements of the algebra under consideration. Let, like in (1.7)*

$$B^{(j)}=(A-a_j)B(A-a_j)^{-1}, \quad j=1,2,3. \tag{1.10}$$

Find analogs of Eqs. (1.3) and (1.8). Hint: elements a_i do not commute with B.

Problem 2 *Let A, B_1 and B_2 be three elements of an associative algebra. Let*

$$B=\begin{pmatrix} 0 & B_1\\ B_2 & 0\end{pmatrix} \text{ and we understand } A \text{ as } \begin{pmatrix} A & 0\\ 0 & A\end{pmatrix}. \tag{1.11}$$

Parameters a_1, a_2 and a_3 are in $\mathbb{C}$ and we denote

$$\sigma=\sigma_3=\begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}. \tag{1.12}$$

Let us introduce dependence on m_i by means of relations (1.5) and

$$\begin{aligned}&B^{(1)}=(A-a_1)B(A-a_1)^{-1}, \qquad B^{(2)}=(A-a_2)B(A-a_2)^{-1},\\ &B^{(3)}=(A-a_3\sigma)B(A-a_3\sigma)^{-1}.\end{aligned} \tag{1.13}$$

Find commutator identity and corresponding evolution equation.

Problem 3 *Under the same conditions as in Problem 2 but in the case of evolution*

$$\begin{aligned}&B^{(1)}=(A-a_1)B(A-a_1)^{-1}, \qquad B^{(2)}=(A-a_2\sigma)B(A-a_2\sigma)^{-1},\\ &B^{(3)}=(A-a_3\sigma)B(A-a_3\sigma)^{-1},\end{aligned} \tag{1.14}$$

find commutator identity and corresponding evolution equation.

Problem 4 *Under the same conditions as in Problem 2 but in the case of evolution*

$$\begin{aligned}&B^{(1)}=(A-a_1\sigma)B(A-a_1\sigma)^{-1}, \qquad B^{(2)}=(A-a_2\sigma)B(A-a_2\sigma)^{-1},\\ &B^{(3)}=(A-a_3\sigma)B(A-a_3\sigma)^{-1},\end{aligned} \tag{1.15}$$

find commutator identity and corresponding evolution equation.

2 Lecture.

2.1 Operator realization of elements of an associative algebra.

In order to arrive to nonlinear evolution equation we need a so called "dressing procedure", that in its turn require specific realization of elements of associative algebra. Taking that we are working here with discrete variables running through $\mathbb{Z}$ into account, we consider (infinite) matrices F, G, ets. Let T denotes shift matrix $T_{m_1,m_1'} = \delta_{m_1,m_1'+1}$. For any matrix $F = \{F_{ij}\}_{i,j\in\mathbb{Z}}$ we introduce $f_n(m_1) = F_{m_1,m_1-n}$, so that matrix F can be written as

$$F = \sum_{n\in\mathbb{Z}} f_n T^n, \tag{2.1}$$

where all matrices $f_n = \operatorname{diag}\{f_n(m_1)\}_{m_1\in\mathbb{Z}}$ are diagonal, i.e., mutually commuting ones. Notice, that this is leading consideration only, so we do not discuss convergence of the above series. With this accuracy we uniquely associate to every matrix F its symbol

$$\widetilde{F}(m_1, z) = \sum_{n\in\mathbb{Z}} f_n(m_1) z^n, \tag{2.2}$$

where $m_1 \in \mathbb{Z}$, $z = z_{\rm Re} + iz_{\rm Im} \in \mathbb{C}$. It is easy to see that the standard product of matrices F and G in terms of their symbols takes the form

$$\widetilde{FG}(m_1, z) = \oint_{|\zeta|=1} \frac{d\zeta}{2\pi i\zeta} \widetilde{F}(m_1, z\zeta) \sum_{m_1'\in\mathbb{Z}} \zeta^{m_1-m_1'} \widetilde{G}(m_1', z). \tag{2.3}$$

Here and below we use relations

$$\oint_{|\zeta|=1} \frac{d\zeta\,\zeta^n}{2\pi i\zeta} = \delta_{n,0}, \qquad \delta_c(\zeta_j) = \sum_{n=-\infty}^{\infty} \zeta_j^n, \tag{2.4}$$

where the latter one gives the delta-function on the contour, i.e.,

$$\oint\limits_{|\zeta_1|=1} \frac{d\zeta_1}{2\pi i\zeta_1} f(\zeta_1)\delta_c(\zeta_1) = f(1) \tag{2.5}$$

for a test-function $f(\zeta)$. In other words, if function $\varphi(\zeta)$ on the unit circle $\zeta \in \mathbb{C}$, $|\zeta| = 1$, admits decomposition in the Fourier series, i.e.,

$$\varphi(\zeta) = \sum_{n\in\mathbb{Z}} \varphi_n\,\zeta^{-n}, \text{ then } \varphi_n = \oint\limits_{|\zeta|=1} \frac{d\zeta\,\zeta^{n-1}}{2\pi i}\,\varphi(\zeta), \tag{2.6}$$

that means direct and inverse Fourier transforms correspondingly. We see that the composition law (2.3) gives a kind of "deformed" Fourier transform. Moreover, in the case where symbol $\widetilde{F}(m_1, z)$ of operator F is independent of z (due to (2.1) this means that infinite matrix F is diagonal) this law reduces to the composition of the direct and inverse Fourier transforms, so that

$$\widetilde{FG}(m_1, z) = \widetilde{F}(m_1)\widetilde{G}(m_1, z) \tag{2.7}$$

for any operator G (see Problem 5). Thus operators with z-independent symbols play the role of multiplication ones.

As useful examples we mention that for the unit and shift matrices (I and T correspondingly) $i_n(m_1) = \delta_{m_1,0}$ and $t_n(m_1) = \delta_{m_1,1}$, so by the above definition we have for the symbols:

$$\widetilde{I}(m,z) = 1, \qquad \widetilde{T}(m,z) = z. \tag{2.8}$$

Relation (2.1) shows that we use an analog of the normal order: all shift operators are placed to the right from multiplication ones, that is confirmed by (2.7). Correspondingly, let G be a function of the shift operator only, i.e., due to (2.1) and (2.2) its symbol is independent of the discrete variable, $\widetilde{G}(m_1,z) \equiv \widetilde{G}(z)$. Then by (2.3) we get

$$\widetilde{FG}(m_1,z) = \widetilde{F}(m_1,z)\widetilde{G}(z) \tag{2.9}$$

for an arbitrary F (see Problem 6). Similarity transformation by means of operator T, as follows from (2.3) and (2.9), gives a shift of the discrete variable

$$\widetilde{TFT^{-1}}(m_1,z) = \widetilde{F}(m_1+1,z), \text{ i.e., } TFT^{-1} = F^{(1)}, \tag{2.10}$$

where notation (1.5) was used. This relation is essential for construction below.

On the set of these functions $F(m,z)$ we define the following linear operations:

complex conjugation: $\quad \widetilde{F^*}(m,z) = \overline{\widetilde{F}(m,\overline{z})}, \quad (2.11)$

transposition: $\quad \widetilde{F^{\mathrm{T}}}(m,z) = \oint\limits_{|\zeta|=1} \frac{d\zeta}{2\pi i\zeta} \widetilde{F}(m-n,\zeta)(z\zeta^n), \quad (2.12)$

Hermitian conjugation: $\quad F^{\dagger} = (F^{\mathrm{T}})^*. \quad (2.13)$

In what follows we consider set of "pseudo-matrix" operators F, G, ... given by their symbols $\widetilde{F}$, $\widetilde{G}$, ... with the above composition law. We impose condition that these symbols are tempered distributions with respect to their variables, or Fourier coefficients of distributions. But in generic situation we do not expect any relation of the kind (2.1) of these operators with matrices, in particular, we do not expect any analyticity property of the symbols of operators with respect to the variable z. Because of this one can introduce on this set operations that are well defined in terms of symbols, but have no analog on the set of matrices. In particular, we define operation of $\bar{\partial}$-differentiation: $F \to \bar{\partial}F$:

$$(\widetilde{\bar{\partial}F})(m,z) = \frac{\partial\widetilde{F}(m,z)}{\partial\overline{z}}. \tag{2.14}$$

This derivative is the measure of departure of the symbol of operator from analyticity, so it also give a measure of departure of operator from the infinite matrix, i.e., from situation when series in (2.1) converges. In particular, unit and shift operators, as follows from (2.8) obey

$$\bar{\partial}I = 0, \qquad \bar{\partial}T = 0. \tag{2.15}$$

We consider operators A and B as operators of the above kind with symbols $\widetilde{A}$ and $\widetilde{B}$. Dependence of the symbol of B on m_1, $B^{(1)} = (A-a_1)B(A-a_1)^{-1}$ is exactly as the one under the similarity transformation (2.10) by means of operator T. Thus we can set

$$A = T + a_1, \text{ i.e., } \widetilde{A}(m,z) = z + a_1. \tag{2.16}$$

Then in correspondence to (1.7)

$$\begin{gathered} B^{(1)} = TBT^{-1}, \quad B^{(2)} = (T + a_{12})B(T + a_{12})^{-1}, \\ B^{(3)} = (T + a_{13})B(T + a_{13})^{-1}, \end{gathered} \tag{2.17}$$

where notations (1.5) and (1.2) were used. Symbol $\widetilde{B}(m_1, m_2, m_3, z)$ of operator B depends now on the three discrete variables, $m_1, m_2, m_3 \in \mathbb{Z}$, besides the variable z. In what follows we use notation $\widetilde{B}(m, z)$, setting $m = \{m_1, m_2, m_3\}$. Dependence on m_2 and m_3 does not affect (2.3), where product of symbols must be considered as pointwise with respect to these variables, so that

$$(FG)^{(i)} = F^{(i)}G^{(i)}, \quad i = 1, 2, 3, \tag{2.18}$$

where for $i = 1$ this equality follows from (2.3).

Problem 5 *Let F be a diagonal matrix, i.e., its symbol is z-independent of z: $\widetilde{F}(m_1, z) \equiv \widetilde{F}(m_1)$. Prove that under this condition Eq. (2.3) takes the form (2.7) for any operator G.*

Problem 6 *Let G be a function of the shift operator only, i.e., its symbol is independent of the discrete variable, $\widetilde{G}(m_1, z) \equiv \widetilde{G}(z)$. Prove that under this condition Eq. (2.3) takes the form (2.9) for an arbitrary operator F.*

3 – 4 Lectures

3.1 Symbol of operator B.

Thanks to m-dependence of operator B specified in (2.17), its symbol can be presented as

$$\widetilde{B}(m,z)=\oint\limits_{|\zeta|=1}\frac{d\zeta}{2\pi i\zeta}\zeta^{m_1}\left(\frac{z\zeta+a_{12}}{z+a_{12}}\right)^{m_2}\left(\frac{z\zeta+a_{13}}{z+a_{13}}\right)^{m_3}b(\zeta,z), \tag{3.1}$$

where $b(\zeta,z)$ is some function. It is reasonable to exclude its exponential growth with respect to m_2 and m_3. So we impose conditions $|z\zeta+a_{12}|=|z+a_{12}|$, $|z\zeta+a_{13}|=|z+a_{13}|$, that are equivalent to either $\zeta=1$, or $\overline{z}/z=\zeta\overline{a}_{12}/a_{12}=\zeta\overline{a}_{13}/a_{13}$. The first condition leads to a trivial constant operator in (3.1), so we consider the second one only. Because of it: $\overline{a}_{12}/a_{12}=\overline{a}_{13}/a_{13}$, and thus (shifting phase of z, if necessary) we can choose all a_j to be real. This means that function $b(\zeta,z)$ has support on the surface $\zeta=\overline{z}/z$. In the simplest case $b(\zeta,z)=\delta_c(\zeta z/\overline{z})f(z)$, where δ_c is the δ-function on the unit circle and $\widetilde{R}(z)$ is an arbitrary function of $z\in\mathbb{C}$. Then representation (3.1) for the symbol of B becomes

$$\widetilde{B}(m,z)=\left(\frac{\overline{z}}{z}\right)^{m_1}\left(\frac{\overline{z}+a_{12}}{z+a_{12}}\right)^{m_2}\left(\frac{\overline{z}+a_{13}}{z+a_{13}}\right)^{m_3}f(z), \tag{3.2}$$

Taking property of the m-dependent factor here into account, it is reasonable to input condition that $\widetilde{R}(\overline{z})=\overline{\widetilde{R}(z)}$. Then also

$$\widetilde{B}(m,\overline{z})=\overline{\widetilde{B}(m,z)},\ \text{i.e., } B^*=B, \tag{3.3}$$

where notation (2.11) was used. In generic situation $b(\zeta,z)$ in (3.1) can be proportional to the finite sum of derivatives of $\delta_c(\zeta)$, that we do not consider here in order to avoid asymptotic growth of $\widetilde{B}(m,z)$ by m.

3.2 Cauchy–Green formula

Below we use terminology of the theory of functions of complex variables and notation $z=z_{\text{Re}}+iz_{\text{Im}}\in\mathbb{C}$. We also write $d^2z=dz_{\text{Re}}dz_{\text{Im}}\equiv 2idz\wedge d\overline{z}$ and derivatives $\partial_z=\frac{1}{2}(\partial_{z_{\text{Re}}}-i\partial_{z_{\text{Im}}})\equiv\partial$, $\partial_{\overline{z}}=\frac{1}{2}(\partial_{z_{\text{Re}}}+i\partial_{z_{\text{Im}}})\equiv\overline{\partial}$. Function $f(z)$ is analytic if it is differentiable and obeys Cauchy conditions, that in these terms can be written as $\partial_{\overline{z}}f(z)=0$. Here we do not assume analyticity of functions under consideration and use notation $f(z)$ as short for $f(z_{\text{Re}},z_{\text{Im}})$, i.e., function of two real variables. Under proper assumptions on smoothness of a function $f(z)$ of complex variable and border ∂D of a simply connected domain D on $\mathbb{C}$ one has Green's formulas:

$$2i\int\limits_D d^2z\,\overline{\partial}f(z)=\oint\limits_{\partial D}dz\,f(z),\qquad 2i\int\limits_D d^2z\,\partial f(z)=-\oint\limits_{\partial D}d\overline{z}\,f(z),$$

where domain D is to the left from the contour ∂D in process of integration by it.

Useful relation is given by means of the formula from the theory of distributions:

$$\overline{\partial}\frac{1}{z}=\pi\delta(z)\equiv\pi\delta(z_{\text{Re}})\delta(z_{\text{Im}}),$$

where $\delta(z_{\rm Re})$ and $\delta(z_{\rm Im})$ are delta-functions of their arguments. In order to prove this relation we let function $f(z)$ to be infinitely differentiable and to decay at $z \to \infty$ faster than any power of z (both these conditions are too strong, in fact) and use definition of derivative of a distribution:

$$\int d^2\left(\overline{\partial}\,\frac{1}{z}\right)f(z) = -\int d^2\,\frac{1}{z}\overline{\partial}f(z) =$$
$$= -\lim_{\varepsilon\to 0}\int\limits_{|z|>\varepsilon} d^2z\,\frac{1}{z}\,\overline{\partial}f(z) = -\lim_{\varepsilon\to 0}\int\limits_{|z|>\varepsilon} d^2z\,\overline{\partial}\,\frac{f(z)}{z} =$$

where we used that in this domain function $1/z$ is analytic; then by the Green's formula

$$= \frac{-1}{2i}\lim_{\epsilon\to 0}\oint\limits_{|z|=\varepsilon} dz\,\frac{f(z)}{z} = \frac{-f(0)}{2i}\lim_{\epsilon\to 0}\oint\limits_{|z|=\varepsilon} dz\,\frac{1}{z} = \pi f(0),$$

where $f(0)$ was substituted for $f(z)$ for ε small enough and the last integral was calculated explicitly. By means of these relations we can prove the Cauchy–Green formula:

$$f(z) = -\frac{1}{2\pi i}\oint\limits_{\partial D} dz'\,\frac{f(z')}{z-z'} + \frac{1}{\pi}\int\limits_{D}\frac{d^2z'}{z-z'}\,\overline{\partial}'f(z'),$$

when $z \in D \subset \mathbb{C}$ and $f(z) = 0$ otherwise. This formula generalizes to the non-holomorphic case the standard Cauchy formula. Here we denoted $\overline{\partial}' = \partial_{\overline{z}'}$

3.3 Dressing procedure

The main object of our construction, **dressing operator** K with symbol $\widetilde{K}(m,z)$, is introduced as solution of the $\overline{\partial}$-problem:

$$\begin{aligned}&\overline{\partial}K = KB,\\ &\lim_{z\to\infty}\widetilde{K}(m,z) = 1,\end{aligned} \tag{3.4}$$

where product in the r.h.s. is understood in the sense of (2.3). Differential $\overline{\partial}$-equation here due to (2.3) and (3.2) in terms of symbols sounds as

$$\overline{\partial}\,\widetilde{K}(m,z) = \widetilde{K}(m,\overline{z})\left(\frac{\overline{z}}{z}\right)^{m_1}\left(\frac{\overline{z}+a_{12}}{z+a_{12}}\right)^{m_2}\left(\frac{\overline{z}+a_{13}}{z+a_{13}}\right)^{m_3} f(z), \tag{3.5}$$

so we can use the Cauchy–Green formula and write that inside any domain D

$$\widetilde{K}(m,z) = -\frac{1}{2\pi i}\oint\limits_{\partial D} dz'\,\frac{\widetilde{K}(m,z')}{z-z'} + \frac{1}{\pi}\int\limits_{D}\frac{d^2z}{z-z'}\,\overline{\partial}'\widetilde{K}(m,z').$$

In order to get integral equation on $\widetilde{K}(m,z)$ we have to expand the domain D on the whole complex plane, so that we have to impose some asymptotic condition on behavior of this symbol at z-infinity. Notice that if it tends to some constant value f_∞, then the first term here equals this constant. So setting asymptotic condition in (3.4) we can extent domain D to the whole complex plain $\mathbb{C}$, so that in terms of symbols we get integral equation

$$\widetilde{K}(m,z) = 1 + \frac{1}{\pi}\int \frac{d^2z'}{z-z'}\,\overline{\partial}'\widetilde{K}(m,z'). \tag{3.6}$$

This integral equation is equivalent to the problem (3.4) and can be used to prove existence and uniqueness of solution of this problem. Here we do not go in this details and we assume this unique solvability. This assumption is crucial for our construction, but not essential for its results. As the first result of this assumption we get that because of conjugation property of B (see (3.3)) we also have conjugation property for the dressing operator:

$$K^{*} = K. \tag{3.7}$$

Let us consider an operator F in our class of operators, such that its symbol $\widetilde{F}(m,z)$ is entire function of z, i.e. $\overline{\partial}F = 0$, see (2.14). Thanks to (2.3) we have that $\overline{\partial}FK = F\overline{\partial}K = FKB$, so that FK obey the same differential equation in (3.4). Then we get instead of (3.6) integral equation

$$\widetilde{FK}(m,z) = \widetilde{F}(m,z) + \frac{1}{\pi}\int \frac{d^2z'}{z-z'}\,\overline{\partial}'\widetilde{K}(m,z'), \quad \overline{\partial}F = 0, \tag{3.8}$$

assuming that the integral converges. Thus asymptotic behavior of the composed operator FK is determined by the asymptotic behavior of the symbol of operator F. Vice verse, due to assumption on the unique solvability of the problem (3.4) we see that any solution of the equation $\overline{\partial}G = GB$ with asymptotic behavior determined by entire function $\widetilde{F}(m,z)$ can be written as

$$G = FK. \tag{3.9}$$

Indeed, difference $G - FK$ obeys the same differential equation $\overline{\partial}(G - FK) = (G - FK)B$ but with zero inhomogeneous term, as asymptotics of this difference equals to zero.

Dependence of operator K on variables m is introduced by means of the same $\overline{\partial}$-problem:

$$\overline{\partial}K^{(j)} = K^{(j)}B^{(j)}, \qquad \lim_{z\to\infty}\widetilde{K^{(j)}}(m,z) = 1, \quad j = 1,2,3, \tag{3.10}$$

where (2.18) was taken into account. But we have to check that evolutions of K defined in this way are mutually compatible. We use here that compatibility of evolution equations of operator B is obvious by construction. Then by (2.18) and (3.10) $\overline{\partial}K^{(i,j)} = K^{(i,j)}B^{(i,j)}$ and $\overline{\partial}K^{(j,i)} = K^{(j,i)}B^{(i,j)}$ for any $i,j = 1,2,3$. Thus difference $K^{(i,j)} - K^{(j,i)}$ obeys $\overline{\partial}$-equation in (3.4), but with zero asymptotic behavior. So this difference equals to zero due to the assumption on the unique solvability. Let us consider consequences of the equality (2.17) for operator K. Notice that this operator, as any operator of the class under consideration obeys (2.10),

$$K^{(1)} = TKT^{-1}, \tag{3.11}$$

that is compatible with (3.10) for $j = 1$ because of of the first equality in (2.17). Consider now $j = 2$. Thanks to (2.15) and (2.17) we derive

$$\overline{\partial}(K^{(2)}(T + a_{12})) = (K^{(2)}(T + a_{12}))B,$$

i.e., product $K^{(2)}(T + a_{12})$ obeys the same $\overline{\partial}$-equation but with asymptotics that growth linearly at z-infinity. Thus thanks to observation in (3.9) there exists multiplication operator X—operator with symbol independent of the variable z—such that $K^{(2)}(T + a_{12}) = (T + X)K$. In order to determine this operator we have to specify asymptotic condition in (3.4) by means of the next term of expansion,

$$K = I + uT^{-1} + \dots, \quad z \to \infty, \tag{3.12}$$

where dots denote terms with symbols decaying faster than z^{-1}, and where u is a multiplication operator. So in terms of symbols this can be written as

$$\widetilde{K}(m, z) = 1 + \frac{u(m)}{z} + \dots, \quad z \to \infty. \tag{3.13}$$

We assume below that $u(m)$ decays rapidly enough at m-infinity:

$$\lim_{m_i \to \infty} u(m) = 0, \tag{3.14}$$

while in fact it would be enough to impose condition that it tends to an arbitrary constant.

Thus we get (see Problem 9) that due to (3.11)

$$K^{(2)}(T + a_{12}) = K^{(1)}T + (a_{12} + u^{(2)} - u^{(1)})K, \tag{3.15}$$

Analogous consideration shows that the evolution with respect to m_3 is given by equation

$$K^{(3)}(T + a_{13}) = K^{(1)}T + (a_{13} + u^{(3)} - u^{(1)})K. \tag{3.16}$$

Problem 7 *By means of (2.3) derive the Leibnitz rule for $\overline{\partial}(FG)$. (Hint: this rule is valid in the standard form for symbols, but it is necessary to write it in terms of operators and their derivatives).*

Problem 8 *Find property of operator u defined in (3.12) with respect to operation (2.13). What does it mean for its symbol?*

Problem 9 *Give details of derivation of Eq. (3.15).*

5 Lecture

5.1 Hirota difference equation

Thanks to (2.7) and (2.9) in terms of symbols relations (3.15) and (3.16) sound as

$$(z+a_{12})\widetilde{K}^{(2)}(m,z)=z\widetilde{K}^{(1)}(m,z)+\bigl(u^{(2)}(m)-u^{(1)}(m)+a_{12}\bigr)\widetilde{K}(m,z), \tag{5.1a}$$

$$(z+a_{13})\widetilde{K}^{(3)}(m,z)=z\widetilde{K}^{(1)}(m,z)+\bigl(u^{(3)}(m)-u^{(1)}(m)+a_{13}\bigr)\widetilde{K}(m,z), \tag{5.1b}$$

so that variable $z\in\mathbb{C}$ plays the role of a spectral parameter. Equations (5.1) are compatible by construction:

$$K^{(2,3)}=K^{(3,2)}, \tag{5.2}$$

as we have proved on the previous lecture. Thanks to (3.15) and (3.16) direct check of this compatibility gives

$$\begin{aligned}
&K^{(2,3)}(T+a_{13})(T+a_{12})=(K^{(2)}(T+a_{1,2}))^{(3)}(T+a_{1,3})=K^{(1,1)}T^2+\\
&\qquad+(a_{1,3}+a_{1,2}-u^{(1,1)}+u^{(2,3)})K^{(1)}T+(a_{1,2}+u^{(2,3)}-u^{(1,3)})(a_{1,3}+u^{(3)}-u^{(1)})K,\\
&K^{(2,3)}(T+a_{13})(T+a_{12})=(K^{(3)}(T+a_{1,3}))^{(2)}(T+a_{1,2})=K^{(1,1)}T^2+\\
&\qquad+(a_{1,3}+a_{1,2}-u^{(1,1)}+u^{(2,3)})K^{(1)}T+(a_{1,3}+u^{(2,3)}-u^{(1,2)})(a_{1,2}+u^{(2)}-u^{(1)})K.
\end{aligned}$$

Summarizing, we get (Problem 10) that function $u(m)$ obeys

$$(a_{1,2}+u^{(2,3)}-u^{(1,3)})(a_{1,3}+u^{(3)}-u^{(1)})=(a_{1,3}+u^{(2,3)}-u^{(1,2)})(a_{1,2}+u^{(2)}-u^{(1)}),$$

that can be simplified say as

$$\begin{aligned}
&u^{(12)}(u^{(2)}-u^{(1)}+a_{12})+a_{12}u^{(3)}+u^{(23)}(u^{(3)}-u^{(2)}+a_{23})+a_{23}u^{(1)}+\\
&+u^{(31)}(u^{(1)}-u^{(3)}+a_{31})+a_{31}u^{(2)}=0,
\end{aligned} \tag{5.3}$$

so that the original Eq. (1.8) is its linearized version. This is one of forms of the Hirota difference equation. Thus by means of our dressing procedure we arrived to nonlinear counterpart of the original linear equation on operator B. Moreover, we constructed Lax representation (here it is better to use term “zero-curvature condition”): equations (5.1a), (5.1b) on an auxiliary (in a sense that it does not participate in (5.3)) function $\widetilde{K}(m,z)$. It is easy to check that now we can forget about condition of unique solvability of the problem (3.4) that was so essential in derivation. Indeed, equivalence of (5.3) and compatibility condition does needs no any assumption and can be checked directly.

For the following it would be reasonable to simplify notations. For shortness we introduce a new dependent variable

$$v(m)=u(m)-m_1a_1-m_2a_2-m_3a_3, \tag{5.4}$$

so that

$$v^{(i)}-v^{(j)}=u^{(i)}-u^{(j)}+a_{ji}, \tag{5.5}$$

that substitute combination that appeared in equations above. In particular for the Hirota difference equation instead of (5.3) we get

$$v^{(1,2)}(v^{(1)}-v^{(2)})+v^{(2,3)}(v^{(2)}-v^{(3)})+v^{(3,1)}(v^{(3)}-v^{(1)})=0, \tag{5.6}$$

that is the more standard way to write down the Hirota difference equation. Notice that while constants a_i are absent in (5.6), by (5.4) they determine the asymptotic behavior of $v(m)$: this function grows linearly with respect to m at infinity. Thanks to (5.5) this means that asymptotically

$$\lim_{m_i, m_j \to \infty} \big(v^{(i)} - v^{(j)}\big) = a_{ji}. \tag{5.7}$$

Let us mention that this asymptotic behavior cancels ill definiteness of (5.6). Indeed, consider the Cauchy problem for the (5.6):

$$v(m_1, m_2, 0) = v_0(m_1, m_2), \tag{5.8}$$

where v_0 is some given function. This Cauchy problem has two trivial solutions:

$$\text{either } v^{(3)} \equiv v^{(1)}, \text{ or } v^{(3)} \equiv v^{(2)}. \tag{5.9}$$

But it is just condition (5.7) that forbids this equalities thanks to (1.2). Below we show that in our case such Cauchy problem is uniquely solvable.

5.2 Jost solution.

We continue to change notations. We define the spectral parameter k as

$$k = z + a_1, \tag{5.10}$$

(cf. (2.16)) that makes relation above more symmetric. And we introduce two more functions:

$$\chi(m, k) = \widetilde{K}(m, k - a_1), \tag{5.11}$$

$$\varphi(m, k) = E(m, k)\chi(m, k), \qquad k \in \mathbb{C}, \tag{5.12}$$

where

$$E(m, k) = (k - a_1)^{m_1}(k - a_2)^{m_2}(k - a_3)^{m_3}, \tag{5.13}$$

Function $\varphi(m, k)$ is called Jost solution—Jost was the first to realize that in study of the spectral problem for the Schrödinger equation it is useful to introduce functions that admit analytic continuation in the complex domain. In our case instead of analyticity we have $\overline{\partial}$-problem (3.4). Formally it (strictly speaking, (3.5)) can be written for the Jost solution as

$$\frac{\partial \varphi(m, k)}{\partial \overline{k}} = r(k)\varphi(m, \overline{k}), \tag{5.14}$$

where

$$r(k) = f(k + a_1). \tag{5.15}$$

Notice that (5.14) does not contain dependence on variables m_i: it appears only due to the asymptotic condition in (3.4). So it is better to write equations on χ, that also follow from (3.5):

$$\frac{\partial \chi(m, k)}{\partial \overline{k}} = \frac{E(m, \overline{k})}{E(m, k)} r(k)\chi(m, \overline{k}), \tag{5.16}$$

$$\lim_{k \to \infty} \chi(m, k) = 1. \tag{5.17}$$

Equations in terms of this function looks to be more complicated then in terms of the function φ, but in the discrete case considered here the latter one, Eq. (5.14), is correct only for all $m_i \geq 0$, Problem 11. Analog of function χ was also introduced in the study of the spectral problem for the Schrödinger equation by Faddeev, so it is often referred to as Faddeev function.

We list here some properties of these functions that follow from the previous results. Thus for any $k \in \mathbb{C}$ we have that

$$\overline{\varphi(m,k)} = \varphi(m,\overline{k}), \qquad \overline{\chi(m,k)} = \chi(m,\overline{k}), \qquad \overline{r(k)} = r(\overline{k}), \tag{5.18}$$

$$\chi(m,k) = 1 + \frac{u(m)}{k} + o(k^{-1}), \quad k \to \infty, \tag{5.19}$$

$$\overline{u(m)} = u(m). \tag{5.20}$$

In terms of the function $\chi(m,k)$ equations of the Lax pair take the form

$$(k-a_2)\chi^{(2)}(m,k) = (k-a_1)\chi^{(1)}(m,k) + \bigl(u^{(2)}(m) - u^{(1)}(m) + a_{12}\bigr)\chi(m,k), \tag{5.21a}$$

$$(k-a_3)\chi^{(3)}(m,k) = (k-a_1)\chi^{(1)}(m,k) + \bigl(u^{(3)}(m) - u^{(1)}(m) + a_{13}\bigr)\chi(m,k), \tag{5.21b}$$

$$(k-a_3)\chi^{(3)}(m,k) = (k-a_2)\chi^{(2)}(m,k) + \bigl(u^{(3)}(m) - u^{(2)}(m) + a_{23}\bigr)\chi(m,k), \tag{5.21c}$$

thus preserving invariance with respect to the cycle permutations of the indexes $\{1,2,3\}$. For the Jost solutions themselves we have Lax pair of HDE being given by any two of the following three equations

$$\varphi^{(2)} = \varphi^{(1)} + \bigl(v^{(2)} - v^{(1)}\bigr)\varphi, \tag{5.22a}$$

$$\varphi^{(3)} = \varphi^{(2)} + \bigl(v^{(3)} - v^{(2)}\bigr)\varphi, \tag{5.22b}$$

$$\varphi^{(1)} = \varphi^{(3)} + \bigl(v^{(1)} - v^{(3)}\bigr)\varphi. \tag{5.22c}$$

Thus passage from χ to φ cancel explicit dependence on k, it appears due to the normalization condition (5.17) only. It is easy to check that the HDE is condition of compatibility for any pair of equations with respect to all three variables m_i. At the same time Eq. (5.6) can be considered as an evolution equation, where, say, m_1 and m_2 play role of the space variables, and m_3 is the time one.

Problem 10 *Give details of derivation of Eq. (5.1) by means of (3.11), (3.15) and (3.16).*

Problem 11 *Why Eq. (5.14) is valid when all $m_i \geq 0$?*

6 Lecture

6.1 Direct problem: Green's function and Jost solution

Let us prove that equation (5.21a) can be written in integrable form like:

$$\chi(m,k) = 1 + \sum_{n_1,n_2\in\mathbb{Z}} G(m-n,k)(u^{(2)}(n) - u^{(1)}(n))\chi(n,k), \quad k\in\mathbb{C}, \tag{6.1}$$

where the Green's function is equal to

$$G(m,k) = \oint\limits_{|\zeta_1|=1} \frac{d\zeta_1}{2\pi i} \oint\limits_{|\zeta_2|=1} \frac{d\zeta_2}{2\pi i} \frac{\zeta_1^{m_1-1}\zeta_2^{m_2-1}}{(k-a_2)\zeta_2 - (k-a_1)\zeta_1 + a_2 - a_1}. \tag{6.2}$$

For this sake we use (6.1) to write down

$$\begin{aligned}
&(k-a_2)\chi^{(2)}(m,k) - (k-a_1)\chi^{(1)}(m,k) = a_{12} + \\
&\quad + \sum_{n_1,n_2} \oint_{|\zeta_1|=1} \frac{d\zeta_1}{2\pi i} \oint_{|\zeta_2|=1} \frac{d\zeta_2}{2\pi i} \frac{\big((k-a_2)\zeta_2 - (k-a_1)\zeta_1\big)\zeta_1^{m_1-n_1-1}\zeta_2^{m_2-n_2-1}}{(k-a_2)\zeta_2 - (k-a_1)\zeta_1 + a_2 - a_1} \times \\
&\quad \times (u^{(2)}(n) - u^{(1)}(n))\chi(n,k) = \\
&\quad = a_{12} + \sum_{n_1,n_2} \oint\limits_{|\zeta_1|=1} \frac{d\zeta_1}{2\pi i} \oint\limits_{|\zeta_2|=1} \frac{d\zeta_2}{2\pi i} \zeta_1^{m_1-n_1-1}\zeta_2^{m_2-n_2-1}(u^{(2)}(n) - u^{(1)}(n))\chi(n,k) + \\
&\quad + a_{12} \sum_{n_1,n_2} \oint_{|\zeta_1|=1} \frac{d\zeta_1}{2\pi i} \oint_{|\zeta_2|=1} \frac{d\zeta_2}{2\pi i} \frac{\zeta_1^{m_1-n_1-1}\zeta_2^{m_2-n_2-1}}{(k-a_2)\zeta_2 - (k-a_1)\zeta_1 + a_2 - a_1} \times \\
&\quad \times (u^{(2)}(n) - u^{(1)}(n))\chi(n,k) = \\
&\quad = a_{12} + a_{12}(\chi(m,k) - 1) + (u^{(2)}(m) - u^{(1)}(m))\chi(m,k)).
\end{aligned}$$

Denominator of the integral in the r.h.s. of (6.2) has zeros in the two cases only: where

$$\zeta_1 = \zeta_2 = 1 \quad \text{or} \quad \zeta_1 = \frac{\overline{k} - a_1}{k - a_1}, \quad \zeta_2 = \frac{\overline{k} - a_2}{k - a_2}, \tag{6.3}$$

so the integral converges and defines $G(m,k)$ as distribution with respect to k. Any of these representations show that the Green's function has properties of conjugation

$$\overline{G(m,k)} = G(m,\overline{k}) = \left(\frac{k-a_1}{\overline{k}-a_1}\right)^{m_1} \left(\frac{k-a_2}{\overline{k}-a_2}\right)^{m_2} G(m,k) \tag{6.4}$$

and antisymmetry

$$G(m_1,m_2,k) = -G(m_2,m_1,k)\Big|_{a_1\leftrightarrow a_2}. \tag{6.5}$$

Say, we use here:

$$G(m,\overline{k}) = \oint_{|\zeta_1|=1} \frac{d\zeta_1}{2\pi i} \oint_{|\zeta_2|=1} \frac{d\zeta_2}{2\pi i} \frac{\zeta_1^{m_1-1}\zeta_2^{m_2-1}}{(\overline{k}-a_2)\zeta_2 - (\overline{k}-a_1)\zeta_1 + a_1 - a_2} =$$

$$= \left(\frac{k-a_1}{\overline{k}-a_1}\right)^{m_1-1} \left(\frac{k-a_2}{\overline{k}-a_2}\right)^{m_2-1} \times$$

$$\times \oint_{|\zeta_1|=1} \frac{d\zeta_1}{2\pi i} \oint_{|\zeta_2|=1} \frac{d\zeta_2}{2\pi i} \frac{\left(\zeta_1 \frac{\overline{k}-a_1}{k-a_1}\right)^{m_1-1} \left(\zeta_2 \frac{\overline{k}-a_2}{k-a_2}\right)^{m_2-1}}{(k-a_2)\frac{\overline{k}-a_2}{k-a_2}\zeta_2 - (k-a_1)\frac{\overline{k}-a_1}{k-a_1}\zeta_1 + a_1 - a_2} =$$

$$= \left(\frac{k-a_1}{\overline{k}-a_1}\right)^{m_1} \left(\frac{k-a_2}{\overline{k}-a_2}\right)^{m_2} G(m,k).$$

6.2 Properties of the Jost solutions

Here we study properties of the function $\chi(m,k)$ defined by equation (6.1), in which connection we assume below unique solvability of this equation. Because of (6.4) reality of the potential $u(m)$ is equivalent to condition

$$\overline{\chi(m,k)} = \chi(m,\overline{k}), \tag{6.6}$$

while second equality in (6.4) shows that function

$$\widetilde{\chi}(m,k) = \left(\frac{\overline{k}-a_1}{k-a_1}\right)^{m_1} \left(\frac{\overline{k}-a_2}{k-a_2}\right)^{m_2} \chi(m,\overline{k}) \tag{6.7}$$

obeys integral equation

$$\widetilde{\chi}(m,k) = \left(\frac{\overline{k}-a_1}{k-a_1}\right)^{m_1} \left(\frac{\overline{k}-a_2}{k-a_2}\right)^{m_2} + \\ + \sum_{n_1,n_2\in\mathbb{Z}} G(m-n,k)\big(u^{(2)}(n) - u^{(1)}(n)\big)\widetilde{\chi}(n,k), \tag{6.8}$$

i.e., equation with the same kernel as in (6.1).

Asymptotic behavior of $\chi(m,k)$ follows thanks to equations (6.1):

$$\lim_{k\to\infty} \chi(m,k) = 1, \qquad \lim_{|m_1|+|m_2|\to\infty} \chi(m,k) = 1, \tag{6.9}$$

and for the second term of $1/k$ expansion we get

$$k(\chi(m,k)-1) \to$$

$$\to \sum_{n_1,n_2} \oint_{|\zeta_1|=1} \frac{d\zeta_1}{2\pi i} \oint_{|\zeta_2|=1} \frac{d\zeta_2}{2\pi i} \frac{\zeta_1^{m_1-n_1-1}\zeta_2^{m_2-n_2-1}}{\zeta_2-\zeta_1} \big(u^{(2)}(n) - u^{(1)}(n)\big) =$$

$$= \sum_{n_1,n_2} \oint_{|\zeta_1|=1} \frac{d\zeta_1}{2\pi i} \oint_{|\zeta_2|=1} \frac{d\zeta_2}{2\pi i} \frac{\zeta_1^{m_1-n_1-1}\zeta_2^{m_2-n_2-1}}{\zeta_2-\zeta_1} \big(u(n)\zeta_2 - u(n)\zeta_1\big),$$

so that

$$\lim_{k\to\infty} k(\chi(m,k)-1) = u(m). \tag{6.10}$$

This limiting values is independent of the half plane where $k \to \infty$. It is worth to mention that from the difference equation (5.21a) we get the asymptotics behavior in the form $k(\chi^{(2)}(m,k) - \chi^{(1)}(m,k)) \to u^{(2)}(m) - u^{(1)}(m)$ only. In fact it is equivalent to (6.2) thanks to the asymptotic decaying of the potential and the second equality in (6.9).

6.3 Time evolution and Inverse problem

Time evolution, i.e., dependence of $\chi(m,k)$ on m_3 is switched on by means of (5.21b) and for the Jost solution itself it follows by (5.12). Let us introduce scattering data and find out their evolution. The departure from analyticity of $\chi(m,k)$ is given by the $\overline{\partial}$-differentiation of Eq. (6.1), so that we have (see Problem 12)

$$\frac{\partial\chi(m,k)}{\partial\overline{k}} = \left(\frac{\overline{k}-a_1}{k-a_1}\right)^{m_1}\left(\frac{\overline{k}-a_2}{k-a_2}\right)^{m_2} r(k,m_3) + \\ + \sum_{n_1,n_2\in\mathbb{Z}} G(m-n,k)(u^{(2)}(n) - u^{(1)}(n))\frac{\partial\chi(n,k)}{\partial\overline{k}}. \tag{6.11}$$

Here we introduced scattering data $r(m_3,k)$ defined by the equality (Problem 12)

$$r(m_3,k) = -\frac{\operatorname{sgn}\operatorname{Im}k}{2\pi i(\overline{k}-a_1)(\overline{k}-a_2)}\times \\ \times \sum_{m_1,m_2\in\mathbb{Z}}\left(\frac{k-a_1}{\overline{k}-a_1}\right)^{m_1}\left(\frac{k-a_2}{\overline{k}-a_2}\right)^{m_2}(u^{(2)}(m)-u^{(1)}(m))\chi(m,k). \tag{6.12}$$

Because of Eq. (6.6) (i.e., because of reality of the potential $u(m)$) we have that $r(k,m_3)$ obeys

$$\overline{r(m_3,k)} = r(m_3,\overline{k}). \tag{6.13}$$

Under assumption of the unique solvability of the problem (6.8) we get by (6.11) that $\partial\chi(m,k)/\partial\overline{k} = r(m_3,k)\widetilde{\chi}(m,k)$, or thanks to (6.7) that

$$\frac{\partial\chi(m,k)}{\partial\overline{k}} = \left(\frac{\overline{k}-a_1}{k-a_1}\right)^{m_1}\left(\frac{\overline{k}-a_2}{k-a_2}\right)^{m_2} r(m_3,k)\chi(m,\overline{k}). \tag{6.14}$$

Time evolution of the spectral data, i.e., dependence on m_3 trivially follows (Problem 13) from $\overline{\partial}$-differentiation of the equation (5.21b) of the Lax pair, and (6.11):

$$r(m_3,k) = \left(\frac{\overline{k}-a_3}{k-a_3}\right)^{m_3} r(k), \tag{6.15}$$

where function $r(k)$ is independent of m_3 and by (6.12) is uniquely defined by the initial data.

Summarizing, the inverse problem to determine $\chi(m,k)$ is given by the equation

$$\frac{\partial\chi(m,k)}{\partial\overline{k}} = R(m,k)\chi(m,\overline{k}) \tag{6.16}$$

with normalization condition (5.17). Here we denoted

$$R(m,k) = \left(\frac{\overline{k}-a_1}{k-a_1}\right)^{m_1}\left(\frac{\overline{k}-a_2}{k-a_2}\right)^{m_2}\left(\frac{\overline{k}-a_3}{k-a_3}\right)^{m_3} r(k), \quad k\in\mathbb{C}. \tag{6.17}$$

For any $r(k)$ this function obeys the linearized version of the Hirota difference equation. We also mention that because of (6.15)

$$|R(m,k)| = |r(k,m_3)| = |r(k)|, \tag{6.18}$$

i.e., $|R(m,k)|$ is independent of m.

Problem 12 *Proof representations (6.11) and (6.12).*

Problem 13 *Let us consider (5.21b). Differentiating it with respect to $\overline{k}$ we get (6.14). Then (6.15) follows, say, as asymptotics with respect to $m_1 \to \infty$. Proof details of this consideration.*

7 Lecture.

7.1 Integrals of motion

Let us introduce function

$$\rho(k)=\sum_{m_1,m_2\in\mathbb{Z}}(u^{(2)}(m)-u^{(1)}(m))\chi(m,k). \tag{7.1}$$

Thanks to the asymptotic decaying of the potential $u(m)$ and boundedness of the function $\chi(m,k)$ by m this series converge and function $\rho(k)$ decays when $k\to\infty$. It obeys conjugation property

$$\overline{\rho(k)}=\rho(\overline{k}) \tag{7.2}$$

thanks to reality of the potential. For the $\overline{\partial}$-derivative of this function we get

$$\frac{\partial\rho(k)}{\partial\overline{k}}=r(k)\Bigl(\frac{\overline{k}-a_3}{k-a_3}\Bigr)^{m_3}\sum_{m_1,m_2\in\mathbb{Z}}\Bigl(\frac{\overline{k}-a_1}{k-a_1}\Bigr)^{m_1}\Bigl(\frac{\overline{k}-a_2}{k-a_2}\Bigr)^{m_2}\bigl(u^{(2)}(m)-u^{(1)}(m)\bigr)\bigr)\chi(m,\overline{k}).$$

Thanks to (6.12) and (6.15) we have

$$\begin{aligned}r(\overline{k})=&-\frac{\operatorname{sgn}\operatorname{Im}\overline{k}}{2\pi i(k-a_1)(k-a_2)}\Bigl(\frac{\overline{k}-a_3}{k-a_3}\Bigr)^{m_3}\times\\&\times\sum_{m_1,m_2\in\mathbb{Z}}\Bigl(\frac{\overline{k}-a_1}{k-a_1}\Bigr)^{m_1}\Bigl(\frac{\overline{k}-a_2}{k-a_2}\Bigr)^{m_2}\bigl(u^{(2)}(m)-u^{(1)}(m)\bigr)\bigr)\chi(m,\overline{k}),\end{aligned}$$

so that combining results of these two relations we get

$$\frac{\partial\rho(k)}{\partial\overline{k}}=-2\pi i(k-a_1)(k-a_2)\operatorname{sgn}\bigl(\operatorname{Im}k\bigr)|r(k)|^2. \tag{7.3}$$

Now taking (5.17) into account we get that in terms of the scattering data function $\rho(k)$ is given by equality

$$\rho(k)=-2i\int d^2k'\frac{(k'-a_1)(k'-a_2)}{k-k'}\operatorname{sgn}\bigl(\operatorname{Im}k'\bigr)|r(k')|^2, \tag{7.4}$$

where $dk^2=d\mathrm{Re}\,k\,\mathrm{Im}\,dk$. Thanks to Eq. (7.3) this proves that $\rho(k)$ is independent of time m_3 and it is the generating function of the infinite set of integrals of motion. Thus thanks to relation (6.10) the first nontrivial integral (the first coefficient of $1/k$ expansion) is

$$\begin{aligned}\rho_1=&\sum_{n_1,n_2\in\mathbb{Z}}(u^{(2)}(m)-u^{(1)}(m))u(m)=\\&=-2i\int d^2k'(k'-a_1)(k'-a_2)\operatorname{sgn}\bigl(\operatorname{Im}k'\bigr)|r(k')|^2.\end{aligned} \tag{7.5}$$

7.2 Higher Hirota difference equations.

An obvious way to introduce new discrete independent variables in HDE is to enlarge number of evolution equations of the kind (1.7), i.e., to introduce in addition to the discrete variables $\{m_1,m_2,m_3\}$ as many another variables $\{m_4,m_5,\ldots\}$ as one wants, so that dynamics with respect to any of them is given by means of $B^{(i)} = (A - a_i)B(A - a_i)^{-1}$, where $a_4, a_5, \ldots$ are different (real) parameters. All these evolutions are mutually compatible and compatible with the original variables, but their definition shows that for any i, j, k we have an analog of (1.8) (see also (1.2)):

$$a_{ij}\{B^{(ij)} + B^{(k)}\} + \text{cycle}\{i, j, k\} = 0.$$

Then we get that with respect to any three variables m_i, m_j and m_k function $u(m_1, \ldots)$ defined in (3.12) and function $v(m) = u(m) - \sum_i a_i m_i$ (cf. (5.4)) obey the same HDE. Thus this "extension" is trivial one and can be interesting only for the study of symmetries of the HDE.

Thus in order to get higher analogs of the HDE, we have to consider higher analogs of the similarity transformations (1.7). Let $p_i = p_i(T)$, $i = 1, 2, 3$, be polynomials of operator T of the orders n_i with constant coefficients, i.e., symbols $\widetilde{p}_i(m, z) = p_i(z)$ are polynomials of $z \in \mathbb{C}$. We set also that all these polynomials has simple and mutually different zeros and that the coefficients of the highest powers equal to I. As before, we consider operator B with symbol $\widetilde{B}(m_1, m_2, m_3, z)$ depending on discrete variables $m_i \in \mathbb{Z}$, but now dependence on these variables is given by

$$B^{(i)} = p_i B p_i^{-1}, \quad i = 1, 2, 3, \tag{7.6}$$

instead of (1.7). Notice, that due to condition on the polynomials p_i we can write every of them as

$$p_i(T) = \prod_{j=1}^{n_i}(T - x_{ij}), \tag{7.7}$$

so that shift with respect to i-th variable by (7.6) is equivalent to the n_i shifts in the sense of (1.7). Nevertheless, derivation of evolution equations (7.6) by means of such multidimensional reductions is very complicated even in the linear case, so we construct nonlinear equations on the base of (7.6) directly. To be consistent with the shift with respect to $p_1(T)$ we choose

$$p_1(T) = T. \tag{7.8}$$

The dressing operator K is defined by the same $\overline{\partial}$-problem (3.4) and its dependence on m_i is given by (7.6). Then, as before, under assumption of the unique solvability of the (3.4), there exist polynomials $P_i(T)$ such that

$$K^{(i)} p_i = P_i K, \quad i = 1, 2, 3. \tag{7.9}$$

Let us write

$$p_i(T) = \sum_{j=0}^{n_i} y_{ij} T^j, \qquad P_i(T) = \sum_{j=0}^{n_i} Y_{ij} T^j, \tag{7.10}$$

where $y_{i,n_i} = Y_{i,n_i} \equiv 1$ and where all y_{ij} are constants, while Y_{ij} are multiplication operators, $\widetilde{Y}_{ij}(m, z) = \widetilde{Y}_{ij}(m)$. Then (7.9) takes the form

$$K^{(i)} \sum_{j=0}^{n_i} y_{ij} T^j = \sum_{j=0}^{n_i} Y_{ij} K^{(1\times j)} T^j, \quad i = 1, 2, 3. \tag{7.11}$$

Here we introduced notation (cf. (2.10)):

$$\widetilde{K}^{(1\times j)}(m_1, m_2, \ldots, z) = \widetilde{K}(m_1 + j, m_2, \ldots, z). \tag{7.12}$$

Equation (7.11) can be simplified being written in terms of the Jost solution (cf. (5.12))

$$\varphi(m,z)=\widetilde{K}(m,z)p_1(z)^{m_1}p_2(z)^{m_2}p_3(z)^{m_3}, \tag{7.13}$$

that gives by (2.8), (7.11) and (7.12)

$$\varphi^{(i)}(m_1,m_2,m_3,z)=\sum_{j=0}^{m_i}Y_{i,j}(m)\varphi(m_1+j,m_2,m_3,z). \tag{7.14}$$

Representation of the symbol of operator B follows from (7.6) in analogy to (3.1):

$$\widetilde{B}(m,z)=\oint\limits_{|\zeta|=1}\frac{d\zeta}{2\pi i\zeta}\zeta^{m_1}\left(\frac{p_2(\zeta z)}{p_2(z)}\right)^{m_2}\left(\frac{p_3(\zeta z)}{p_3(z)}\right)^{m_3}b(\zeta,z),$$

where $b(\zeta,z)$ is some function. In order to prevent growth of the symbol with respect to m_2 or m_3, we impose condition $|p_i(\zeta z)|=|p_i(z)|$. Moreover, for simplicity we take that polynomials $p_i(z)$ has real coefficients and $b(\zeta,z)=\delta_c(\zeta z/\overline{z})f(z)$, where $\delta_c(\zeta)$ is δ-function on the unit contour and $f(z)$ is an arbitrary function of $z\in\mathbb{C}$. Then in analogy to (3.2) we get representation

$$\widetilde{B}(m,z)=\left(\frac{\overline{z}}{z}\right)^{m_1}\left(\frac{p_2(\overline{z})}{p_2(z)}\right)^{m_2}\left(\frac{p_3(\overline{z})}{p_3(z)}\right)^{m_3}f(z). \tag{7.15}$$

By assumption of unique solvability of problem (3.4) we derive that evolution equations (7.9) (or (7.11)) are compatible:

$$K^{(i,j)}=K^{(j,i)} \tag{7.16}$$

for any i and j. This compatibility enables to derive discrete version of the Zakharov–Shabat system. Indeed, thanks to (2.18) and (7.9)

$$K^{(i,j)}p_ip_j=P_i^{(j)}K^{(j)}p_j=P_i^{(j)}P_jK,\quad i,j=1,2,3. \tag{7.17}$$

Taking that polynomials p_i and p_j with constant coefficients commute into account (see (2.9)), we get that the l.h.s. is symmetric with respect to i and j thanks to (7.16). Then the r.h.s. gives

$$P_i^{(j)}P_j=P_j^{(i)}P_i.\quad 1\leq i,j\leq 3, \tag{7.18}$$

Vice verse, (7.16) follows from (7.18). Discrete version (7.18) of the Zakharov–Shabat system enables derivation of evolution equations on coefficient functions of polynomials P_i.

7.3 An example of the higher Hirota difference equation.

Here we consider an example of the higher equation closest to HDE. Let dynamics of operator B in (7.6) be given by means of polynomials

$$p_1(T)=T,\qquad p_2(T)=T+a_{12},\qquad p_3(T)=(T+a_1)^2-a_3^2, \tag{7.19}$$

where (7.8) was taken into account and where a_1, a_2 and a_3^2 are real constants, $a_{12} = a_1 - a_2 \neq 0$, $a_3 \neq 0, \pm a_2$. Let us denote the first difference of operators as $\Delta_i B = B^{(i)} - B$. Then operator B obeys difference equation

$$\begin{aligned} &\big[(\Delta_1 a_1 - \Delta_2 a_2)^2 - a_3^2(\Delta_1 - \Delta_2)^2\big]\Delta_3 B = \\ &\qquad = a_{12}\Delta_1\Delta_2\big(a_{12}\Delta_1\Delta_2 + 2\Delta_1 a_1 - 2\Delta_2 a_2\big)B, \end{aligned} \tag{7.20}$$

that follows from a corresponding commutator identity. It also can be checked directly since here (7.15) takes the form

$$\widetilde{B}(m,z) = \left(\frac{\overline{z}}{z}\right)^{m_1}\left(\frac{\overline{z}+a_{12}}{z+a_{12}}\right)^{m_2}\left(\frac{(\overline{z}+a_1)^2 - a_3^2}{(z+a_1)^2 - a_3^2}\right)^{m_3} f(z). \tag{7.21}$$

Problem 14 *Proof that relations (7.19) give equation (7.20).*

8 Lecture.

8.1 An example of the higher Hirota difference equation (continuation).

The dressing operator K is defined as always by (3.4), so that by (7.10):

$$P_2(T) = T + Y_{20}, \tag{8.1}$$
$$P_3(T) = T^2 + Y_{31}T + Y_{30}, \tag{8.2}$$

where symbols of operators Y_{ij} are independent of z. Then by (7.11) the Lax pair is given in the form

$$K^{(2)}(T + a_{12}) = K^{(1)}T + Y_{20}K, \tag{8.3}$$
$$K^{(3)}[(T + a_1)^2 - a_3^2] = K^{(1,1)}T^2 + Y_{31}K^{(1)}T + Y_{30}K, \tag{8.4}$$

where coefficients obey

$$\begin{gathered} Y_{31}^{(1)} + Y_{20}^{(3)} = Y_{20}^{(1,1)} + Y_{31}^{(2)}, \\ Y_{30}^{(1)} + Y_{20}^{(3)}Y_{31} = Y_{30}^{(2)} + Y_{31}^{(2)}Y_{20}^{(1)}, \\ Y_{20}^{(3)}Y_{30} = Y_{30}^{(2)}Y_{20}, \end{gathered} \tag{8.5}$$

due to (7.18) and (8.1), (8.2).

Taking symmetry of this reduction with respect to variables m_1 and m_2 into account it is reasonable to rewrite (8.4) in the explicitly symmetric form by means of (8.3). Thus we get

$$K^{(2)}(A - a_2) = K^{(1)}(A - a_1) + Y_{20}K, \tag{8.6}$$
$$\begin{aligned} K^{(3)}[A^2 - a_3^2] &= K^{(1,2)}(A - a_1)(A - a_2) + \\ &+ X_{31}\big(K^{(1)}(A - a_1) + K^{(2)}(A - a_2)\big) + X_{30}K, \end{aligned} \tag{8.7}$$

where again for the sake of symmetry we used (2.16) and where new coefficients equal

$$X_{31} = \frac{1}{2}\big(Y_{31} - Y_{20}^{(1)}\big), \qquad X_{30} = Y_{30} + X_{31}Y_{20}. \tag{8.8}$$

In these terms relations (8.5) also take symmetric form

$$Y_{20}^{(3)} = Y_{20}^{(1,2)} + 2X_{31}^{(2)} - 2X_{31}^{(1)}, \tag{8.9}$$
$$2Y_{20}^{(3)}X_{31} = X_{30}^{(2)} - X_{30}^{(1)} + X_{31}^{(2)}Y_{20}^{(2)} + X_{31}^{(1)}Y_{20}^{(1)}, \tag{8.10}$$
$$2Y_{20}^{(3)}X_{30} = \big[X_{30}^{(2)} + X_{30}^{(1)} + X_{31}^{(2)}Y_{20}^{(2)} - X_{31}^{(1)}Y_{20}^{(1)}\big]Y_{20}. \tag{8.11}$$

Coefficients Y_{ij} (or X_{ij}) must be defined by substitution of asymptotic expansion in (8.3) and (8.4), or (8.6), (8.7). In order to preserve above mentioned symmetry, we use here the latter two equations and taking (2.16) into account we write the expansion in the form

$$K = I + uA^{-1} + wA^{-2} + \cdots, \tag{8.12}$$

where symbols of operators u and w depend on variables m only. We omit here details of computations and in order to present their results introduce functions

$$\begin{aligned} v(m) &= u(m) - m_1 a_1 - m_2 a_2, \qquad (8.13)\\ f(m) &= w(m) - (m_1 a_1 + m_2 a_2) u(m) + \\ &+ \frac{1}{2}(m_1 a_1 + m_2 a_2)^2 - \frac{m_1 a_1^2}{2} - \frac{m_2 a_2^2}{2} - m_3 a_3^2. \qquad (8.14) \end{aligned}$$

Then inserting (8.12) in (8.6) and (8.7) we get

$$Y_{20} = v^{(2)} - v^{(1)}, \qquad (8.15a)$$

$$f^{(2)} - f^{(1)} = Y_{20} v, \qquad (8.15b)$$

$$X_{31} = \frac{1}{2}(v^{(3)} - v^{(1,2)}), \qquad (8.15c)$$

$$X_{30} = f^{(3)} - f^{(1,2)} - X_{31}(v^{(1)} + v^{(2)}). \qquad (8.15d)$$

Problem 15 *Prove (8.5) due to (8.3) and (8.4).*

Problem 16 *Prove (8.8)–(8.11).*

9 Lecture.

9.1 An example of the higher Hirota difference equation (continuation-2).

Thus three functions Y_{20}, X_{30} and X_{31} are given in terms of two functions v and f and must obey three equations (8.9)–(8.11). As we mentioned above this system is compatible. In particular, it is easy to check that (8.9) and (8.10) become identities due to (8.15a)–(8.15c), and (8.15d) reduces to

$$\begin{aligned} 2v^{(2,3)}\bigl[f^{(3)} - f^{(1,2)} - v^{(3)}v^{(2)} + v^{(2)}v^{(1,2)}\bigr] - \\ -2v^{(1,3)}\bigl[f^{(3)} - f^{(1,2)} - v^{(3)}v^{(1)} + v^{(1)}v^{(1,2)}\bigr] = \\ = (v^{(2)} - v^{(1)})\bigl[(f^{(2)} - v^{(2)}v)^{(3)} + (f^{(1)} - v^{(1)}v)^{(3)} - \\ -(f^{(2)} - v^{(2)}v)^{(1,2)} - (f^{(1)} - v^{(1)}v)^{(1,2)}\bigr], \end{aligned} \tag{9.1}$$

that gives one equation on two functions. These functions are not independent, as thanks to (8.15a) and (8.15b)

$$f^{(2)} - v^{(2)}v = f^{(1)} - v^{(1)}v. \tag{9.2}$$

Equations (9.1) and (9.2) are equations of the integrable system, that give an example of the higher HDE. This system follows as condition of compatibility of the Lax pair (8.6), (8.7) that in terms of the Jost solution (cf. (7.13)),

$$\varphi(m,k) = \widetilde{K}(m,z)z^{m_1}(z+a_{12})^{m_2}[(z+a_1)^2 - a_3^2]^{m_3}, \tag{9.3}$$

reads as

$$\varphi^{(2)} = \varphi^{(1)} + (v^{(2)} - v^{(1)})\varphi, \tag{9.4}$$

$$\begin{aligned} \varphi^{(3)} = \varphi^{(1,2)} + (v^{(3)} - v^{(1,2)})\frac{\varphi^{(1)} + \varphi^{(2)}}{2} + \\ + \bigl[f^{(3)} - f^{(1,2)} - \frac{1}{2}(v^{(1)} + v^{(2)})(v^{(3)} - v^{(1,2)})\bigr]\varphi, \end{aligned} \tag{9.5}$$

where (8.15) was used. Omitting details we mention that thanks to (9.4) equation (9.5) can be written in the form

$$\begin{aligned} \varphi^{(3)} = \varphi^{(1,1)} + (v^{(3)} - v^{(1,1)})\varphi^{(1)} + \\ + \bigl[f^{(3)} - f^{(1,2)} - v^{(1)}(v^{(3)} - v^{(1,2)})\bigr]\varphi, \end{aligned} \tag{9.6}$$

that together with (9.4) gives the equivalent Lax pair.

We considered a method of derivation of nonlinear (difference) integrable equations and their Lax pairs. Our construction was not free of assumptions, first of all the assumption on unique solvability of the $\bar{\partial}$-problem (3.4) and assumption of existence of the asymptotic expansions (3.12). These assumptions were extremely essential for our derivation. On the other side, when Lax pairs are derived check of compatibility of its equations is purely algebraic operation, that needs no any assumptions and lead to integrable nonlinear equation. Say, higher HDE, i.e., system (9.1), (9.2) is condition of compatibility of (9.4), (9.5), that can be checked directly.

In a general situation considered in (7.6) existence of the corresponding commutator identity is equivalent to existence of a polynomial $Q(x_1, x_2, x_3)$, such that

$$Q(\mathrm{Ad}_1, \mathrm{Ad}_2, \mathrm{Ad}_3) = 0, \tag{9.7}$$

where we denoted adjoint action of operator T on the associative algebra discussed in Introduction as

$$\mathrm{Ad}_i B = p_i(T) B p_i(T)^{-1}, \quad i = 1, 2, 3. \tag{9.8}$$

Here B is an arbitrary element of this algebra, but if we switch on its dependence on variables m_i by means of (7.6), $B^{(i)} = \mathrm{Ad}_i B$, we get by (9.7) closed linear equation on $B(m_1, m_2, m_3)$, cf. (7.20). This argumentation and construction presented in this article show that it is natural to suppose that the only linear difference equations in $(2+1)$ dimensions that can be lifted to nonlinear integrable ones are those that can be presented in the form of commutator identities. Notice, that in this discussion relation (7.8) was not used.

9.2 (1+1)-dimensional reductions of the HDE

We demonstrate that approach based on commutator relations leads to integrable equations in $(2+1)$ dimensions. In order to get $(1+1)$-dimensional integrable systems one has to perform reductions. Following idea of our approach, we start with construction of reductions of linear equation (1.8) on B and then apply dressing procedure to get nonlinear integrable systems. Thus in this case dimensional reduction is understood as a relation between values of operator B given by some shifts of independent variables m_i. Such relation must be compatible with (3.2) and must preserve dependence of B on two independent variables.

Thanks to (3.2) it is easy to see that any such reduction leads to an equation on the spectral parameter z: it had to belong to a some curve on $\mathbb{C}$. This is possible only if function $f(z)$ in (3.2), and then $\widetilde{B}(m, z)$ itself, have support on this curve, that here for simplicity we consider as proportionality to a corresponding δ-function. But then (3.5) means that symbol $\widetilde{K}(m, z)$ is analytic function outside this curve, so the inverse problem (3.4) must be substituted by the standard Riemann–Hilbert problem.

9.3 Reduction $B^{(3)} = B$.

The system (9.1), (9.2) admits (1+1)-dimensional reductions. Indeed, thanks to (7.21) this reduction means that symbol $\widetilde{B}(m, z)$ is different from zero if $(\overline{z} + a_1)^2 = (z + a_1)^2$, i.e., $z_{\mathrm{Re}} = -a_1$, so that function $f(z)$ in (7.21) must be proportional to $\delta(z_{\mathrm{Re}} + a_1)$:

$$\widetilde{B}(m, z) = \left(\frac{a_1 + iz_{\mathrm{Im}}}{a_1 - iz_{\mathrm{Im}}}\right)^{m_1} \left(\frac{a_1 + iz_{\mathrm{Im}}}{a_1 - iz_{\mathrm{Im}}}\right)^{m_2} \delta(z_{\mathrm{Re}} + a_1) r(z_{\mathrm{Im}}). \tag{9.9}$$

Then the inverse problem (3.4) shows that the dressing operator is not only independent of m_3, but its symbol $\widetilde{K}(m_1, m_2, z)$ is analytic function of z when $z_{\mathrm{Re}} \neq -a_1$.

In order to get reduced Lax pair and nonlinear equation, notice that coefficients of asymptotic expansion (8.12) are independent of m_3, i.e., $u(m) = u(m_1, m_2)$, $w(m) = w(m_1, m_2)$. Correspondingly, by (8.13) and (8.14) $v(m) = v(m_1, m_2)$,

$f(m) = g(m_1, m_2) - a_3^3 m_3$, where $g(m_1, m_2) = f(m_1, m_2, 0)$. Inserting these relations in (9.1) and (9.2) we get nonlinear integrable system

$$
\begin{aligned}
&(g^{(2)} - v^{(2)}v)^{(1,2)} + (g^{(1)} - v^{(1)}v)^{(1,2)} - \\
&- (g^{(2)} - v^{(2)}v)^{(1)} - (g^{(1)} - v^{(1)}v)^{(2)} - \\
&- (g^{(2)} - v^{(2)}v) - (g^{(1)} - v^{(1)}v) + 2g - 2v(v^{(1)} + v^{(2)}) = 0, \qquad (9.10)\\
&g^{(2)} - v^{(2)}v = g^{(1)} - v^{(1)}v. \qquad (9.11)
\end{aligned}
$$

Taking that now symbol $\widetilde{K}(m,z)$ is independent of m_3 into account we define the Jost solution by means of equality

$$\psi(m_1, m_2, z + a_1) = \widetilde{K}(m,z) z^{m_1} (z + a_{12})^{m_2} \equiv \frac{\varphi(m,z)}{[(z + a_1)^2 - a_3^2]^{m_3}}, \qquad (9.12)$$

see (9.3). Thus we get from (9.5), (9.3) the reduced Lax pair:

$$
\begin{aligned}
&\psi^{(1,1)} + (v - v^{(1,1)})\psi^{(1)} + \big[g - g^{(1,2)} - v^{(1)}(v - v^{(1,2)})\big]\psi = \lambda^2 \psi, \qquad (9.13)\\
&\psi^{(2)} = \psi^{(1)} + (v^{(2)} - v^{(1)})\psi, \qquad (9.14)
\end{aligned}
$$

where the spectral parameter $\lambda = z + a_1$, see (10.7), was used.

10 Lecture.

10.1 Reduction $B^{(1,3)} = B$.

We start with condition $B^{(1,3)} = B$. In terms of symbols this reduction gives:

$$\widetilde{B}(m_1, m_2, m_3, z) = \widetilde{B}(m_1 - m_3, m_2, 0, z), \tag{10.1}$$

that due to (3.2) is possible only if $z_{\mathrm{Re}} = -a_{13}/2$ (we omit the trivial case $z_{\mathrm{Im}} = 0$). Setting here for simplicity

$$a_3 = -a_1, \tag{10.2}$$

we see that the above reduction require proportionality of a symbol B to δ-function $\delta(z_{\mathrm{Re}} + a_1)$, so that by (3.2)

$$\begin{aligned}\widetilde{B}(m_1,&m_2, 0, z) = \\ &= \left(\frac{a_1 + iz_{\mathrm{Im}}}{a_1 - iz_{\mathrm{Im}}}\right)^{m_1} \left(\frac{a_2 + iz_{\mathrm{Im}}}{a_2 - iz_{\mathrm{Im}}}\right)^{m_2} b(z_{\mathrm{Im}})\delta(z_{\mathrm{Re}} + a_1),\end{aligned} \tag{10.3}$$

where $b(z_{\mathrm{Im}})$ is an arbitrary function of its argument (Scattering Data). Operator B with this symbol obviously obeys equation

$$a_{12}(B^{(1,2)} - B) + (a_1 + a_2)(B^{(1)} - B^{(2)}) = 0, \tag{10.4}$$

while the corresponding reduction of the original Eq. (1.8) gives

$$\begin{aligned}a_{12}(B^{(1,2)} - B) + (a_1 + a_2)(B^{(1)} - B^{(2)}) = \\ = \left[a_{12}(B^{(1,2)} - B) + (a_1 + a_2)(B^{(1)} - B^{(2)})\right]^{-1}.\end{aligned}$$

Both sides of this equation are independent of m_1, so (10.4) appears as result of its summation.

Let us emphasize that because of (3.4) symbol $\widetilde{K}(m, z)$ of the dressing operator is analytic function of $z \in \mathbb{C}$ in half planes $z_{\mathrm{Re}} \gtrless -a_1$.

Thanks to (2.3), (3.4) and (10.1) we get that also $K^{(1,2)} = K$, i.e.,

$$\widetilde{K}(m_1, m_2, m_3, z) = \widetilde{K}(m_1 - m_2, 0, m_3, z), \quad z \in \mathbb{C}.$$

Thus equation (5.1a) of the Lax pair is unchanged, while for (5.1b) we have

$$\begin{aligned}(z + 2a_1)\widetilde{K}(m, z) = z\widetilde{K}^{(1,1)}(m, z) + \\ + (v(m) - v^{(1,1)}(m))\widetilde{K}^{(1)}(m, z).\end{aligned} \tag{10.5}$$

Thanks to (3.12) also $u(m_1, m_2, m_3, z) = u(m_1 - m_2, 0, m_3, z)$. Relation (10.2) gives the same dependence of $v(m)$ on $m_1 - m_3$ and m_2. Because of this specific dependence on m we have to modify definition (5.12) of the Jost solution:

$$\psi(m_1 - m_3, m_3, k) = \widetilde{K}(m, z)z^{m_1 - m_3}(z + a_{13})^{m_2}, \tag{10.6}$$

where we denoted

$$k = z + a_1, \tag{10.7}$$

that in fact is the symbol of operator A, see (2.17). Thus setting now $m_3 = 0$ we write:

$$v(m) \equiv v(m_1, m_2) = v(m_1, m_2, 0) - a_1 m_1 - a_2 m_2, \tag{10.8}$$

so that equation (5.22c) is left unchanged, $\psi^{(2)} = \psi^{(1)} + (v^{(2)} - v^{(1)})\psi$, and (10.5) and the Lax pair itself takes the form

$$\psi^{(1,1)} = (v^{(1,1)} - v)\psi^{(1)} + (k - a_1^2)\psi, \tag{10.9}$$

$$\psi^{(1,2)} = (v^{(1,2)} - v)\psi^{(1)} + (k - a_1^2)\psi, \tag{10.10}$$

where $\psi^{(1)}$ in the second equality was substituted by (10.9).

Equation of compatibility of this pair can be derived either directly, or as reduction of (5.6) and reads as

$$\big((v^{(1,2)} - v)(v^{(2)} - v^{(1)})\big)^{(1)} = (v^{(1,2)} - v)(v^{(2)} - v^{(1)}). \tag{10.11}$$

Thus multiplication operator in the r.h.s. (or l.h.s.) has symbol independent of m_1. Taking (10.8) and decay of function $u(m)$ at $m_1 \to \infty$ into account we have that

$$v^{(2)}(m) - v^{(\pm 1)}(m) \to \pm a_{12} \tag{10.12}$$

in this limit. Thus (10.11) gives

$$(v^{(1,2)} - v)(v^{(2)} - v^{(1)}) = a_2^2 - a_1^2. \tag{10.13}$$

Eq. (10.13) is known as the discrete potential KdV equation. It was derived by F. Nijhoff et al (1984) and was discussed in detail in literature together with its non-Abelian generalizations. Here we provide derivation of this equation as an example of dimensional reduction in the framework of our approach.

11 Lecture.

11.1 Reduction $B^{(3)} = B^{(1,2)}$.

This $1+1$-dimensional reduction of HDE preserves its specific property: symmetry with respect to independent variables. Let for simplicity

$$a_3 = a_1 + a_2, \tag{11.1}$$

then thanks to (3.2) this reduction means that z must obey condition $z\overline{z} - (z + \overline{z})a_2 - a_2 a_{12} = 0$, i.e.,

$$|z - a_2|^2 = a_1 a_2. \tag{11.2}$$

In other words, symbol $\widetilde{B}(m, z)$ must be proportional to δ-function on the circle (11.2), so here $a_1 a_2 > 0$, and symbol of the dressing operator is analytic inside and outside of the circle (11.2). Notice also that thanks to this reduction symbols of operators B and K obey conditions

$$\begin{aligned} \widetilde{B}(m_1, m_2, m_3, z) &= \widetilde{B}(m_1 + m_3, m_2 + m_3, 0, z), \\ \widetilde{K}(m_1, m_2, m_3, z) &= \widetilde{K}(m_1 + m_3, m_2 + m_3, 0, z), \end{aligned} \tag{11.3}$$

so by (3.12) the same is dependence of $u(m)$ on variables m_i, and due to (5.4) and (11.1) the same is valid for function v:

$$v(m_1, m_2, m_3) = v(m_1 + m_3, m_2 + m_3, 0). \tag{11.4}$$

We see that equation (3.15) is unchanged under this reduction and (3.16) reduces to

$$(z - a_2)\widetilde{K}^{(1,2)}(m, z) = z\widetilde{K}^{(1)}(m, z) + (v^{(1,2)}(m) - v^{(1)}(m))\widetilde{K}(m, z),$$

where now $m_3 = 0$. We introduce the Jost solution (cf. (5.12)) by means of relation

$$\psi(m_1, m_2, k) = \widetilde{K}(m_1, m_2, 0, z) z^{m_1} (z + a_{12})^{m_2}, \tag{11.5}$$

where

$$k = \frac{2}{a_{12}} \left(\frac{a_1}{z + a_{12}} - \frac{a_2}{z} \right) \tag{11.6}$$

is the spectral parameter. Finally, for the Lax pair we get

$$\psi^{(2)} - \psi^{(1)} = (v^{(2)} - v^{(1)})\psi, \tag{11.7}$$

$$k\psi^{(1,2)} = \psi^{(1)} + \psi^{(2)} + (2v^{(1,2)} - v^{(1)} - v^{(2)})\psi, \tag{11.8}$$

and corresponding nonlinear integrable equation reads as

$$\left(v^{(1,2)}(v^{(2)} - v) + vv^{(2)}\right)^{(1)} = \left(v^{(1,2)}(v^{(1)} - v) + vv^{(1)}\right)^{(2)}, \tag{11.9}$$

that is equation of a 1+1-dimensional chain with discrete time evolutions, symmetric with respect to both independent variables.

11.2 Reduction $B^{(3)} = B^{(-1,-2)}$

This is another reduction that also leads to the symmetric chain. Repeating the same argumentation as above, we get that

$$\widetilde{B}(m,z) = \widetilde{B}(m_1 - m_3, m_2 - m_3, 0, z), \tag{11.10}$$

that means the symbol $\widetilde{B}(m,z)$ is proportional to δ-function on the hyperbola given by the equation $3(z_{\rm Re} + a_1)^2 - z_{\rm Im}^2 = (a_1 + a_2)^2 - a_1 a_2$. Omitting other details we present here the corresponding nonlinear equation only:

$$v^{(1,2)}(v^{(1)} - v^{(2)}) - v^{(-1,-2)}(v^{(-1)} - v^{(-2)}) = v^{(1)}v^{(-2)} - v^{(-1)}v^{(2)}. \tag{11.11}$$

11.3 Soliton solutions

Soliton solutions for the Hirota difference equation are well known in the literature. Let we have two numbers $N_a, N_b \geq 1$, and set of $N = N_a + N_b$ real parameters $\varkappa_n$ that we can choose to be ordered: $\varkappa_1 < \varkappa_2 < \cdots < \varkappa_N$. Let $\chi(m,k)$ be a meromorphic function of k that has poles at points $k = \varkappa_{n_1}, \ldots, \varkappa_{n_{N_b}}$, where $\{n_1, \ldots, n_{N_b}\}$ is a subset of $\{1, \ldots, N\}$. Let us rescale the Jost solution,

$$\chi(m,k) \to \chi(m,k) \prod_{j=1}^{N_b} (k - \varkappa_{n_j}), \tag{11.12}$$

so that the new one is a polynomial of order k^{N_b} with the unity coefficient at higher power. Thanks to (6.10) we have

$$\frac{\chi(m,k)}{k^{N_b}} = 1 + \frac{1}{k}\Big(u(m) - \sum_{j=1}^{N_b} \varkappa_{n_j}\Big) + \ldots. \tag{11.13}$$

Thus

$$\chi(m,k) = k^{N_b} + \sum_{l=1}^{N_b} k^{l-1} X(l,m), \tag{11.14}$$

where $X(l,m)$ are some coefficients to be determined. For this aim we use (5.12) with function $\chi(x,k)$ substituted from the latter equality. Then on values of the Jost solution at points $k = \varkappa_n$ impose N_b conditions:

$$\big(\varphi(m,\varkappa_1), \ldots, \varphi(m,\varkappa_N)\big) D = 0. \tag{11.15}$$

where D is matrix of the size $N \times N_b$ with at least two nonzero maximal minors. This condition gives linear system of equations to determine uniquely $X(l,m)$. To describe solution of this system we use here the following notation: let V be incomplete Vandermond matrix of the size $(N_b + 1) \times N$,

$$V = \begin{pmatrix} 1 & \ldots & 1 \\ \vdots & & \vdots \\ \varkappa_1^{N_b} & \ldots & \varkappa_N^{N_b} \end{pmatrix}, \tag{11.16}$$

and $V(l)$ is matrix V with removed l-th row (i.e., matrix of the size $N_b \times N$). We also need two diagonal $(N \times N)$-matrices (see (5.13)):

$$E(m) = \operatorname{diag}\{E(m, \varkappa_1), \ldots, E(m, \varkappa_N)\} \tag{11.17}$$
$$k - \varkappa = \operatorname{diag}\{k - \varkappa_1, \ldots, k - \varkappa_N\}. \tag{11.18}$$

Let also $Y(l, m)$ denote determinant of $(N_b \times N_b)$-matrix

$$Y(l, m) = (-1)^{N_b+1-l} \det(V(l)E(m)D). \tag{11.19}$$

Then it is easy to see that

$$X(l, m) = -\frac{Y(l, m)}{Y(m)}, \qquad Y(m) = Y(N_b + 1, m) \tag{11.20}$$

Now using (11.14) we readily get that

$$\chi(m, k) = \frac{Z(m, k)}{Y(m)}, \tag{11.21}$$

where

$$Z(m, k) = \det\bigl(V(N_b + 1)(k - \varkappa)E(m)D\bigr), \tag{11.22}$$

and notation (11.18) was used. Thanks to definition (11.13) we get

$$u(m) = \sum_{j=1}^{N_b} \varkappa_{n_j} - \frac{Y(N_b, m)}{Y(m)}. \tag{11.23}$$

As an example of this generic construction we present one-soliton solution:

$$u(m) = \frac{\varkappa_2 - \varkappa_1}{1 + cf(m)}, \tag{11.24}$$

where c a real constant and

$$f(m) = \frac{E(m, \varkappa_2)}{E(m, \varkappa_1)} \equiv \left(\frac{\varkappa_2 - a_1}{\varkappa_1 - a_1}\right)^{m_1} \left(\frac{\varkappa_2 - a_2}{\varkappa_1 - a_2}\right)^{m_2} \left(\frac{\varkappa_2 - a_3}{\varkappa_1 - a_3}\right)^{m_3}. \tag{11.25}$$

Already this example shows that the consideration here was formal in the sense that denominator in (11.23) (i.e., τ-function) can take zero values, so solution can be singular for some values of m. Strictly speaking soliton solutions do not fit in the class of solutions for which the IST was developed in the previous sections. Soliton solutions interpolate between different constants on the m-infinity and one has to develop version of the IST that enables consideration of such solutions. Another property, specific for the soliton solutions of the Hirota difference equation is existence of a resonant solitons, i.e., solitons where parameters $\varkappa_i$ coincide with some of parameters a_1, a_2, a_3. One soliton solution (11.24) shows that in the corresponding limit solution exists, but its properties can be rather strange.