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Parametrizations of Canonical Bases and Totally Positive Matrices

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1. INTRODUCTION AND MAIN RESULTS

Let N be the maximal unipotent subgroup of a semisimple group of simply-laced type; let $m = \dim N$. Recently, Lusztig [29, 30] discovered a remarkable parallelism between

(1) labellings (by m-tuples of nonnegative integers) of the canonical basis B of the quantum group corresponding to N, and

(2) parametrizations of the variety $N_{>0}$ of totally positive elements in N by m-tuples of positive real numbers.

In each case, there is a natural family of parametrizations, one for each reduced word of the element of maximal length in the Weyl group.

In this paper, the following two problems are solved for the type A.

Problem I. For any pair of reduced words, find an explicit formula for the transition map that relates corresponding parametrizations of B or $N_{>0}$.

Problem II. Each parametrization in (2) is a restriction of a birational isomorphism $\mathbb{C}^m \to N$. For any reduced word, find an explicit formula for the inverse map $N \to \mathbb{C}^m$.

Our results have the following applications:

(i) a new proof and generalization of piecewise-linear minimization formulas for quivers of type A given in [23, 24], and

(ii) a new family of criteria for total positivity generalizing and refining the classical Fekete criterion [13].

In order to treat Problems I and II simultaneously, we develop a general framework for studying Lusztig's transition maps, in which the role of scalars is played by an arbitrary zerosumfree semifield. Our methods rely on an interpretation of reduced words via pseudo-line arrangements. The solutions of the above problems are then obtained by means of a special substitution that we call the *Chamber Ansatz*.

We now begin a systematic account of our main results. Since we shall only treat the A_r type, N will be the group of unipotent upper triangular matrices of order $r+1$; its dimension is equal to $m = \binom{r+1}{2}$. Then Problem II can be reformulated as the following problem in linear algebra.

1.1. *Problem.* For a generic unipotent upper-triangular matrix x of order $r+1$, find explicit formulas for the factorizations of x into the minimal number of elementary Jacobi matrices.

To be more precise, by an elementary Jacobi matrix we mean a matrix of the form

$$1 + te_i = \begin{pmatrix} 1 & & & & & \\ & \ddots & & & & \\ & & 1 & t & & \\ & & & 1 & & \\ & & & & \ddots & \\ & & & & & 1 \end{pmatrix}$$

where e_i is an $(i, i+1)$ matrix unit. The minimal possible number of such matrices needed for factoring a generic $x \in N$ is easily seen to be $\dim N = m = \binom{r+1}{2}$. Then, for a sequence $\mathbf{h} = (h_1, ..., h_m)$ of indices, we consider factorizations

$$x = (1 + t_1 e_{h_1}) \cdots (1 + t_m e_{h_m}). \tag{1.1}$$

It is not hard to show that such a factorization of a generic matrix $x \in N$ exists and is unique if and only if $\mathbf{h}$ is a reduced word of the element w_0 of maximal length in the symmetric group S_{r+1}, i.e., $w_0 = s_{h_1} \cdots s_{h_m}$ where the $s_i = (i, i+1)$ are adjacent transpositions.

Moreover, for a given reduced word $\mathbf{h}$, each coefficient t_k in (1.1) is a rational function in the matrix entries x_{ij} that we denote by $t_k^{\mathbf{h}}$ to emphasize the role of $\mathbf{h}$. Thus the map $t \mapsto x = x^{\mathbf{h}}(t)$ defined by (1.1) is a birational isomorphism $\mathbb{C}^m \to N$. In this notation, Problem 1.1 asks for explicit computation of the inverse rational map $t^{\mathbf{h}}: N \to \mathbb{C}^m$.

1.2. Example. Let $r = 2$. In this case, there are two reduced words for w_0:

$$\mathbf{h} = 121, \qquad \mathbf{h}' = 212.$$

Denote $t_k^{\mathbf{h}}(x) = t_k$, $t_k^{\mathbf{h}'}(x) = t'_k$ for $k = 1, 2, 3$. Thus we are interested in the factorizations

$$x = (1 + t_1 e_1)(1 + t_2 e_2)(1 + t_3 e_1) = (1 + t'_1 e_2)(1 + t'_2 e_1)(1 + t'_3 e_2) \tag{1.2}$$

of a generic unitriangular 3×3-matrix x. Multiplying the matrices in (1.2), we obtain

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t_1 + t_3 & t_1 t_2 \\ 0 & 1 & t_2 \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & t'_2 & t'_2 t'_3 \\ 0 & 1 & t'_1 + t'_3 \\ 0 & 0 & 1 \end{pmatrix}. \tag{1.3}$$

Solving these equations for the t_k and t'_k, we find

$$t_1 = \frac{x_{13}}{x_{23}}, \qquad t_2 = x_{23}, \qquad t_3 = \frac{x_{12}x_{23} - x_{13}}{x_{23}} \tag{1.4}$$

and

$$t'_1 = \frac{x_{12}x_{23} - x_{13}}{x_{12}}, \qquad t'_2 = x_{12}, \qquad t'_3 = \frac{x_{13}}{x_{12}}. \tag{1.5}$$

Our general solution of Problem 1.1 will require two ingredients. The first is the following combinatorial construction. Let $\mathbf{h} = (h_1, ..., h_m)$ be a reduced word for w_0. For each entry h_k of $\mathbf{h}$, define the set $L \subset \{1, ..., r+1\}$ and two integers i and j by

$$\begin{aligned} L &= s_{h_m} \cdots s_{h_{k+1}}(\{1, ..., h_k - 1\}) \\ i &= s_{h_m} \cdots s_{h_{k+1}}(h_k) \\ j &= s_{h_m} \cdots s_{h_{k+1}}(h_k + 1) \end{aligned} \tag{1.6}$$

For example, for $\mathbf{h} = 213231$ and $k = 3$, we obtain $(i, j) = (1, 3)$ and $L = \{2, 4\}$.

As k ranges from 1 to m, the pair (i, j) given by formulas (1.6) ranges over the m pairs of integers (i, j) satisfying $1 \leqslant i < j \leqslant r+1$, each of these pairs appearing exactly once. This is a special case of a result in [5, VI.1.6], if one identifies such pairs (i, j) with the positive roots of type A_r.

Another ingredient of our solution of Problem 1.1 is a certain birational transformation of the group N which is defined as follows. For a matrix g, let $[g]_+$ denote the last factor u in the Gaussian LDU-decomposition $g = v^T \cdot d \cdot u$ where $u \in N$, $v \in N$, and d is diagonal. (Such decomposition exists and is unique for a sufficiently generic $g \in GL_{r+1}$.) The matrix entries of $[g]_+$ are rational functions of g; explicit formulas can be written in terms of the minors of g. The following linear-algebraic fact is not hard to prove.

1.3. Lemma. *The map $y \mapsto x = [w_0 y^T]_+$ is a birational automorphism of N. (Here w_0 is identified with the corresponding permutation matrix.) The inverse birational automorphism $x \mapsto y$ is given by*

$$y = w_0^{-1} [x w_0^{-1}]_+^T \, w_0.$$

For example, in the case $r=2$, the birational automorphism $x \mapsto y$ is given by

$$x = \begin{pmatrix} 1 & x_{12} & x_{13} \\ 0 & 1 & x_{23} \\ 0 & 0 & 1 \end{pmatrix} \mapsto \begin{pmatrix} 1 & \dfrac{x_{23}}{x_{12}x_{23}-x_{13}} & \dfrac{1}{x_{13}} \\ 0 & 1 & \dfrac{x_{12}}{x_{13}} \\ 0 & 0 & 1 \end{pmatrix} = y.$$

In what follows, $[i, j]$ will stand for $\{a \in \mathbb{Z} : i \leqslant a \leqslant j\}$. For a subset $J \subset [1, r+1]$ and a matrix $g \in GL_{r+1}$, let $\Delta^J(g)$ denote the minor of g that occupies several first rows and whose column set is J.

We are now in a position to present a solution to Problem 1.1 (cf. Theorem 3.1.1).

1.4. THEOREM. *For $x \in N$, define the matrix y as in Lemma* 1.3. *Then the coefficients $t_k = t_k^{\mathbf{h}}$ in the factorization* (1.1) *of x are given by*

$$t_k^{\mathbf{h}} = \frac{\Delta^L(y)\, \Delta^{L \cup \{i, j\}}(y)}{\Delta^{L \cup \{i\}}(y)\, \Delta^{L \cup \{j\}}(y)} \tag{1.7}$$

where L, i, and j are defined by (1.6).

Formula (1.7) is a particular case of what we call the "Chamber Ansatz." This terminology comes from the pseudo-line arrangement that naturally corresponds to $\mathbf{h}$. Instead of defining this arrangement formally (which is done in Section 2.3), we give a self-explanatory example for the reduced word 213231—see Fig. 1. To every chamber of our arrangement (there are

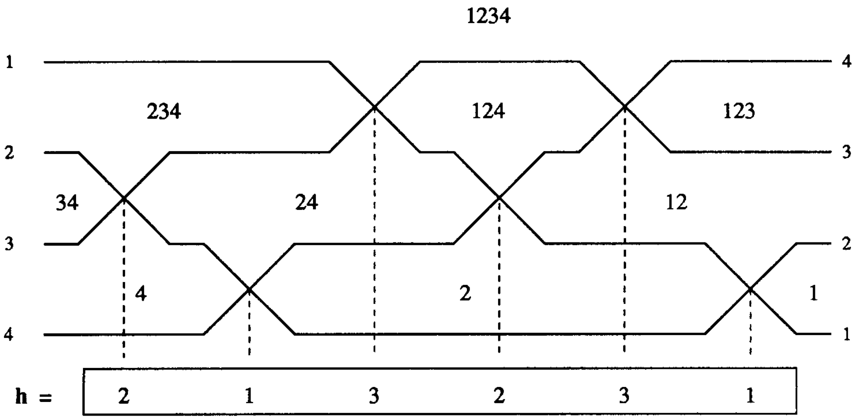


FIG. 1. Pseudo-line arrangement and chamber sets.

$\binom{r+2}{2}+1$ such chambers), we associate the set of labels of the pseudo-lines that pass *below* this chamber. These *chamber sets* are shown in the figure.

In this geometric language, the indices i and j in (1.6) are the labels of the pseudo-lines which intersect at the kth crossing, counting from the left. The four sets L, $L \cup \{i\}$, $L \cup \{j\}$, and $L \cup \{i, j\}$ participating in (1.7) are exactly the chamber sets associated with the four chambers surrounding this crossing; see Fig. 2.

The minors of y that participate in (1.7) are rational functions of $x \in N$. In the next theorem, we express each coefficient t_k as a ratio of products of irreducible polynomials in the matrix entries x_{ij}.

For a matrix $x \in N$, define

$$Z_a(x) = \Delta^{[a+1,\, r+1]}(x), \qquad a = 1, ..., r \tag{1.8}$$

and

$$T_J(x) = \Delta^J(y) \prod_{a \notin J,\, a+1 \in J} Z_a(x), \qquad J \subset [1, r+1]. \tag{1.9}$$

1.5. THEOREM. *The coefficients $t_k = t_k^{\mathbf{h}}$ in the factorization* (1.1) *of a matrix $x \in N$ are given by the following version of the Chamber Ansatz*;

$$t_k^{\mathbf{h}} = Z_i(x)^{\delta_{i+1,j}} \frac{T_L(x)\, T_{L \cup \{i,j\}}(x)}{T_{L \cup \{i\}}(x)\, T_{L \cup \{j\}}(x)}, \tag{1.10}$$

where, as before, L, i, and j are given by (1.6), *and $\delta_{i+1,j}$ is the Kronecker symbol. All factors Z_a and T_J are irreducible polynomials in the matrix entries x_{ij}.*

This theorem is proved in Proposition 3.6.4 (see also (2.9.8)). In Section 3.3 we provide formulas which express the T_J directly as polynomials in the x_{ij}.

The last theorem has the following immediate consequences.

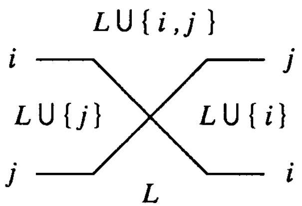


FIG. 2. Chamber sets at the intersection of pseudo-lines i and j.

1.6. Corollary. 1. *Regardless of the value of* r, *every rational function* $t_k^{\mathbf{h}}$ *has at most three irreducible factors in the numerator and at most two in the denominator.*

2. *For a given* $\mathbf{h}$, *there are exactly* $m = \binom{r+1}{2}$ *polynomials in the matrix entries* x_{ij} *that appear as irreducible factors for* $t_1^{\mathbf{h}}, ..., t_m^{\mathbf{h}}$.

3. *The total number of irreducible factors for all* $t_k^{\mathbf{h}}$ (*with both* $\mathbf{h}$ *and* k *varying*) *is equal to* $2^{r+1} - r - 2$.

For example, when $r = 2$, the $2^3 - 2 - 2 = 4$ irreducible factors participating in (1.4)–(1.5) are:

$$Z_1(x) = \Delta^{[2,3]}(x) = x_{12}x_{23} - x_{13}, \qquad Z_2(x) = x_{13},$$

$$T_{\{1,3\}}(x) = x_{12}, \qquad T_2(x) = x_{23}.$$

In accordance with statement 2 above, each of (1.4) and (1.5) involves three of these four polynomials.

Theorems 1.4, 1.5 have applications in the theory of totally positive matrices. Recall that a matrix is *totally positive* if all its minors are positive real numbers. These matrices play an important role in different ares of mathematics, from differential equations to combinatorics (see, e.g., [1, 6]). For a matrix $x \in N$, we modify the definition of total positivity by saying that x is *totally positive with respect to* N if every minor that does not identically vanish on N has a positive value for x. We denote by $N_{>0} \subset N$ the variety of all matrices in N that are totally positive with respect to N.

The following result is presented in [30] as a corollary of the proof of A. Whitney's reduction theorem [35].

1.7. Proposition. *For any reduced word* $\mathbf{h}$ *for* w_0, *the birational isomorphism* $x^{\mathbf{h}}: \mathbb{C}^m \to N$ *defined by* (1.1) *restricts to a bijection* $\mathbb{R}^m_{>0} \to N_{>0}$ *between the set of m-tuples of positive real numbers and the variety of totally positive unipotent upper-triangular matrices.*

Using this proposition and Theorems 1.4 and 1.5, we obtain the following set of criteria for total positivity (cf. Theorem 3.2.1 and Proposition 3.3.3).

1.8. Theorem. *Let* $\mathbf{h}$ *be a reduced word for* w_0. *Then, for a matrix* $x \in N$, *the following conditions are equivalent*:

(1) x *is totally positive* (*with respect to* N);

(2) $\Delta^J(x) > 0$ *for all chamber sets* J *of* $\mathbf{h}$;

(3) $T_J(x) > 0$ *and* $Z_a(x) > 0$ *for all chamber sets* J *of* $\mathbf{h}$ *and all* $a \in [1, r]$.

The total number of non-constant polynomials appearing in each of the conditions (2) and (3) is equal to m. Thus the positivity of all minors of x is equivalent to positivity of certain $m = \binom{r+1}{2}$ irreducible polynomials.

The study of general (not necessarily upper-triangular) totally positive matrices can be reduced to the study of $N_{>0}$, in view of the following well-known result [11]: a square matrix g of order $r+1$ is totally positive if and only if it is (uniquely) represented in the form $g = v^T \cdot d \cdot u$ where $u, v \in N_{>0}$ and d is a diagonal matrix with positive diagonal entries. In other words, a matrix is totally positive if and only if all three factors in its Gaussian (or LDU-) factorization are totally positive. Theorem 1.8 now implies the following total positivity criterion.

1.9. Corollary. *Let* $\mathbf{h}$ *and* $\mathbf{h}'$ *be reduced words for* w_0. *An* $(r+1) \times (r+1)$*-matrix* g *is totally positive if and only if* $\Delta^J(g) > 0$ *and* $\Delta^{J'}(g^T) > 0$ *for all chamber sets* J *of* $\mathbf{h}$ *and all chamber sets* J' *of* $\mathbf{h}'$.

In the special case $\mathbf{h} = \mathbf{h}' = (1, 2, 1, ..., r, r-1, ..., 1)$, we obtain the classical Fekete criterion [13]. Namely, a square matrix g of order $r+1$ is totally positive if and only if the following $(r+1)^2$ minors of g are positive: all minors occupying several initial rows and several consecutive columns, and all minors occupying several initial columns and several consecutive rows.

We now turn to the discussion of Problem I. Let U_+ be the q-deformation of the universal enveloping algebra of the Lie algebra of N, and let B be the canonical basis in U_+ (see [28]). In this paper, we will only study the combinatorial properties of B. The main problem and its solution will be formulated in a form that will not assume any familiarity with quantum groups on the part of the reader.

According to Lusztig [29, 30], there is a total similarity between the parametrizations of $N_{>0}$ given in Proposition 1.7, and certain parametrizations of the canonical basis B by m-tuples of nonnegative integers. For every reduced word $\mathbf{h}$ for w_0, there is a natural bijective parametrization $t^{\mathbf{h}}: B \to \mathbb{Z}_+^m$. (It will soon become clear why we use the same notation $t^{\mathbf{h}}$ for this bijection and the bijection $N_{>0} \to \mathbb{R}_{>0}^m$ discussed above.)

The key role in the combinatorial understanding of the canonical basis B is played by the transition maps $R_{\mathbf{h}}^{\mathbf{h}'}: \mathbb{Z}_+^m \to \mathbb{Z}_+^m$ defined by

$$R_{\mathbf{h}}^{\mathbf{h}'} = t^{\mathbf{h}'} \circ (t^{\mathbf{h}})^{-1} \tag{1.11}$$

for any pair $\mathbf{h}$ and $\mathbf{h}'$ of reduced words of w_0. Lusztig [29] described $R_{\mathbf{h}}^{\mathbf{h}'}$ as a composition of some simple piecewise-linear transformations. To explain his description, let us first recall [5, 22] that any two reduced words of the same element can be obtained from each other by a sequence of elementary moves of the following two types:

2-*move*. Replace two consecutive entries ij by ji provided $|i-j| \geqslant 2$.

3-*move*. Replace three consecutive entries iji by jij if $|i-j|=1$.

If $\mathbf{h}'$ differs from $\mathbf{h}$ by a 2-move applied to the entries h_k and h_{k+1}, then $t' = R_{\mathbf{h}}^{\mathbf{h}'}(t)$ is obtained from $t=(t_1, ..., t_m)$ by switching the components t_k and t_{k+1}. If $\mathbf{h}'$ is obtained from $\mathbf{h}$ by a 3-move applied to the entries h_{k-1}, h_k, and h_{k+1}, then t' is given by

$$\begin{aligned} t'_{k-1} &= t_k + t_{k+1} - \min(t_{k-1}, t_{k+1}), \\ t'_k &= \min(t_{k-1}, t_{k+1}), \\ t'_{k+1} &= t_k + t_{k-1} - \min(t_{k-1}, t_{k+1}). \end{aligned} \tag{1.12}$$

An arbitrary transition map $R_{\mathbf{h}}^{\mathbf{h}'}$ can be computed as a composition of transformations corresponding to elementary moves. In general, as many as $\binom{r+1}{3}$ such steps may be needed, and the computations become very involved (see, e.g., [9]). This leads us to the following more precise version of Problem I.

1.10. *Problem.* For arbitrary $\mathbf{h}$ and $\mathbf{h}'$, find a closed formula for the transition map $R_{\mathbf{h}}^{\mathbf{h}'}$ that does not involve the iteration process.

As observed in [30], Problem 1.10 has a natural linear-algebraic counterpart. For any reduced word, consider the bijection $t^{\mathbf{h}}\colon N_{>0} \to \mathbb{R}^m_{>0}$. These bijections are related to each other via transition maps $R_{\mathbf{h}}^{\mathbf{h}'}\colon \mathbb{R}^m_{>0} \to \mathbb{R}^m_{>0}$ defined by $R_{\mathbf{h}}^{\mathbf{h}'} = t^{\mathbf{h}'} \circ (t^{\mathbf{h}})^{-1}$, in complete analogy with (1.11). To compute transition maps in this setting, we can use the same approach as before, namely, iterate elementary transformations which correspond to 2- and 3-moves:

(2-*move*) if $\mathbf{h}'$ differs from $\mathbf{h}$ by a 2-move that switches the kth and $k+1$st entries, then $t' = R_{\mathbf{h}}^{\mathbf{h}'}(t)$ is obtained from $t \in \mathbb{R}^m_{>0}$ by simply switching the corresponding components:

$$t'_k = t_{k+1}, \qquad t'_{k+1} = t_k; \tag{1.13}$$

(3-*move*) if $\mathbf{h}'$ is obtained from $\mathbf{h}$ by a 3-move applied at positions $k-1$, k, and $k+1$, then $t' = R_{\mathbf{h}}^{\mathbf{h}'}(t)$ is given by

$$t'_{k-1} = \frac{t_k t_{k+1}}{t_{k-1} + t_{k+1}}, \qquad t'_k = t_{k-1} + t_{k+1}, \qquad t'_{k+1} = \frac{t_k t_{k-1}}{t_{k-1} + t_{k+1}}. \tag{1.14}$$

The relation (1.13) follows from the fact that e_i and e_j commute for $|i-j| \geqslant 2$, and (1.14) is a consequence of the matrix identity (1.3).

The formulas (1.12) and (1.14) which describe elementary transition maps in two different settings, are strikingly similar. In fact, (1.12) becomes identical to (1.14) if one uses an exotic semifield structure on $\mathbb{Z}$, where the usual addition plays the role of multiplication, and taking the minimum plays the role of addition. These operations have the usual associativity, commutativity and distributivity properties, the latter being a rephrasing of the identity

$$a + \min(b, c) = \min(a+b, a+c).$$

In this semifield, one can divide but *not* subtract, since addition is idempotent: $\min(a, a) = a$.

The semifield $(\mathbb{Z}, \min, +)$ is known under various names. We will use the term *tropical semifield*, which we learned from M.-P. Schützenberger. A detailed study of its algebraic properties, along with numerous applications, can be found in [3]. (We thank D. Krob for providing this reference.)

The piecewise-linear transition maps $R_{\mathbf{h}}^{\mathbf{h}'}: \mathbb{Z}_+^m \to \mathbb{Z}_+^m$ can now be expressed as *rational* mappings in the sense of the tropical semifield. Moreover, the parallelism described above shows that these maps can be given by exactly the same formulas as the transition maps for the parametrization of $N_{>0}$.

It is then natural to consider a common generalization of the two settings by defining transition maps over an arbitrary ground semifield P. (See Section 2.1 for exact axiomatic description of P.) The main object of study is the set $\mathscr{L}_r(P)$ of vectors $(t_k^{\mathbf{h}})$ whose components are elements of P which satisfy the 2-move and 3-move relations (1.13)–(1.14); these components are double-indexed by reduced words $\mathbf{h}$ for w_0 and integers $k = 1, ..., m$. We call $\mathscr{L}_r(P)$ *the Lusztig variety*. The previous discussion shows that, canonically,

$$\mathscr{L}_r(\mathbb{Z}_+) = B,$$

where B is the canonical basis in U_+, and $\mathbb{Z}_+$ is equipped with the tropical semiring structure. On the other hand,

$$\mathscr{L}_r(\mathbb{R}_{>0}) = N_{>0},$$

where $\mathbb{R}_{>0}$ is the semifield of positive real numbers with the usual operations.

Problem 1.10 can be formulated for a general Lusztig variety $\mathscr{L}_r(P)$, where it amounts to finding explicit formulas (in terms of the semifield operations) for the transition maps

$$(t_1^{\mathbf{h}}, ..., t_m^{\mathbf{h}}) \xmapsto{R_{\mathbf{h}}^{\mathbf{h}'}} (t_1^{\mathbf{h}'}, ..., t_m^{\mathbf{h}'}).$$

Such formulas can be written in a universal form, i.e., in a form independent of the choice of P. Moreover, it is easy to show (cf. Proposition 2.1.7) that subtraction-free formulas for the components of $R_{\mathbf{h}}^{\mathbf{h}'}$ in the "geometric" case $P=\mathbb{R}_{>0}$ will necessarily be universal. Thus it is enough to solve Problem 1.10 in the special case of the Lusztig variety $\mathscr{L}_r(\mathbb{R}_{>0})=N_{>0}$, provided the answer is expressed in a subtraction-free form.

Our approach to the last problem is as follows: we compute the transition maps

$$R_{\mathbf{h}}^{\mathbf{h}'}: \mathbb{R}^m_{>0} \to \mathbb{R}^m_{>0}$$

directly from the definition $R_{\mathbf{h}}^{\mathbf{h}'}=t^{\mathbf{h}'}\circ(t^{\mathbf{h}})^{-1}=t^{\mathbf{h}'}\circ x^{\mathbf{h}}$, using our solution to Problem 1.1. Note that

$$t^{\mathbf{h}'}: N_{>0} \to \mathbb{R}^m_{>0}$$

can be obtained from (1.7) or (1.10), while the map

$$x^{\mathbf{h}}: \mathbb{R}^m_{>0} \to N_{>0}$$

is explicitly given by (1.1). Thus the only ingredients needed to complete the solution of Problem 1.10 are the subtraction-free formulas expressing the rational functions $\Delta^J(y)$ (or the polynomials $Z_a(x)$ and $T_J(x)$) in terms of the variables $t_1^{\mathbf{h}}, ..., t_m^{\mathbf{h}}$. Such formulas do exist, as the following result shows.

1.11. Theorem. *Let* $\mathbf{h}$ *be a reduced word for* w_0, *and let the matrix* x *be defined by the factorization* (1.1). *Let* y *be related to* x *as in Lemma* 1.3. *Then, for any* $a\in[1, r]$ *and any* $J\subset[1, r+1]$,

(1) $\Delta^J(y)$ *is a Laurent polynomial in the variables* t_k, *with nonnegative integer coefficients*;

(2) $Z_a(x)$ *is a monomial in the* t_k;

(3) $T_J(x)$ *is a polynomial in the* t_k *with nonnegative integer coefficients.*

In view of (1.9), statement (1) above follows immediately from (2) and (3). The monomial in (2) is given by the formula

$$Z_a(x)=\prod_{k:\, i\leqslant a<j} t_k \tag{1.15}$$

where i and j are defined as in (1.6); see (2.9.9). Statement (3) is proved in Theorem 3.7.4.

Let $T^{\mathbf{h}}_J(t_1, ..., t_m)$ be the polynomials in (3), i.e.,

$$T^{\mathbf{h}}_J(t_1, ..., t_m) = T_J((1+t_1 e_{h_1}) \cdots (1+t_m e_{h_m})). \tag{1.16}$$

In view of (1.10) and (1.15), in order to obtain a general formula for the transition maps $R^{\mathbf{h}'}_{\mathbf{h}}$, we only need subtraction-free formulas for the polynomials $T^{\mathbf{h}}_J$. Finding such a formula for general J and $\mathbf{h}$ remains an open problem.

To bypass this problem, we observe that a computation of a particular map $R^{\mathbf{h}'}_{\mathbf{h}}$ via (1.10) only involves the polynomials $T^{\mathbf{h}}_J$, where J is a chamber set for $\mathbf{h}'$. We will now solve Problem 1.10 as follows:

(1) express a general transition map $R^{\mathbf{h}'}_{\mathbf{h}}$ as a composition

$$R^{\mathbf{h}'}_{\mathbf{h}} = R^{\mathbf{h}'}_{\mathbf{h}^0} \circ R^{\mathbf{h}^0}_{\mathbf{h}}, \tag{1.17}$$

where $\mathbf{h}_0$ is the lexicographically minimal reduced word given by

$$\mathbf{h}^0 = (1, 2, 1, 3, 2, 1, ..., r, r-1, ..., 1); \tag{1.18}$$

(2) compute $R^{\mathbf{h}^0}_{\mathbf{h}}$, by obtaining a subtraction-free formula for $T^{\mathbf{h}}_J$, J being a chamber set for $\mathbf{h}^0$, and $\mathbf{h}$ arbitrary (see Proposition 2.10.2 and Theorem 2.4.6);

(3) compute $R^{\mathbf{h}'}_{\mathbf{h}^0}$, by obtaining a subtraction-free formula for $T^{\mathbf{h}^0}_J$, for an arbitrary J (see Theorem 2.10.3 and Theorem 2.8.2).

Formula (1.17) replaces multiple iterations of transformations (1.13)–(1.14) by a two-step computation that uses the minimal reduced word $\mathbf{h}^0$ as a "hub."

In the case related to the canonical basis, the semifield operations are "tropical," and each polynomial $T^{\mathbf{h}}_J$ becomes the minimum of certain linear forms in the variables $t_1, ..., t_m$. Then Lusztig's piecewise-linear transition maps $R^{\mathbf{h}'}_{\mathbf{h}}\colon \mathbb{Z}^m_+ \to \mathbb{Z}^m_+$ are expressed in terms of these minima.

As an application of our formulas, we obtain a new proof and generalization of piecewise-linear minimization formulas for quivers of type A given in [23, 24].

A key role in our proofs of the above results is played by the following observation (cf. Proposition 2.5.1).

1.12. THEOREM. *Let $\{M_J\colon J \subset [1, r+1]\}$ be a family of elements of a semifield P. Let the elements $t^{\mathbf{h}}_k$ be defined by the Chamber Ansatz substitution*

$$t^{\mathbf{h}}_k = \frac{M_L M_{L\cup\{i,j\}}}{M_{L\cup\{i\}} M_{L\cup\{j\}}} \tag{1.19}$$

where L, i, and j are related to k via (1.6). *Then the point* $\mathbf{t} = (t_k^{\mathbf{h}})$ *belongs to the Lusztig variety* $\mathscr{L}_r(P)$ *if and only if the* M_J *satisfy*

$$M_{L \cup \{i,k\}} M_{L \cup \{j\}} = M_{L \cup \{i,j\}} M_{L \cup \{k\}} + M_{L \cup \{j,k\}} M_{L \cup \{i\}} \tag{1.20}$$

whenever $i < j < k$ *and* $L \cap \{i, j, k\} = \phi$.

In other words, the Chamber Ansatz (1.19) translates the 2- and 3-move relations (1.13)–(1.14) into the 3-term relations (1.20). Relations (1.20) can be viewed as a semifield analogue of some of the classical Plücker relations.

We then refine Theorem 1.12 by giving the following alternative description of the Lusztig variety (cf. Theorem 2.7.1).

1.13. THEOREM. *There is a natural bijection between the Lusztig variety* $\mathscr{L}_r(P)$ *and the variety* $\mathscr{M}_r(P)$ *of vectors* (M_J) *satisfying the* 3*-term relations* (1.20), *together with the normalization condition*

$$M_{[1,b]} = 1, \qquad b = 0, 1, ..., r+1. \tag{1.21}$$

In one direction, the bijection $\mathscr{M}_r(P) \to \mathscr{L}_r(P)$ *is given by the Chamber Ansatz* (1.19). *The inverse bijection is given by*

$$M_J = \prod_{k:\, i \notin J,\, j \in J} (t_k^{\mathbf{h}})^{-1} \tag{1.22}$$

whenever J is a chamber set for $\mathbf{h}$; *in the last formula, the product is over all k such that* $i \notin J$ *and* $j \in J$, *for i and j defined by* (1.6).

We conclude this long introduction by describing the structure of the paper. In Chapter 2, we present a general study of the Lusztig variety $\mathscr{L}_r(P)$ over an arbitrary semifield P. In addition to the results stated above, we would like to mention two combinatorial constructions that play an important role in our theory. In Section 2.4, we construct an embedding of $\mathscr{L}_r(P)$ into the nil–Temperley–Lieb algebra [14, 17]. (Although the map $x^{\mathbf{h}}$ can be defined for any P, it is not injective in general.) In Section 2.6, the solutions of the 3-term relations (1.20) are given in terms of vertex-disjoint path families in acyclic planar graphs, in the spirit of [27, 19]. Our formulas for the transition maps $R_{\mathbf{h}^0}^{\mathbf{h}'}$ and $R_{\mathbf{h}}^{\mathbf{h}^0}$ are also written in the language of vertex-disjoint paths.

Chapter 3 is devoted to the "geometric" case $P = \mathbb{R}_{>0}$. The main results of this chapter have already been discussed. In contrast to the relatively elementary tools used in the rest of the paper, the proof of some properties of the polynomials T_J, such as statement (3) of Theorem 1.11, requires techniques from representation theory of quantum groups (see Sections 3.5–3.7).

Chapter 4 concentrates on the "tropical" case $P = \mathbb{Z}_+$. In particular, we show that our results imply piecewise-linear minimization formulas [24, 23] for the multi-segment duality and more general transformations related to representations of quivers of type A. Combinatorics of quivers proves to be related to two interesting special classes of reduced words and corresponding arrangements (see Sections 4.3–4.4).

In Chapter 5, we extend some of our results to the situation where the maximal permutation w_0 is replaced by an arbitrary permutation $w \in S_{r+1}$. We mostly deal with the case $P = \mathbb{R}_{>0}$. In this case, the natural generalization $\mathscr{L}^w(\mathbb{R}_{>0})$ of the Lusztig variety has the following realization inside the group N (see [30]). It can be identified with the intersection $N^w_{>0}$ of the subset $N_{\geqslant 0} \subset N$ of totally nonnegative matrices with the Bruhat cell $B_- w B_-$ where B_- is the Borel subgroup of lower-triangular matrices. Our main contribution is an explicit description of the set $N^w_{>0}$ that only uses $l(w) = \dim N^w_{>0}$ algebraic inequalities (see Section 5.4). This generalizes the total positivity criteria of Theorem 1.8. An important role in our description of $N^w_{>0}$ is played by the change of variables $x \mapsto y$, which extends the one in Lemma 1.3. Here y has the following geometric meaning: its matrix elements form the natural system of affine coordinates in the Schubert cell corresponding to w.

The connections between the 3-move relations (1.14) and the Yang–Baxter equation are discussed in the Appendix.

2. LUSZTIG VARIETY AND CHAMBER ANSATZ

Throughout the paper, we use the following notation. For i and j integers, $[i, j]$ denotes the set $\{a \in \mathbb{Z} : i \leqslant a \leqslant j\}$. The set of all reduced words for an element $w \in S_{r+1}$ is denoted by $R(w)$. Recall that a reduced word for w is a sequence of indices $\mathbf{h} = (h_1, ..., h_l)$ such that $l = l(w)$ is the number of inversions of w, and $s_{h_1} \cdots s_{h_l} = w$ where s_h denotes a simple transposition $(h, h+1)$.

2.1. *Semifields and Subtraction-Free Rational Expressions*

In what follows, by a *semifield* we will mean a set K endowed with two operations, addition and multiplication, which have the following properties:

(1) addition in K is commutative and associative;

(2) multiplication makes K an abelian group;

(3) distributivity: $(a + b)\, c = ac + bc$ for $a, b, c \in K$.

Note that, in view of (2), a semifield in our sense does not contain zero. This terminology is not quite standard: what we call a semifield would be called in [20] the group of units of a zerosumfree semifield. In our calculus, the elements of K will play the role of scalars.

A subset $P \subset K$ is called a *semiring* if it is closed under the operations of addition and multiplication (but not necessarily division). Most of the semirings we consider also satisfy the following additional condition:

$$\text{if} \quad a, b \in P \qquad \text{then} \quad \frac{a}{a+b} \in P \tag{2.1.1}$$

(cf. [29, 42.2.2]). In particular, the ambient semifield K is itself a semiring satisfying (2.1.1).

The following two examples will be of most importance to us.

2.1.1. Example. Let $K = \mathbb{R}_{>0}$ be the set of positive real numbers, with the usual operations.

2.1.2. Example (Tropical Semifield and Semiring). The tropical semifield is $K = \mathbb{Z}$, with multiplication and addition given by

$$a \odot b = a + b, \qquad a \oplus b = \min(a, b). \tag{2.1.2}$$

These operations make K a semifield whose unit is $0 \in \mathbb{Z}$. The distributivity property is a rephrasing of the identity

$$\min(a, b) + c = \min(a + c, b + c).$$

In this example, addition $\oplus$ is idempotent: $a \oplus a = a$, so K cannot be embedded into a field.

The *tropical semiring* is the set of nonnegative integers $P = \mathbb{Z}_+ \subset K$, with addition $\oplus$ and multiplication $\odot$. It is indeed a semiring, and (2.1.1) is satisfied.

2.1.3. Example. Let K consist of rational functions over $\mathbb{R}$ whose values on a given subset of $\mathbb{R}$ are positive. The operations are usual.

2.1.4. Example [29, 42.2.2(c)]. Let K be the set of Laurent series in one variable over $\mathbb{R}$ whose leading coefficient is positive.

In Examples 2.1.1, 2.1.3, and 2.1.4, $\mathbb{R}$ can be replaced by an arbitrary totally ordered field.

2.1.5. Example (Universal Semifield). Consider the field $\mathbb{Q}(z_1, ..., z_m)$ of rational functions in the variables $z_1, ..., z_m$ with coefficients in $\mathbb{Q}$. Let $\mathbb{Q}_{>0}(z_1, ..., z_m)$ denote the minimal sub-semifield of $\mathbb{Q}(z_1, ..., z_m)$ that contains $z_1, ..., z_m$. The elements of $\mathbb{Q}_{>0}(z_1, ..., z_m)$ are *subtraction-free rational expressions* in $z_1, ..., z_m$. Equivalently, $\mathbb{Q}_{>0}(z_1, ..., z_m)$ consists of all rational functions which can be represented as a ratio of two polynomials in $z_1, ..., z_m$ with nonnegative integer coefficients. For example, $x^2 - x + 1 \in \mathbb{Q}_{>0}(x)$ because $x^2 - x + 1 = (x^3 + 1)/(x + 1)$.

The semifield of the last example is universal in the following sense.

2.1.6. Lemma. *For any semifield K and any elements $t_1, ..., t_m \in K$, there is a unique homomorphism of semifields $\mathbb{Q}_{>0}(z_1, ..., z_m) \to K$ such that $z_1 \mapsto t_1, ..., z_m \mapsto t_m$.*

Following the common abuse of notation, we will denote by $f(t_1, ..., t_m) \in K$ the image of a rational function $f \in \mathbb{Q}_{>0}(z_1, ..., z_m)$ under the homomorphism of Lemma 2.1.6.

Proof of Lemma 2.1.6. The uniqueness of a homomorphism in question is obvious. To prove the existence, we only need to show that if $P_1/Q_1 = P_2/Q_2$ where P_1, P_2, Q_1, and Q_2 are polynomials in $z_1, ..., z_m$ with nonnegative integer coefficients, then

$$P_1(t_1, ..., t_m)/(t_1, ..., t_m) = P_2(t_1, ..., t_m)/Q_2(t_1, ..., t_m). \tag{2.1.3}$$

The equality $P_1/Q_1 = P_2/Q_2$ means that, in the expansions of $P_1 Q_2$ and $P_2 Q_1$, each monomial in $z_1, ..., z_m$ appears with the same coefficient. This implies that

$$P_1(t_1, ..., t_m)\, Q_2(t_1, ..., t_m) = P_2(t_1, ..., t_m)\, Q_1(t_1, ..., t_m).$$

Dividing both sides by $Q_1(t_1, ..., t_m)\, Q_2(t_1, ..., t_m)$ (a legitimate operation in a semifield K), we obtain (2.1.3), as desired. ∎

For a semifield K, let $\mathrm{Map}(K^m, K)$ denote the set of all maps $K^m \to K$. This set is a semifield itself, under pointwise addition and multiplication. The homomorphism of Lemma 2.1.6 gives rise to a semifield homomorphism $\mathbb{Q}_{>0}(z_1, ..., z_m) \to \mathrm{Map}(K^m, K)$, which we will denote by $f \mapsto f_K$. For f a rational expression, f_K is simply the function $(t_1, ..., t_m) \mapsto f(t_1, ..., t_m)$. For instance, if $f = x^2 - xy + y^2 \in \mathbb{Q}_{>0}(x, y)$ and $K = \mathbb{Z}$ is the tropical semifield, then $f_K \colon \mathbb{Z}^2 \to \mathbb{Z}$ is given by $f_K(n, m) = \min(3n, 3m) - \min(n, m) = 2\min(n, m)$. Note that the right-hand side is also g_K for $g = (x + y)^2$, which should be no surprise, since, indeed, in the tropical semifield, $(x^3 + y^3)/(x + y) = (x + y)^2$.

The following simple observation will play an important role in the sequel.

2.1.7. PROPOSITION. *Let $f, f' \in \mathbb{Q}_{>0}(z_1, ..., z_m)$ be two subtraction-free rational expressions. Then the following are equivalent*:

(i) $f = f'$;

(ii) $f_{\mathbb{R}_{>0}} = f'_{\mathbb{R}_{>0}}$;

(iii) $f_K = f'_K$ *for any semifield K.*

Proof. The implications (i) $\Rightarrow$ (iii) $\Rightarrow$ (ii) are obvious, and (ii) $\Rightarrow$ (i) simply states that a rational function which vanishes at all tuples of positive real numbers, is zero. ▮

2.2. *Lusztig Variety*

2.2.1. DEFINITION. Let K be a semifield, and $P \subset K$ a semiring satisfying (2.1.1). We define the *Lusztig variety* $\mathscr{L} = \mathscr{L}_r(P)$ as follows. An element $\mathbf{t}$ of $\mathscr{L}_r(P)$ is, by definition, a tuple

$$\mathbf{t} = (t^{\mathbf{h}})_{\mathbf{h} \in R(w_0)}$$

where each $t^{\mathbf{h}} = (t_1^{\mathbf{h}}, ..., t_m^{\mathbf{h}})$ is a "vector" in P^m, and these vectors satisfy the 2-move and 3-move relations (1.13)–(1.14).

For example, if $r = 2$, then

$$\mathscr{L}_2(P) = \left\{ \mathbf{t} = \begin{pmatrix} t^{121} \\ t^{212} \end{pmatrix} = \begin{pmatrix} (t_1, t_2, t_3) \\ (t'_1, t'_2, t'_3) \end{pmatrix} : t'_1 = \frac{t_2 t_3}{t_1 + t_3},\ t'_2 = t_1 + t_3,\ t'_3 = \frac{t_2 t_1}{t_1 + t_3} \right\}$$

where $t_1, t_2, t_3, t'_1, t'_2, t'_3$ are elements of P. Note that a point $\mathbf{t} = (t^{\mathbf{h}})$ of the Lusztig variety is uniquely defined by a single vector $t^{\mathbf{h}} \in P^m$ where $\mathbf{h}$ is an arbitrary reduced word for w_0. We will soon show (see Theorem 2.2.6 below) that, in fact, $t^{\mathbf{h}}$ can be chosen arbitrarily.

2.2.2. EXAMPLE. Let $P = \mathbb{R}_{>0}$ (see Example 2.1.1). Following [30], we will show that, in this case, the Lusztig variety is in a natural bijection with the set of totally positive unitriangular matrices. More precisely, let $N = N_r(\mathbb{R})$ be the group of real unipotent upper-triangular matrices of order $r+1$. A matrix $x = (x_{ij}) \in N$ is called *totally positive* if every minor which does not identically vanish on N is positive when evaluated at x.

Let us now restate, using the terminology just introduced, the classical results on total positivity that have been already discussed in the introduction. As before, let e_i, for $i = 1, ..., r$, denote the matrix whose $(i, i+1)$-entry is 1 and all other entries are 0. The following theorem is a restatement of Proposition 1.7.

2.2.3. THEOREM. *For an element* $\mathbf{t} = (t_k^{\mathbf{h}}) \in \mathscr{L}_r(\mathbb{R}_{>0})$, *the product*

$$x(\mathbf{t}) = (1 + t_1^{\mathbf{h}} e_{h_1}) \cdots (1 + t_m^{\mathbf{h}} e_{h_m}) \tag{2.2.1}$$

does not depend on the choice of a reduced word $\mathbf{h} = (h_1, ..., h_m) \in R(w_0)$. *The map* $t \mapsto x(\mathbf{t})$ *is a bijection between the Lusztig variety* $\mathscr{L}_r(\mathbb{R}_{>0})$ *and the variety* $N_{>0}$ *of totally positive matrices in N. Furthermore: for a fixed* $\mathbf{h}$, *the map*

$$t = (t_1, ..., t_m) \mapsto x^{\mathbf{h}}(t) = (1 + t_1 e_{h_1}) \cdots (1 + t_m e_{h_m}) \tag{2.2.2}$$

is a bijection from $\mathbb{R}^m_{>0}$ *to* $N_{>0}$.

This result essentially appears in [30]. The fact that the map $\mathbf{t} \mapsto x(\mathbf{t})$ is well defined is justified by the way formulas (1.13)–(1.14) were originally obtained. Total positivity of the matrix $x(\mathbf{t})$ can be proved by a direct combinatorial argument (see Section 2.4). To prove bijectivity, it suffices to give explicit formulas for the inverse map $x^{\mathbf{h}}(t) \mapsto t$ in the special case $\mathbf{h} = \mathbf{h}^0$ (see (1.18)). Such formulas can indeed be obtained, in terms of the minors of the matrix $x = x^{\mathbf{h}}(t)$; see (2.4.15).

2.2.4. *Remark.* Theorem 2.2.3 can be used for parametrizing the variety of all totally positive matrices in GL_{r+1} It is well known [11] that a matrix $g \in GL_{r+1}(\mathbb{R})$ is totally positive if and only if g has a totally positive Gaussian decomposition, that is, iff $g = v^T du$ where $u, v \in N_{>0}$ and d is a diagonal matrix with positive diagonal entries. Applying Theorem 2.2.3 to u and v, we obtain a family of parametrizations of our totally positive matrix g by $2m + r + 1 = (r+1)^2$ positive numbers.

2.2.5. EXAMPLE. Let $P = \mathbb{Z}_+$ be the tropical semiring from Example 2.1.2. According to the Introduction, the Lusztig variety $\mathscr{L}_r(P)$ can be identified with the canonical basis B in $U_q(\mathrm{Lie}(N))$ (see [28]).

We now return to the general case of an arbitrary ground semiring P, as described in Definition 2.2.1.

2.2.6. THEOREM. *For any* $\mathbf{h} \in R(w_0)$, *the projection* $\mathbf{t} \mapsto t^{\mathbf{h}}$ *is a bijection between the Lusztig variety* $\mathscr{L}$ *and* P^m.

Proof. We need to show the commutativity of the diagram whose objects are copies of P^m labelled by reduced words $\mathbf{h} \in R(w_0)$, and the morphisms are elementary transition maps given by (1.13)–(1.14), for all possible 2- and 3-moves. In particular, the composition of elementary transition maps along any oriented cycle of this diagram needs to be the identity map.

In the notation of Section 2.1, this amounts to verifying a collection of identities of the form $f_K = f'_K$ where f and f' are some subtraction-free rational expressions. By Proposition 2.1.7, it is sufficient to consider the special case $K = \mathbb{R}_{>0}$. In this case, according to Theorem 2.2.3, any composition of morphisms in our diagram that goes from $\mathbf{h}$ to $\mathbf{h}'$ is equal to $(x^{\mathbf{h}'})^{-1} \circ x^{\mathbf{h}}$, and commutativity follows. ∎

Theorem 2.2.6 implies that, for any two reduced words $\mathbf{h}$ and $\mathbf{h}'$, there is a well-defined bijection $R_{\mathbf{h}}^{\mathbf{h}'}: P^m \mapsto P^m$ given by

$$R_{\mathbf{h}}^{\mathbf{h}'}(t^{\mathbf{h}}) = t^{\mathbf{h}'} \tag{2.2.3}$$

where $t^{\mathbf{h}}$ and $t^{\mathbf{h}'}$ are the $\mathbf{h}$- and $\mathbf{h}'$-components of the same element of the Lusztig variety. Bijections $R_{\mathbf{h}}^{\mathbf{h}'}$ are called *transition maps*. They are one of the main objects of study in this paper. Note that, according to the proof of Theorem 2.2.6, the components of any transition map are given by subtraction-free rational expressions which do not depend on the choice of a ground semiring P.

2.3. *Pseudo-line Arrangements*

In this section we will describe a geometric representation of reduced words by pseudo-line arrangements on the plane. This representation has now become a folklore in low dimensional topology, the study of the Yang–Baxter equation, and geometric combinatorics. See [21, 36] and references therein.

Fix a vertical strip on the plane. By a *pseudo-line arrangement* (see Fig. 3) we will mean a configuration of $r+1$ pseudo-lines $\mathrm{Line}_1, \ldots, \mathrm{Line}_{r+1}$ in the strip with the following three properties:

(i) each vertical line in the strip intersects each of the pseudo-lines at exactly one point;

(ii) every two pseudo-lines cross each other exactly once within the strip and do not have other meeting points;

(iii) the configuration is generic in the following sense: no three lines meet at a point; no two crossing points lie on the same vertical line.

We will label the pseudo-lines so that their right endpoints are numbered 1 through $r+1$ bottom-up; thus the left endpoints will be numbered top to bottom. One can always draw such an arrangement as shown in Fig. 3, by combining segments taken from a collection of $r+1$ horizontal lines with X-shaped switches between them. This produces a wiring diagram of a reduced word; cf. [21, p. 111].

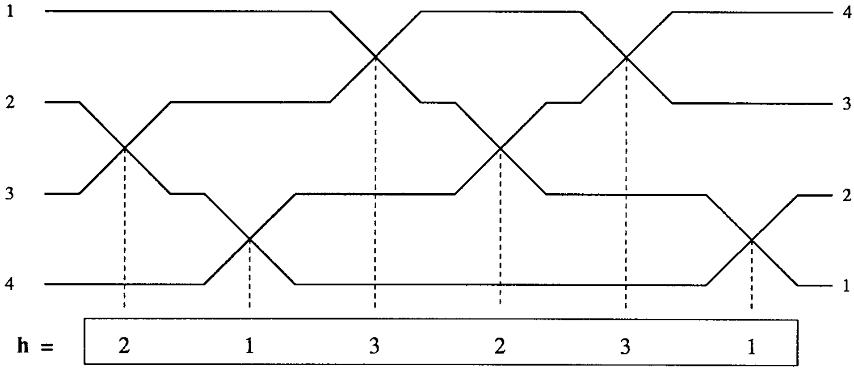


FIG. 3. Pseudo-line arrangement for $\mathbf{h} = 213231$.

Such arrangements (modulo natural isotopy equivalence) are in a natural bijection with reduced words of w_0. Let A be an arrangement. By the property (ii) above, the total number of crossing points in A is $m = \binom{r+1}{2}$; let us order them from left to right. For $k = 1, ..., m$, let $h_k - 1$ be the number of pseudo-lines passing strictly below the kth crossing point. Then $\mathbf{h} = (h_1, ..., h_m)$ is the sequence of levels where the X-switches are located. These switches can be viewed as adjacent transpositions, and thus $\mathbf{h}$ is a reduced word for w_0 (cf. Fig. 3). We call $\mathbf{h}$ the reduced word associated with A and denote $A = \mathrm{Arr}(\mathbf{h})$.

Let us label the crossing point of pseudo-lines Line_i and Line_j, with $i < j$, by the pair (i, j). The left-to-right ordering of the crossing points in A results in a total ordering on the set of pairs

$$\Pi = \Pi_r = \{(i, j): 1 \leqslant i < j \leqslant r + 1\}. \tag{2.3.1}$$

Note that such pairs naturally correspond to positive roots of type A_r. The ordering of positive roots thus obtained is known as the normal (or total reflection) ordering (see [5, VI.1.6]) associated to the corresponding reduced word. By traversing our arrangement from right to left, one can easily see that, in the interval between the kth and the $(k+1)$'st crossing point, the jth pseudo-line from the bottom is $\mathrm{Line}_{s_{h_m} \cdots s_{h_{k+1}}(j)}$. Hence the kth positive root, with respect to this ordering, is given by

$$\theta_k = s_{h_m} \cdots s_{h_{k+1}}(\alpha_{h_k}), \tag{2.3.2}$$

where $\alpha_1, ..., \alpha_r$ are the simple roots of type A_r, in the standard notation.

We denote by $\mathbf{n} = \mathbf{n}(\mathbf{h})$ the total ordering of Π_r that corresponds to a reduced word $\mathbf{h}$. The following characterization of such orderings is well known (see, e.g., [25] and references therein).

2.3.1. PROPOSITION. *The correspondence* $\mathbf{h} \mapsto \mathbf{n}(\mathbf{h})$ *is a bijection between the set* $R(w_0)$ *of reduced words for* w_0 *and the set of total orderings of* Π_r *which have the following betweenness property: for any three indices* $i, j, k \in [1, r+1]$ *with* $i<j<k$, *the pair* (i, k) *lies between* (i, j) *and* (j, k).

One can translate the 2- and 3-moves on reduced words into the language of normal orderings, where they become the following:

2-*move.* In a normal ordering $\mathbf{n}$, interchange two consecutive (with respect to $\mathbf{n}$) entries (i, j) and (k, l) provided all of i, j, k, l are distinct.

3-*move.* Replace three consecutive entries $(i, j)\,(i, k)\,(j, k)$ such that $i<j<k$ by $(j, k)\,(i, k)\,(j, k)$ (or vice versa).

For example, for $\mathbf{h}=213231$, the normal ordering is $\mathbf{n}(\mathbf{h})=23\ 24\ 13\ 14\ 34\ 12$ (cf. Fig. 3). There are two possible 2-moves, one of them swapping 24 and 13, another one swapping 34 and 12. There is only one possible 3-move, transforming $\mathbf{n}(\mathbf{h})$ into 23 24 34 14 13 12.

Passing from reduced words to normal orderings allows us to simplify the definition of the Lusztig variety. The components $t_k^{\mathbf{h}}$ of a vector $t^{\mathbf{h}}$ representing an element $\mathbf{t} \in \mathscr{L}_r(P)$ are naturally associated with the crossing points in the arrangement $\mathrm{Arr}(\mathbf{h})$. This makes it natural to change the notation for the variables $t_k^{\mathbf{h}}$ as follows. Suppose a pair (i, j) appears in position k in the ordering $\mathbf{n}=\mathbf{n}(\mathbf{h})$. Then we write $t_{ij}^{\mathbf{n}}$ instead of $t_k^{\mathbf{h}}$. For instance, for $\mathbf{h}=213231$, the vector $t^{\mathbf{h}}=(t_1, ..., t_6)$ becomes $t^{\mathbf{n}}=(t_{23}, t_{24}, t_{13}, t_{14}, t_{34}, t_{12})$, in accordance with Fig. 4.

In this new notation, the defining relations (1.13) and (1.14) take the following form:

(2-*move relations*) if $\mathbf{n}'$ differs from $\mathbf{n}$ by a 2-move, then $t^{\mathbf{n}'}=t^{\mathbf{n}}$;

(3-*move relations*) if $\mathbf{n}'$ is obtained from $\mathbf{n}$ by a 3-move that

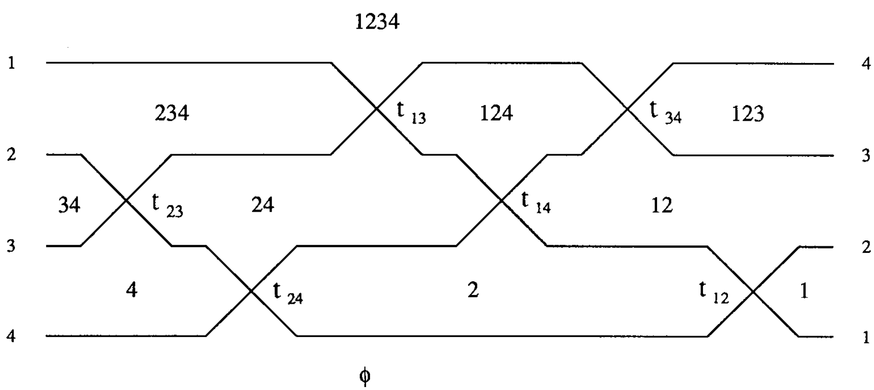


FIG. 4. Chamber sets for $\mathbf{h}=213231$.

transforms consecutive entries (i, j), (i, k), and (j, k) into (j, k), (i, k), and (i, j) (or vice versa), then $t' = t^{\mathbf{n}'}$ is obtained from $t = t^{\mathbf{n}}$ by

$$t'_{ij} = \frac{t_{ij} t_{ik}}{t_{ij} + t_{jk}}, \qquad t'_{ik} = t_{ij} + t_{jk}, \qquad t'_{jk} = \frac{t_{ik} t_{jk}}{t_{ij} + t_{jk}}. \tag{2.3.3}$$

2.3.2. *Remark.* The modified definition has an important advantage that the 2-moves do not affect the vector $t^{\mathbf{n}}$. This means that $t^{\mathbf{n}}$ only depends on the *commutation class* of the corresponding reduced decomposition $\mathbf{h}$, i.e., on the equivalence class of $\mathbf{h}$ under 2-moves.

The 3-move relations can be restated in "topological" terms, since $t^{\mathbf{n}}$ only depends on the isotopy class of the pseudo-line arrangement $\mathrm{Arr}(\mathbf{n}) = \mathrm{Arr}(\mathbf{h})$. (Here we relax the genericity condition, allowing two or more crossing points to lie on the same vertical line; we then consider isotopies within this class of pseudo-line arrangements.) In this language, a 3-move $\mathbf{n} \to \mathbf{n}'$ corresponds to the transformation of (an isotopy class of) a pseudo-line arrangement shown in Fig. 5.

The relations (2.3.3) can be viewed as certain Yang–Baxter-type rules. We discuss the connections with the Yang–Baxter equation in the Appendix.

Theorem 2.2.6 can be reformulated as follows: for every normal ordering $\mathbf{n}$ of $\Pi = \Pi_r$, the projection $\mathbf{t} \mapsto t^{\mathbf{n}}$ is a bijection between the Lusztig variety $\mathscr{L}$ and the set $P^{\Pi} = \{(t_{ij}) \colon t_{ij} \in P\}$. Therefore, for any two normal orderings $\mathbf{n}$ and $\mathbf{n}'$, there is a well-defined transition map $R_{\mathbf{n}}^{\mathbf{n}'} \colon P^{\Pi} \to P^{\Pi}$ (cf. (2.2.3)). Note that if $\mathbf{n}$ and $\mathbf{n}'$ belong to the same commutation class then $R_{\mathbf{n}}^{\mathbf{n}'}$ is the identity map (cf. Remark 2.3.2).

We conclude this section by introducing some important notation. Let $f \colon \mathscr{L} \to P$ be a function on the Lusztig variety. For any reduced word $\mathbf{h} \in R(w_0)$ (resp. any normal ordering $\mathbf{n}$ of Π) we denote by $f^{\mathbf{h}} \colon P^m \to P$ (resp. $f^{\mathbf{n}} \colon P^{\Pi} \to P$) the function given by $f^{\mathbf{h}}(t) = f(\mathbf{t})$ (resp. $f^{\mathbf{n}}(t) = f(\mathbf{t})$) where $\mathbf{t}$ is the unique element of $\mathscr{L}$ such that $t^{\mathbf{h}} = t$ (resp. $t^{\mathbf{n}} = t$). To illustrate this notation, take $f(\mathbf{t}) = t_k^{\mathbf{h}'}$ for some $\mathbf{h}' \in R(w_0)$ and $k = 1, \ldots, m$.

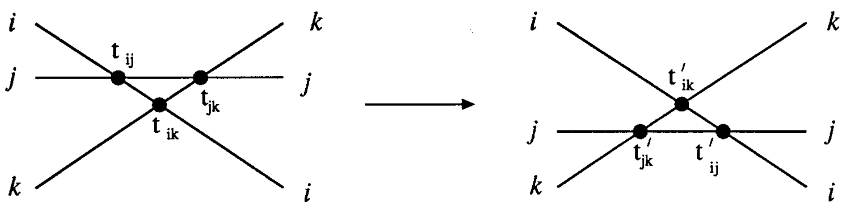


FIG. 5. Geometric interpretation of a 3-move.

Then $f^{\mathbf{h}}(t)=(R_{\mathbf{h}}^{\mathbf{h}'}(t))_k$. Similarly, if $f(\mathbf{t})=t_{ij}^{\mathbf{n}'}$ for some normal ordering $\mathbf{n}'$ and some $(i, j)\in \Pi$, then $f^{\mathbf{n}}(t)=(R_{\mathbf{n}}^{\mathbf{n}'}(t))_{ij}$.

2.4. *Minors and the nil–Temperley–Lieb Algebra*

In this section we will give a "concrete realization" of the Lusztig variety $\mathscr{L}_r(P)$. This will also provide us with a supply of functions on $\mathscr{L}_r(P)$, to be used later in the computation of transition maps.

In the special case $P=\mathbb{R}_{>0}$, such a realization was already given in Theorem 2.2.3, namely, in the assertion that the elements of $\mathscr{L}_r(\mathbb{R}_{>0})$ are faithfully represented by the totally positive matrices. However, for a general Lusztig variety, the same construction fails: although the product (2.2.1) is well defined for any P, the map $\mathbf{t}\mapsto x(\mathbf{t})$ from $\mathscr{L}_r(P)$ to the set of unitriangular matrices with entries in P (cf. (2.2.2)) need not be injective. In particular, it is not injective in the special case of the tropical semiring.

In order to overcome this difficulty, we are going to generalize the notion of a minor of the matrix $x(\mathbf{t})$ to the case of an arbitrary underlying semiring. (To be more precise, we will only need the analogues of those minors that do not identically vanish on the group $N\subset GL_{r+1}(\mathbb{C})$ of upper-triangular matrices.) Of course the usual definition of the determinant involves subtraction, and thus may not be used in a semiring.

Let us first look at the case $P=\mathbb{R}_{>0}$, where defining a minor is not a problem. For an $(r+1)\times(r+1)$-matrix x, and a pair of subsets I, $J\subset[1, r+1]$ of the same size, let $\Delta_I^J(x)$ denote the minor of x, with the row set I and column set J. It is easy to see that if $I=\{i_1<\cdots<i_k\}$ and $J=\{j_1<\cdots<j_k\}$, then $\Delta_I^J(x)$ is not identically equal to zero on N if and only if $i_1\leqslant j_1, ..., i_k\leqslant j_k$. Let us call such a pair of subsets (I, J) an *admissible* pair. An admissible pair (I, J) is *reduced* if $i_s<j_s$ for $s=1, ..., k$. One clearly has $\Delta_I^J(x)=\Delta_{I'}^{J'}(x)$ for all $x\in N$, where (I', J') denotes the reduced pair obtained by removing the set

$$\{j\colon j=i_s=j_s \text{ for some } s=1, ..., k\}$$

from both I and J. Therefore, while studying minors Δ_I^J as functions on N, we may always assume that a pair (I, J) is admissible and reduced. With some abuse of notation, we denote by the same symbol Δ_I^J the function $\mathbf{t}\mapsto\Delta_I^J(x(\mathbf{t}))$ on the Lusztig variety $\mathscr{L}_r(\mathbb{R}_{>0})$.

Our goal now is to define the functions $\Delta_I^J\colon \mathscr{L}_r(P)\to P$ for an arbitrary ground semiring P satisfying the condition (2.1.1). To resolve the difficulties outlined above, we will replace the elementary Jacobi matrices e_i in (2.2.1) by the generators u_i of a certain associative algebra. The commutation relations satisfied by the u_i will ensure that the product (2.2.1) is a well-defined function on $\mathscr{L}_r(P)$, that is, it does not depend on the choice of a reduced word $\mathbf{h}$.

The informal argument presented in this paragraph will motivate subsequent definitions. If we want (2.2.1) to be invariant under the 2-move and 3-move transformations, then we ask for

$$(1+t_1u_i)(1+t_2u_j)=(1+t_2u_j)(1+t_1u_i), \qquad |i-j|\geqslant 2 \tag{2.4.1}$$

and

$$\begin{aligned}(1+t_1u_i)&(1+t_2u_j)(1+t_3u_i)\\ &=\left(1+\frac{t_2t_3}{t_1+t_3}u_j\right)(1+(t_1+t_3)\,u_i)\\ &\quad\times\left(1+\frac{t_1t_2}{t_1+t_3}u_j\right), \qquad |i-j|=1\end{aligned} \tag{2.4.2}$$

to be satisfied for any t_1, t_2, $t_3 \in P$. Equating coefficients of all monomials in t_1 and t_2 on both sides of (2.4.1) yields the relation

$$u_iu_j=u_ju_i, \qquad |i-j|\geqslant 2. \tag{2.4.3}$$

Similarly, multiplying (2.4.2) by $(t_1+t_3)^2$ and equating coefficients of monomials in t_1, t_2, t_3 on both sides yields the relations

$$\begin{aligned}&u_i^2=0\\ &u_iu_ju_i=0, \qquad |i-j|=1\end{aligned} \tag{2.4.4.}$$

Thus the product

$$X(\mathbf{t})=(1+t_1^{\mathbf{h}}u_{h_1})\cdots(1+t_m^{\mathbf{h}}u_{h_m}) \tag{2.4.5}$$

is a well-defined function on the Lusztig variety $\mathscr{L}_r(P)$ if the generators u_i satisfy (2.4.3)–(2.4.4).

The associative algebra defined by (2.4.3)–(2.4.4) is called the *nil–Temperley–Lieb algebra* and denoted by $NTL=NTL_r$ (see [17, 14]). This algebra has a distinguished *monomial* linear basis formed by all distinct (modulo (2.4.3)) nonzero noncommutative monomials in the generators $u_1, ..., u_r$. The dimension of NTL_r is the Catalan number $(1/(r+2))\binom{2r+2}{r+1}$. For example, the monomial basis in NTL_2 is formed by 1, u_1, u_2, u_1u_2, and u_2u_1. For any semiring P, we denote by $NTL_r(P)$ the set of formal linear combinations of the basis monomials in NTL_r, with coefficients in $P\cup\{0\}$. Since the product of two monomials in the u_i is again a monomial (maybe, equal to 0), the multiplication extends to $NTL_r(P)$ without difficulty. Thus $NTL_r(P)$ may be viewed as the nil–Temperley–Lieb algebra "over P." We see that (2.4.5) unambiguously defines a map $\mathscr{L}_r(P)\to NTL_r(P)$.

The monomial basis in NTL_r can be described in several equivalent ways (cf. [4, 14]). For our purposes, the following description is the most convenient. Consider the vector space of formal linear combinations of subsets of the set $[1, r+1]$. Let us represent a generator u_j by the *shift operator* acting in this space by

$$u_j(J) = \begin{cases} J \cup \{j\} \setminus \{j+1\}, & \text{if } \; j+1 \in J, \;\; j \notin J \\ 0, & \text{otherwise.} \end{cases} \tag{2.4.6}$$

This is a faithful representation of NTL_r. The following proposition was essentially proved in [4] (using a different language).

2.4.1. PROPOSITION. *A monomial $u = u_{j_1} \cdots u_{j_s}$ in NTL_r is non-zero if and only if $u(J) = I$ for some reduced admissible pair of subsets (I, J). Furthermore, the pair (I, J) is uniquely determined by u, and the correspondence $u \mapsto (I, J)$ is a bijection between the monomial basis in NTL_r and the set of all reduced admissible pairs (I, J) in $[1, r+1]$.*

For a reduced admissible pair (I, J) we will denote by u_I^J the element of the monomial basis in NTL_r that corresponds to (I, J) via Proposition 2.4.1. For example, the monomial basis in NTL_2 consists of:

$$1 = u_\phi^\phi, \qquad u_1 = u_{\{1\}}^{\{2\}}, \qquad u_2 = u_{\{2\}}^{\{3\}}, \qquad u_1 u_2 = u_{\{1\}}^{\{3\}}, \qquad u_2 u_1 = u_{\{12\}}^{\{23\}}.$$

For an element $X \in NTL_r(P)$ we denote by $\Delta_I^J(X)$ the coefficient of u_I^J in the expansion of X with respect to the monomial basis. Thus, every reduced admissible pair (I, J) gives rise to a function $\mathbf{t} \mapsto \Delta_I^J(X(\mathbf{t}))$ on $\mathscr{L}_r(P)$ with values in $P \cup \{0\}$. With the same abuse of notation as above, we will write $\Delta_I^J(\mathbf{t})$ instead of $\Delta_I^J(X(\mathbf{t}))$. Let us show that this notation is consistent with the one introduced above for $P = \mathbb{R}_{>0}$.

2.4.2. PROPOSITION. *Let $\mathbf{h} = (h_1, ..., h_l)$ be any sequence of indices from $[1, r]$, and let $t_1, ..., t_l$ be positive real numbers. Then*

$$\Delta_I^J((1 + t_1 u_{h_1}) \cdots (1 + t_l u_{h_l})) = \Delta_I^J((1 + t_1 e_{h_1}) \cdots (1 + t_l e_{h_l})) \tag{2.4.7}$$

for every reduced admissible pair (I, J).

Proof. Unraveling the definitions, we obtain the following explicit formula valid for an arbitrary ground semiring P:

$$\Delta_I^J((1 + t_1 u_{h_1}) \cdots (1 + t_l u_{h_l})) = \sum t_{a_1} t_{a_2} \cdots t_{a_s} \tag{2.4.8}$$

where the sum is over all sequences $1 \leqslant a_1 < a_2 < \cdots < a_s \leqslant l$ such that

$$u_{h_{a_1}} u_{h_{a_2}} \cdots u_{h_{a_s}}(J) = I.$$

It remains to show that the minor on the right-hand side of (2.4.7) is given by the same formula. This follows by induction on l from the identity

$$\Delta_I^J(x(1+te_h)) = \Delta_I^J(x) + t\,\Delta_I^{u_h(J)}(x)$$

which is easily checked for any square matrix x over an arbitrary commutative ring. ▮

Now the functions $\Delta_I^J \colon \mathscr{L}_r(P) \to P \cup \{0\}$ are defined unambiguously for any P. Recalling the notation $f^{\mathbf{h}}$ from the last paragraph of Section 2.3, we see that for any $\mathbf{h} \in R(w_0)$, the function $(\Delta_I^J)^{\mathbf{h}}(t_1, ..., t_m)$ is a sum of distinct square-free monomials in $t_1, ..., t_m$ given by (2.4.8). This readily implies that $(\Delta_I^J)^{\mathbf{h}}$ never vanishes, i.e., it is a function $P^m \to P$. For example, in the case when $r = 2$ and $\mathbf{h} = (1, 2, 1)$, the rule (2.4.8) gives

$$(\Delta_\phi^\phi)^{\mathbf{h}} = 1, \qquad (\Delta_{\{1\}}^{\{2\}})^{\mathbf{h}} = t_1 + t_3, \qquad (\Delta_{\{2\}}^{\{3\}})^{\mathbf{h}} = t_2,$$
$$(\Delta_{\{1\}}^{\{3\}})^{\mathbf{h}} = t_1 t_2, \qquad (\Delta_{\{12\}}^{\{23\}})^{\mathbf{h}} = t_2 t_3.$$

These formulas could also be obtained by directly expanding $X(\mathbf{t})$ in the monomial basis:

$$\begin{aligned} X(\mathbf{t}) &= (1 + t_1 u_1)(1 + t_2 u_2)(1 + t_3 u_1) \\ &= 1 + (t_1 + t_3)\, u_1 + t_2 u_2 + t_1 t_2 u_1 u_2 + t_2 t_3 u_2 u_1 \\ &= 1 + (t_1 + t_3)\, u_{\{1\}}^{\{2\}} + t_2 u_{\{2\}}^{\{3\}} + t_1 t_2 u_{\{1\}}^{\{3\}} + t_2 t_3 u_{\{12\}}^{\{23\}}. \end{aligned}$$

Consider then the example of Fig. 3 (that is, $\mathbf{h} = 213231$). For $r = 3$, there are 14 admissible pairs, so we will only write a couple of formulas that can be obtained from (2.4.8):

$$\Delta_{\{1,2\}}^{\{2,3\}} = t_1 t_2 + t_1 t_6 + t_4 t_6;$$
$$\Delta_{\{1,3\}}^{\{2,4\}} = (t_2 + t_6)(t_3 + t_5).$$

Following the strategy outlined in Section 2.3, we will now pass from a reduced word $\mathbf{h} \in R(w_0)$ to the corresponding normal ordering $\mathbf{n} = \mathbf{n}(\mathbf{h})$ of Π, thus replacing the $(\Delta_I^J)^{\mathbf{h}}$ by the polynomials $(\Delta_I^J)^{\mathbf{n}}$ in the variables t_{ij}. For example, in the case of $\mathbf{h} = 213231$, one has $(t_1, t_2, t_3, t_4, t_5, t_6) = (t_{23}, t_{24}, t_{13}, t_{14}, t_{34}, t_{12})$, and therefore

$$\begin{aligned} (\Delta_{\{1,2\}}^{\{2,3\}})^{\mathbf{n}} &= t_{23} t_{24} + t_{23} t_{12} + t_{14} t_{12}; \\ (\Delta_{\{1,3\}}^{\{2,4\}})^{\mathbf{n}} &= (t_{24} + t_{12})(t_{13} + t_{34}). \end{aligned} \tag{2.4.9}$$

We will now obtain an explicit combinatorial formula for each polynomial $(\Delta_I^J)^{\mathbf{n}}$. To do so, let us associate with $\mathbf{n}$ a certain planar acyclic directed graph $\Gamma(\mathbf{n})$ which is constructed as follows. Start with the pseudo-line arrangement $\mathrm{Arr}(\mathbf{n})$ described in Section 2.3. This arrangement is formed by horizontal segments lying on $r+1$ horizontal lines, and X-shaped switches between them. To construct the graph $\Gamma(\mathbf{n})$, let us replace each X-switch by a Z-shaped connector. For example, the arrangement of Fig. 3 will be transformed into the graph shown in Fig. 6. The vertices of $\Gamma(\mathbf{n})$ are the endpoints of all segments of the modified arrangement, and the edges are these segments themselves, oriented left-to-right. The *sources* (resp. *sinks*) of $\Gamma(\mathbf{n})$ are the left (resp. right) endpoints of the horizontal lines. The sources are denoted by $s_1, ..., s_{r+1}$ and the sinks by $S_1, ..., S_{r+1}$ both sets being numbered bottom-up.

We then assign the *weight* $w(e)$ to every edge e of $\Gamma(\mathbf{n})$ as follows: if e is a slanted edge that replaced the crossing of pseudo-lines Line_i and Line_j in the original arrangement $\mathrm{Arr}(\mathbf{n})$, then set $w(e) = t_{ij}$. If e is a horizontal edge, then $w(e) = 1$. Finally, we define the weight $w(\pi)$ of an oriented path π to be the product of the weights $w(e)$ for all edges e of π.

2.4.4. THEOREM. *For any normal ordering* $\mathbf{n}$ *of* Π *and any reduced admissible pair of subsets* (I, J) *of size* l, *the polynomial* $(\Delta_I^J)^{\mathbf{n}}$ *in the variables* t_{ij}, $(i, j) \in \Pi$ *is given by*

$$(\Delta_I^J)^{\mathbf{n}}(t) = \sum_{\pi_1, ..., \pi_l} w(\pi_1) \cdots w(\pi_l) \tag{2.4.10}$$

where the sum is over all families of l *vertex-disjoint paths* $\{\pi_1, ..., \pi_l\}$ *in* $\Gamma(\mathbf{n})$, *each path connecting a source* s_i, $i \in I$, *with a sink* S_j, $j \in J$.

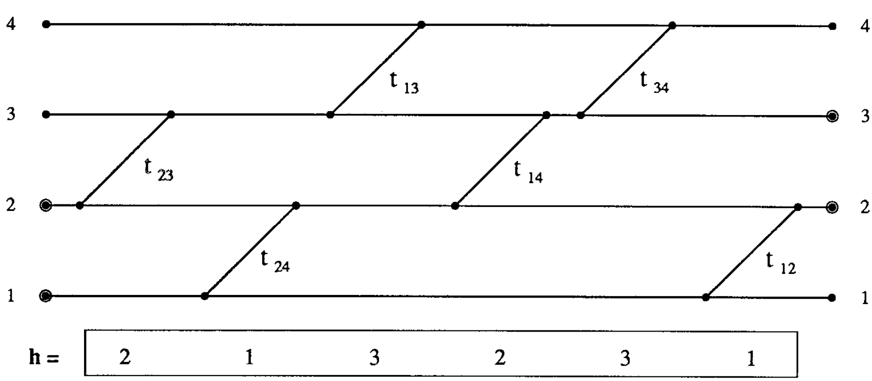


FIG. 6. Graph $\Gamma(\mathbf{n})$.

To illustrate this theorem, consider the example of Fig. 6. Let $I=\{1, 2\}$ and $J=\{2, 3\}$ (the corresponding sources and sinks are circled). Then there are three families of non-intersecting paths which connect I and J, and the weights of them are, respectively, $t_{23}t_{24}$, $t_{23}t_{12}$, and $t_{14}t_{12}$, reproducing (2.4.9).

Proof. Formula (2.4.10) is easily seen to be a reformulation of (2.4.8). More precisely, let us replace every variable t_{ij} in (2.4.10) by the corresponding variable t_k as described in Section 2.3. A straightforward inspection shows that this transforms the summands in (2.4.10) into the monomials in (2.4.8). ∎

For any set $J\subset[1, r+1]$ of size l we shall abbreviate $\Delta^J_{[1,l]}$ to Δ^J. The minors of the form Δ^J will be called *flag minors*. They can be regarded as the analogues of the Plücker coordinates on the flag variety and will be of special importance to us. Among these minors, a special role will be played by the *principal flag minors* $\Delta^{[a+1, r+1]}$, for $a=1, ..., r$. Theorem 2.4.4 implies the following very simple formula for these minors.

2.4.5. Corollary. *For any normal ordering* $\mathbf{n}$ *of* Π *and any* $a=1, ..., r$, *the principal flag minors are given by*

$$(\Delta^{[a+1, r+1]})^{\mathbf{n}} = \prod_{i\leqslant a<j} t_{ij}. \tag{2.4.11}$$

In particular, the polynomial $(\Delta^{[a+1, r+1]})^{\mathbf{n}}$ *does not depend on* $\mathbf{n}$.

Proof. We will derive (2.4.11) from (2.4.10) for $I=[1, r+1-a]$, $J=[a+1, r+1]$, and $l=r+1-a$. The monomial in (2.4.11) corresponds to the following family of vertex-disjoint paths in $\Gamma(\mathbf{n})$. Consider the pseudo-lines $\mathrm{Line}_{a+1}, ..., \mathrm{Line}_{r+1}$ of the arrangement $\mathrm{Arr}(\mathbf{n})$. Whenever two of these pseudo-lines cross each other in $\mathrm{Arr}(\mathbf{n})$, let us replace the corresponding X-switch by a pair of horizontal segments connecting the same points. We then obtain a family $\{\pi^0_1, ..., \pi^0_{r+1-a}\}$ of vertex-disjoint paths in $\Gamma(\mathbf{n})$ that joins I and J. Its total weight is the product of weights t_{ij} which correspond to intersections of pseudo-lines Line_j, $j>a$, with pseudo-lines Line_i, $i\leqslant a$. This yields the right-hand side of (2.4.11).

It remains to show that $\{\pi^0_1, ..., \pi^0_{r+1-a}\}$ is the only family of vertex-disjoint paths in $\Gamma(\mathbf{n})$ that join the sources $s_1, ..., s_{r+1-a}$ with the sinks $S_{a+1}, ..., S_{r+1}$, respectively. Suppose there exists a family of vertex-disjoint paths $\{\pi_1, ..., \pi_{r+1-a}\}$ in $\Gamma(\mathbf{n})$ having the same property but different from $\{\pi^0_1, ..., \pi^0_{r+1-a}\}$. Let A be the first point (looking from the left) where a deviation of π_k from π^0_k occurs, for some k. This implies that immediately to the right of A, the path π_k is below π^0_k and stays on a pseudo-line of

Arr($\mathbf{n}$) that has label $i \leqslant a$. Such pseudo-lines may not intersect pseudo-lines with larger labels in the southwest-northeast direction. Therefore, in order for π_k to eventually converge with π_k^0, the former path has to switch, at some point, to a segment that comes from a pseudo-line with a label $> a$. This segment necessarily lies on a path $\pi_{k'}^0$ with $k' < k$. In order for such a switch to be possible, the path $\pi_{k'}$ should deviate from $\pi_{k'}^0$ (necessarily in the southern direction). Repeating the same argument again, we construct an infinite sequence $k > k' > k'' > \cdots$, thus arriving at a contradiction that proves our claim. ∎

We conclude this section by applying Theorem 2.4.4 to the computation of certain transition maps. Consider the lexicographically minimal reduced word $\mathbf{h}^0 \in R(w_0)$ and the corresponding normal ordering $\mathbf{n}^0 = \mathbf{n}(\mathbf{h}^0)$. They are given by

$$\mathbf{h}^0 = (1, 2, 1, 3, 2, 1, ..., r, r-1, ..., 1); \tag{2.4.12}$$

$$\begin{aligned}\mathbf{n}^0 = (&(r, r+1), (r-1, r+1), (r-1, r), ...,\\ &(1, r+1), (1, r), ..., (1, 2)).\end{aligned} \tag{2.4.13}$$

2.4.6. Theorem. *For any normal ordering* $\mathbf{n}$ *of* Π, *the transition map from* $\mathbf{n}$ *to* $\mathbf{n}^0$ *is given by*

$$(R_{\mathbf{n}}^{\mathbf{n}^0}(t))_{ij} = \frac{(\Delta^{[j,\, r+1]}_{[j-i,\, r+1-i]})^{\mathbf{n}}(t)(\Delta^{[j+1,\, r+1]}_{[j+2-i,\, r+2-i]})^{\mathbf{n}}(t)}{(\Delta^{[j,\, r+1]}_{[j+1-i,\, r+2-i]})^{\mathbf{n}}(t)(\Delta^{[j+1,\, r+1]}_{[j+1-i,\, r+1-i]})^{\mathbf{n}}(t)}. \tag{2.4.14}$$

Note that an explicit combinatorial expression for each minor appearing on the right-hand side of (2.4.14) is given by (2.4.10).

Proof. Let us recall from Section 2.3 the definition of transition maps $R_{\mathbf{n}}^{\mathbf{n}'}: P^{\Pi} \to P^{\Pi}$ and that of functions $f^{\mathbf{n}}$. Then we see that it suffices to prove (2.4.14) for $\mathbf{n} = \mathbf{n}^0$, where it takes the form

$$t_{ij} = \frac{(\Delta^{[j,\, r+1]}_{[j-i,\, r+1-i]})^{\mathbf{n}^0}(t)(\Delta^{[j+1,\, r+1]}_{[j+2-i,\, r+2-i]})^{\mathbf{n}^0}(t)}{(\Delta^{[j,\, r+1]}_{[j+1-i,\, r+2-i]})^{\mathbf{n}^0}(t)(\Delta^{[j+1,\, r+1]}_{[j+1-i,\, r+1-i]})^{\mathbf{n}^0}(t)}. \tag{2.4.15}$$

The minors on the right-hand side of (2.4.15) are given by the following lemma.

2.4.7. Lemma. *For any two indices* a *and* b *such that* $1 \leqslant a < b \leqslant r$, *we have*

$$(\Delta^{[b+1,\, r+1]}_{[b+1-a,\, r+1-a]})^{\mathbf{n}^0}(t) = \prod_{i=1}^{a} \prod_{j=b+1}^{r+1} t_{ij}. \tag{2.4.16}$$

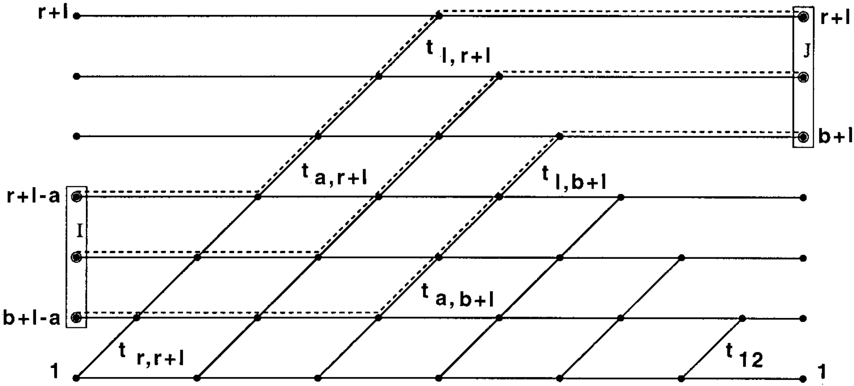


FIG. 7. Graph $\Gamma(\mathbf{n}^0)$.

Proof of Lemma 2.4.7. The proof is very similar to that of Corollary 2.4.5. The graph $\Gamma(\mathbf{n}^0)$ is shown in Fig. 7 (it is drawn in the "compressed" form, with irrelevant horizontal segments contracted).

Direct inspection shows that there is a unique family of $r+1-b$ vertex-disjoint paths $\{\pi_1, ..., \pi_{r+1-b}\}$ in $\Gamma(\mathbf{n}^0)$ that join the sources $s_{b+1-a}, ..., s_{r+1-a}$ with the sinks $S_{b+1}, ..., S_{r+1}$, and the product of weights of these paths is exactly the right-hand side of (2.4.16). ∎

To complete the proof of Theorem 2.4.6, it remains to substitute the expressions given by Lemma 2.4.7 into the right-hand side of (2.4.15), and perform the cancellation. ∎

As an immediate consequence of (2.4.15), we obtain the following statement that shows the relevance of the nil–Temperley–Lieb algebra to the study of the Lusztig variety.

2.4.8. COROLLARY. *The map* $\mathbf{t} \mapsto X(\mathbf{t})$ *from the Lusztig variety* $\mathscr{L}_r(P)$ *to the algebra* $NTL_r(P)$ *is infective. Equivalently, the functions* $\mathbf{t} \mapsto \Delta_I^J(\mathbf{t})$ *separate the points of the Lusztig variety.*

2.5. *Chamber Ansatz*

In this section, we describe how elements of the Lusztig variety can be constructed by means of a special substitution that involves variables indexed by all subsets of the set $[1, r+1]$. To describe this construction, we will need the following notation (cf. (1.6)).

Let $\mathbf{n}$ be a normal ordering of Π. Define

$$L = L^{\mathbf{n}}(i, j)\{k\colon \text{Line}_k \text{ passes below the intersection of } \text{Line}_i \text{ and } \text{Line}_j \text{ in } \operatorname{Arr}(\mathbf{n})\}. \tag{2.5.1}$$

For instance, in Fig. 4, $L^{\mathbf{n}}(1,3)=\{2,4\}$ which we will simply write as 24. One can rewrite (2.5.1) directly in terms of the normal ordering $\mathbf{n}$:

$$L^{\mathbf{n}}(i,j)=\{a\colon i\neq a<j,\, (a,j)\prec_{\mathbf{n}}(i,j)\}\cup\{b\colon j<b,\, (i,j)\prec_{\mathbf{n}}(i,b)\} \tag{2.5.2}$$

where $\prec_{\mathbf{n}}$ stands for "precedes in $\mathbf{n}$." This definition can also be restated in terms of the reduced word $\mathbf{h}$ corresponding to $\mathbf{n}$ (see (1.6)).

Let $M_J (J\subset[1,r+1])$ be a family of variables with values in a semifield K. The *Chamber Ansatz* substitution is defined by

$$t_{ij}^{\mathbf{n}}=\frac{M_L M_{Lij}}{M_{Li}M_{Lj}} \tag{2.5.3}$$

where $L=L^{\mathbf{n}}(i,j)$ is given by (2.5.1), and Li, Lj, and Lij stand for $L\cup\{i\}$, $L\cup\{j\}$, and $L\cup\{i,j\}$, respectively. In our running example (see Fig. 4),

$$t_{23}=\frac{M_4M_{234}}{M_{24}M_{34}},\qquad t_{24}=\frac{M_\phi L_{24}}{M_2M_4},\qquad t_{13}=\frac{M_{24}M_{1234}}{M_{124}M_{234}},$$

$$t_{14}=\frac{M_2M_{124}}{M_{12}M_{24}},\qquad t_{34}=\frac{M_{12}M_{1234}}{M_{123}M_{124}},\qquad t_{12}=\frac{M_\phi M_{12}}{M_1M_2}.$$

2.5.1. Proposition. *The point $\mathbf{t}=(t_{ij}^{\mathbf{n}})$ whose components are defined by the Chamber Ansatz* (2.5.3) *belongs to the Lusztig variety $\mathcal{L}_r(K)$ if and only if the M_J satisfy the relations*

$$M_{Lik}M_{Lj}=M_{Lij}M_{Lk}+M_{Ljk}M_{Li} \tag{2.5.4}$$

for every three indices $i<j<k$ in $[1,r+1]$ and every subset $L\subset[1,r+1]$ such that $L\cap\{i,j,k\}=\phi$.

Proof. The fact that $\mathbf{t}^{\mathbf{n}'}=t^{\mathbf{n}}$ whenever $\mathbf{n}$ and $\mathbf{n}'$ differ by a 2-move, is obvious from (2.5.3). It remains to check that the 3-move relations (2.3.3) translate into (2.5.4). Suppose $\mathbf{n}'$ is obtained from $\mathbf{n}$ by a 3-move applied to consecutive entries (i,j), (i,k), and (j,k) where $i<j<k$. The set

$$L=\{l\colon \text{Line}_l \text{ passes below the triangle formed by } \text{Line}_i,\ \text{Line}_j, \text{ and } \text{Line}_k\} \tag{2.5.5}$$

is clearly the same for $\mathbf{n}$ and $\mathbf{n}'$, as shown in Fig. 8.

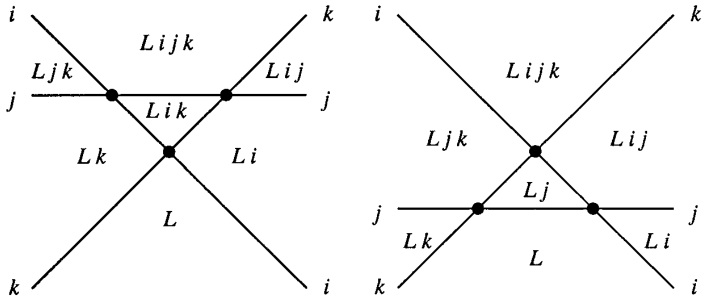


FIG. 8. Transformation of the chamber sets under a 3-move.

Denoting $t = t^{\mathbf{n}}$ and $t' = t^{\mathbf{n}'}$, we see that (2.5.3) takes the form

$$t_{ij} = \frac{M_{Lk} M_{Lijk}}{M_{Lik} M_{Ljk}}, \qquad t_{ik} = \frac{M_L M_{Lik}}{M_{Li} M_{Lk}}, \qquad t_{jk} = \frac{M_{Li} M_{Lijk}}{M_{Lij} M_{Lik}},$$

$$t'_{ij} = \frac{M_L M_{Lij}}{M_{Li} M_{Lj}}, \qquad t'_{ik} = \frac{M_{Lj} M_{Lijk}}{M_{Lij} M_{Ljk}}, \qquad t'_{jk} = \frac{M_L M_{Ljk}}{M_{Lj} M_{Lk}}.$$

Substituting these expressions into (2.3.3) and clearing denominators yields (2.5.4). To complete the proof, it remains to note that, for every i, j, k and L as in Proposition 2.5.1, there exists a normal ordering $\mathbf{n}$, in which (i, j), (i, k), and (j, k) are consecutive, and L is given by (2.5.5). ∎

We shall refer to (2.5.4) as the 3-*Term Relations*. Examples of their solutions will be given in the next section. We will then show in Section 2.7 that every point of the Lusztig variety $\mathscr{L}_r(K)$ (and thus of its subvariety $\mathscr{L}_r(P)$) can be obtained via the Chamber Ansatz.

The name of our Ansatz comes from the following reformulation. *Chambers* of an arrangement $\mathrm{Arr}(\mathbf{n})$ are the connected components of the complement to the union of all pseudo-lines. It is easy to see that every arrangement has $\binom{r}{2}$ bounded and $2r+2$ unbounded chambers. With a chamber C, we associate the *chamber set*

$$L(C) = \{k\colon \mathrm{Line}_k \text{ passes below } C\} \tag{2.5.6}$$

(see Fig. 4). If A, B, C, and D are the four chambers adjacent to the intersection of Line_i and Line_j in $\mathrm{Arr}(\mathbf{n})$ and listed counterclockwise, A being the chamber below t_{ij}, then (2.5.3) can be rewritten as

$$t_{ij}^{\mathbf{n}} = \frac{M_{L(A)} M_{L(C)}}{M_{L(B)} M_{L(D)}}. \tag{2.5.7}$$

2.5.2. *Remark*. The chamber sets of $\mathbf{n}$ only depend on the isotopy class of the pseudo-line arrangement $\mathrm{Arr}(\mathbf{n})$ (cf. Remark 2.3.2). It is not hard to show that, conversely, the isotopy class of $\mathrm{Arr}(\mathbf{n})$ is uniquely determined by its family of chamber sets. A characterization of families of subsets that appear as chamber sets in the same arrangement will be given in a forthcoming paper by B. Leclerc and A. Zelevinsky.

2.6. *Solutions of the* 3-*Term Relations*

Note that the 3-Term Relations (2.5.4) do not involve division and thus make sense when P is any commutative ring or, more generally, any commutative semiring; for this purpose, we do not need the condition (2.1.1).

2.6.1. Definition. Let P be a commutative ring and y be an $(r+1)\times(r+1)$ matrix with entries in P. Let J be a subset of $[1, r+1]$ of size l. As in Section 2.4, $\Delta^J(y)$ will denote the corresponding *flag minor* of y, i.e., the minor with column set J and row set $[1, l]$.

2.6.2. Proposition. *The flag minors $\Delta^J = \Delta^J(y)$ of any square matrix y satisfy the* 3-*Term Relations* (2.5.4):

$$\Delta^{Lik}\Delta^{Lj} = \Delta^{Lij}\Delta^{Lk} + \Delta^{Ljk}\Delta^{Li} \tag{2.6.1}$$

where, as before, $i<j<k$ and $L\cap\{i, j, k\} = \phi$.

Proof. Equation (2.6.1) is a special case of the classical Plücker relations (see, e.g., [18, (15.53)]). ∎

The previous construction may not be directly used for an arbitrary semiring P because the calculation of minors involves subtraction. We will now bypass this problem by introducing "surrogate minors" via vertex-disjoint path families, in the spirit of Section 2.4.

2.6.3. Example. Let Γ be a planar acyclic directed graph with the set of vertices V and the set of edges E. Suppose Γ has sources $s_1, ..., s_{r+1}$ and sinks $S_1, ..., S_{r+1}$. We also assume that, as before, Γ is contained in a vertical strip, with sources and sinks on its left and right boundaries, respectively, both numbered bottom-up. Let $w\colon E\to P$ be a function with values in an arbitrary semiring P. Define the *weight* $w(\pi)$ of an oriented path π as the product of the weights $w(e)$ for all edges e of π. For any subsets $I, J\subset[1, r+1]$ of the same size l, let

$$\Delta_I^J = \Delta_I^J(\Gamma, w) = \sum_{\pi_1, ..., \pi_l} w(\pi_1)\cdots w(\pi_l) \tag{2.6.2}$$

where the sum is over all families of l vertex-disjoint paths $\{\pi_1, ..., \pi_l\}$, each path connecting a source s_i, $i \in I$, with a sink S_j, $j \in J$. As before, we will use the notation

$$\Delta^J = \Delta^J_{[1, l]}. \tag{2.6.3}$$

2.6.4. THEOREM. *For any planar graph Γ and weight function w as above, the elements Δ^J defined by* (2.6.2)–(2.6.3) *satisfy the* 3-*Term Relations* (2.6.1).

Proof. For each L, i, j, and k, both sides of (2.6.1) are *universal* (that is, independent of the ground semiring P) polynomials with nonnegative integer coefficients in the variables $w(e)$, $e \in E$. By Proposition 2.1.7, it is enough to prove (2.6.1) in the case when the underlying semiring is $\mathbb{R}_{>0}$. In this case, the Lindstrom lemma [27] asserts that Δ^J_I is simply a minor (with the row set I and column set J) of the matrix (a_{ij}) defined by

$$a_{ij} = \sum_{\pi} w(\pi) \tag{2.6.4}$$

where the sum is over all paths π connecting the source s_i with the sink S_j. Thus, for $P = \mathbb{R}_{>0}$, Theorem 2.6.4 becomes a special case of Proposition 2.6.2. ∎

Formula (2.6.2) shows that, for a nonnegative weight function w, the matrix (a_{ij}) defined by (2.6.4) has nonnegative minors. This observation is by no means new; it has been used to prove total positivity of various matrices arising in combinatorics (see [6]).

Theorem 2.6.4 can be applied to the graph $\Gamma(\mathbf{n})$ and the weight function w constructed in Section 2.4. By Theorem 2.4.4, the polynomial $\Delta^J_I(\Gamma(\mathbf{n}), w)$ coincides with $(\Delta^J_I)^{\mathbf{n}}$. Thus, we obtain the following corollary.

2.6.5. COROLLARY. *For any normal ordering* $\mathbf{n}$ *of Π, the polynomials* $(\Delta^J)^{\mathbf{n}}$ *satisfy the* 3-*Term Relations* (2.6.1).

2.6.6. EXAMPLE. We will now construct a solution of the 3-Term Relations (2.5.4) from an arbitrary family

$$t = (t_{ij} \colon 1 \leqslant i < j \leqslant r+1) \tag{2.6.5}$$

of elements of the ground semiring P (in the notation of Section 2.3, $t \in P^{\Pi}$).

For a subset $J = \{j_1 < \cdots < j_l\} \subset [1, r+1]$, we define a *J-tableau* as an upper-triangular matrix $A = (a_{pq})_{1 \leqslant p \leqslant q \leqslant l}$ with integer entries satisfying

$$a_{pp} = j_p, \qquad p = 1, ..., l \tag{2.6.6}$$

and the usual monotonicity conditions for semistandard, or column-strict Young tableaux (see, e.g., [31]):

$$a_{pq} \leqslant a_{p,\,q+1}, \qquad a_{pq} < a_{p+1,\,q}. \tag{2.6.7}$$

Then let

$$M_J = M_J(t) = \sum_A \prod_{1 \leqslant p < q \leqslant l} t_{a_{pq},\, a_{pq}+q-p} \tag{2.6.8}$$

where the sum is over all J-tableaux $A = (a_{pq})$.

To illustrate the formula (2.6.8), consider an example $J = \{1, 3, 4\}$. The J-tableaux are:

$$\begin{bmatrix} 1 & 1 & 1 \\ & 3 & 3 \\ & & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 1 & 2 \\ & 3 & 3 \\ & & 4 \end{bmatrix}, \quad \begin{bmatrix} 1 & 2 & 2 \\ & 3 & 3 \\ & & 4 \end{bmatrix}. \tag{2.6.9}$$

To compute M_J, we ignore the diagonal entries in these matrices and replace each entry a in the dth diagonal above the main one by the variable $t_{a,\,a+d}$. Then sum up the products of entries:

$$M_J = t_{12} t_{13} t_{34} + t_{12} t_{24} t_{34} + t_{23} t_{24} t_{34}. \tag{2.6.10}$$

2.6.7. Proposition. *For any family $t \in P^{\Pi}$, the elements M_J defined by* (2.6.8) *satisfy the* 3-*Term Relations* (2.5.4).

Proof. We shall construct a planar graph Γ and weight function w (see Fig. 9) such that the elements Δ^J given by (2.6.3) will coincide with the expressions (2.6.8); the proposition will then follow from Theorem 2.6.4. The vertices of Γ are lattice points (i, j) with $1 \leqslant i \leqslant j \leqslant r+1$. The edges are of two types: the vertical segments directed from (i, j) to $(i+1, j)$ and the diagonal ones from (i, j) to $(i+1, j+1)$. The weight $w(e)$ of any diagonal edge e is 1; the weight of a vertical edge from (i, j) to $(i+1, j)$ is set to be t_{ij}. The sources are $s_i = (1, i)$ and the sinks are $S_j = (j, j)$. Although this graph does not exactly fall under the description of Example 2.6.3 (the location of sources and sinks is somewhat different), one can easily see that this discrepancy is irrelevant.

For $J = \{j_1 < \cdots < j_l\}$, consider a system $\pi_1, ..., \pi_l$ of vertex-disjoint paths connecting the sources $s_1, ..., s_l$ with the sinks $S_{j_1}, ..., S_{j_l}$, respectively. To such a system, we associate a J-tableau $A = (a_{pq})$ as follows:

$$a_{pq} = \max\{i : (i, i+q-p) \in \pi_q\}.$$

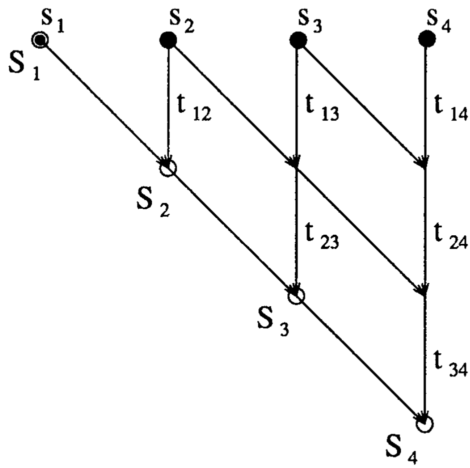


FIG. 9. Graph Γ used in the interpretation of J-tableaux.

It is straightforward to check that this formula establishes a bijective correspondence between systems of paths as above and J-tableaux. Under this bijection, the weight $w(\pi_1) \cdots w(\pi_l)$ of a system of paths equals the product $\prod t_{a_{pq},\, a_{pq}+q-p}$ in (2.6.8) for the corresponding J-tableau A. This completes the proof. ▮

The correspondence described above is illustrated in Fig. 10; the three systems of paths shown correspond to the three tableaux of (2.6.9), in the same order.

2.6.8. *Remark.* The J-tableaux defined above are closely related to the Gelfand–Tsetlin patterns (see, e.g., [7, Section 8]). Replacing each entry a_{pq} by $a_{pq}-p+1$ converts a J-tableau into a matrix that can be viewed as a Gelfand–Tsetlin pattern whose highest weight is

$$\lambda = (j_l - l + 1, ..., j_2 - 1, j_1).$$

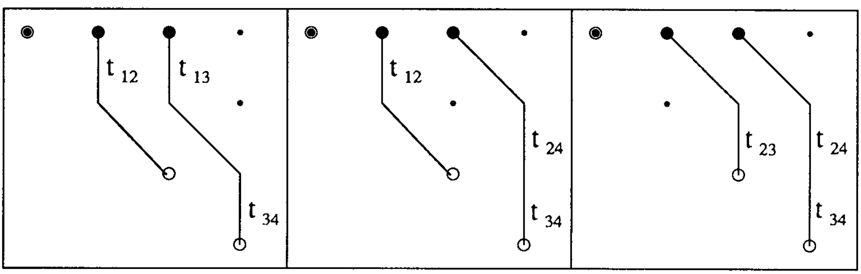


FIG. 10. Path families corresponding to J-tableaux in (2.6.9).

Such GT-patterns are in a natural bijection with semi-standard Young tableaux of shape λ; their number (thus the number of summands in (2.6.8)) is the dimension of the irreducible representation of GL_l with highest weight λ.

2.7. *An Alternative Description of the Lusztig Variety*

Let $\tilde{\mathcal{M}} = \tilde{\mathcal{M}}_r(K)$ be the set of all tuples $\mathbf{M} = (M_J)_{J \subset [1, r+1]}$ of elements of the ground semifield K satisfying the 3-Term Relations (2.5.4). According to Proposition 2.5.1, the Chamber Ansatz (2.5.3) provides a well-defined map $\mathbf{M} \mapsto \mathbf{t}(\mathbf{M})$ from $\tilde{\mathcal{M}}_r(K)$ to $\mathcal{L}_r(K)$.

Let $\mathcal{M} = \mathcal{M}_r(K)$ be the subset of $\tilde{\mathcal{M}}$ formed by those tuples $\mathbf{M} = (M_J)$ that, in addition to (2.5.4), satisfy the normalization condition

$$M_\phi = 1, \qquad M_{[1,b]} = 1, \qquad b = 1, ..., r+1. \tag{2.7.1}$$

2.7.1. Theorem. *The restriction of the Chamber Ansatz map* $\mathbf{M} \mapsto \mathbf{t}(\mathbf{M})$ *onto* $\mathcal{M}_r(K)$ *is a bijection between* $\mathcal{M}_r(K)$ *and the Lusztig variety* $\mathcal{L}_r(K)$. *The inverse bijection* $\mathbf{t} = (t_{ij}^{\mathbf{n}}) \mapsto \mathbf{M}(\mathbf{t}) = (M_J)$ *from* $\mathcal{L}_r(K)$ *to* $\mathcal{M}_r(K)$ *is given by*

$$M_J = M_J(\mathbf{t}) = \left(\prod_{i \notin J,\, j \in J,\, i<j} t_{ij}^{\mathbf{n}} \right)^{-1} \tag{2.7.2}$$

whenever J *is a chamber set for* $\mathbf{n}$.

Proof. First, we observe that each subset $J \subset [1, r+1]$ appears as a chamber set $L(C)$ in some arrangement $\mathrm{Arr}(\mathbf{n})$. To prove Theorem 2.7.1, it is enough to verify the following statements:

(i) the M_J are well defined via (2.7.2), that is, for a given J, the right-hand side of (2.7.2) does not depend on the choice of $\mathbf{n}$ such that J is a chamber set for $\mathbf{n}$;

(ii) the M_J given by (2.7.2) satisfy (2.5.4) and (2.7.1) (i.e., $\mathbf{M} \in \mathcal{M}_r(K)$);

(iii) the maps given by (2.5.3) and (2.7.2) are inverse to each other.

We start with the proof of (i).

2.7.2. Lemma. *Suppose* J *is a chamber set for both* $\mathbf{n}$ *and* $\mathbf{n}'$. *Then one can convert* $\mathbf{n}$ *into* $\mathbf{n}'$ *by a sequence of* 2- *and* 3-*moves, keeping* J *as a chamber set at all times.*

Proof. Let $\mathbf{h}$ be the reduced word associated with $\mathbf{n}$. Consider the pseudo-line arrangement $\mathrm{Arr}(\mathbf{n})$. Draw a vertical line l just before the rightmost point of the chamber whose chamber set is J (we will later refer to it as the J-chamber). Line l partitions the reduced word $\mathbf{h}$ into two parts

$\mathbf{h}_1$ and $\mathbf{h}_2$ which lie, respectively, to the left and to the right of l. For example, if $\mathbf{h} = 213231$, as in Figs. 3 and 4, and $J = \{2, 3, 4\}$, then $\mathbf{h}_1 = 21$ and $\mathbf{h}_2 = 3231$. Let w_1 and w_2 be the permutations whose reduced words are $\mathbf{h}_1$ and $\mathbf{h}_2$.

Let us now apply 2- and 3-moves to the reduced word $\mathbf{h}_2$, keeping $\mathbf{h}_1$ unchanged. In this way, we can transform $\mathbf{h}_2$ into an arbitrary reduced word $\mathbf{h}'_2$ for w_2. The concatenation of $\mathbf{h}_1$ and $\mathbf{h}'_2$ is again a reduced word for w_0, with J as a chamber set.

One can view a reduced word for w_2 as a process of sorting the sequence obtained by reading bottom-up the numbers of pseudo-lines intersecting the vertical line l. This sequence begins with the elements of J (in some order), followed by some permutation of the complement $\bar{J}$. It is clearly possible to begin the sorting process by completely sorting out the elements of J, then sort the elements of the complement, and finally make all necessary switches between the elements of J and $\bar{J}$. Let $\tilde{\mathbf{h}}_2$ denote the reduced word for w_2 that corresponds to the described process.

As shown above, we can convert $\mathbf{h}$ into the concatenation $\tilde{\mathbf{h}}$ of $\mathbf{h}_1$ and $\tilde{\mathbf{h}}_2$, keeping J as a chamber set at all times. In the arrangement for $\tilde{\mathbf{h}}$, let us consider the vertical line $\tilde{l}$ chosen in the same way as l above, i.e., passing just before the rightmost point of the J-chamber. By our construction, the part of $\tilde{\mathbf{h}}$ lying to the right of $\tilde{l}$ is a reduced word of a certain permutation w_J that canonically corresponds to J; namely, w_J sends J to $[1, |J|]$, and $\bar{J}$ to $[|J| + 1, r + 1]$, and is increasing on both J and $\bar{J}$. Separately applying 2- and 3-moves to the parts of $\tilde{\mathbf{h}}$ lying to the left and to the right of $\tilde{l}$, we can transform $\tilde{\mathbf{h}}$ to an arbitrary reduced word for $w_0 w_J^{-1}$ followed by an arbitrary reduced word for w_J; clearly, J will remain as a chamber set under all these operations.

Since the same construction can be applied to the reduced word $\mathbf{h}'$ associated with $\mathbf{n}'$, our statement follows. ∎

2.7.3. Lemma. *Assume* $\mathbf{n}$ *and* $\mathbf{n}'$ *differ by a single* 2- *or* 3-*move, and* J *is a chamber set for both* $\mathbf{n}$ *and* $\mathbf{n}'$. *Then*

$$\prod_{i \notin J,\ j \in J,\ i < j} t_{ij}^{\mathbf{n}} = \prod_{i \notin J,\ j \in J,\ i < j} t_{ij}^{\mathbf{n}'}. \tag{2.7.3}$$

Proof. First note than a 2-move does not change any of the t_{ij}. A 3-move involving pseudo-lines Line_i, Line_j, and Line_k ($i < j < k$) only changes the variables t_{ij}, t_{ik} and t_{jk} according to (2.3.3). Among these three variables, the products in (2.7.3) either contain none, or t_{ij} and t_{ik}, or t_{ik} and t_{jk}. In each of these cases, the identity (2.7.3) holds, since, in (2.3.3), $t_{ij} t_{ik} = t'_{ij} t'_{ik}$ and $t_{ik} t_{jk} = t'_{ik} t'_{jk}$. ∎

Statement (i) is an immediate consequence of the last two lemmas. To prove (ii), note that the normalization condition (2.7.1) is obvious, since the corresponding products in (2.7.2) are empty. To prove (2.5.4), we use the following statement which can be checked by straightforward calculation:

(iv) the M_J given by (2.7.2) satisfy the Ansatz (2.5.3).

Now the 3-Term Relations (2.5.4) follow from Proposition 2.5.1. This proves (ii), i.e., we have verified that (2.7.2) gives a well defined map $\mathscr{L} \to \mathscr{M}$.

According to (iv), the composition $\mathscr{L} \to \mathscr{M} \to \mathscr{L}$ is the identity map on the Lusztig variety. Thus, the Chamber Ansatz map $\mathbf{M} \mapsto \mathbf{t}(\mathbf{M})$ is a surjection $\mathscr{M} \to \mathscr{L}$. To complete the proof of (iii), it remains to prove that the composition $\mathscr{M} \to \mathscr{L} \to \mathscr{M}$ is the identity map on $\mathscr{M}$. Instead of proving this by a direct computation (which is somewhat cumbersome) we can use the following "qualitative" argument. Clearly, it is enough to show that the Chamber Ansatz map $\mathbf{M} \mapsto \mathbf{t}(\mathbf{M})$ is an injection $\mathscr{M} \to \mathscr{L}$. In other words, it is enough to show that the coordinates $M_{L(C)}$ for all chambers C in the arrangement $\mathrm{Arr}(\mathbf{n})$ are uniquely determined by the coordinates $t_{ij}^{\mathbf{n}}$ of $\mathbf{t} = \mathbf{t}(\mathbf{M})$. First note that, in view of (2.7.1), $M_{L(C)} = 1$ for the chambers adjacent to the right boundary. Now the remaining values $M_{L(C)}$ can be computed recursively, by moving from the right boundary to the left one and repeatedly using (2.5.7).

This completes the proof of Theorem 2.7.1. ∎

2.7.4. COROLLARY. *For each* $\mathbf{n}$, *the components* $M_{L(C)}$ *for all chambers* C *in the arrangement* $\mathrm{Arr}(\mathbf{n})$ *that are not adjacent to the right boundary, form a system of independent coordinates on* $\mathscr{M} = \mathscr{M}_r(K)$, *i.e., they can be assigned arbitrary values in* K, *and the remaining components* M_J *of a point* $\mathbf{M} \in \mathscr{M}$ *are expressed through them as subtraction-free rational expressions.*

Proof. Let $\mathbf{t} \in \mathscr{L}$ and $\mathbf{M} \in \mathscr{M}$ correspond to each other as in Theorem 2.7.1. By (2.5.3) and (2.7.2), for each $\mathbf{n}$, the coordinates $t_{ij}^{\mathbf{n}}$ of $\mathbf{t}$ and the components $M_{L(C)}$ of $\mathbf{M}$ for all chambers C in the arrangement $\mathrm{Arr}(\mathbf{n})$ that are not adjacent to the right boundary, are related to each other by an invertible monomial transformation. On the other hand, by Theorem 2.2.6, the components $t_{ij}^{\mathbf{n}}$ for a given $\mathbf{n}$ form a system of independent coordinates on $\mathscr{L}$. This proves the claim. ∎

We shall later refine Corollary 2.7.4 by showing that, for each $\mathbf{n}$, the M_J are Laurent polynomials with nonnegative integer coefficients in the variables $t_{ij}^{\mathbf{n}}$ (or, equivalently, in the corresponding variables $M_{L(C)}$).

We conclude this section with a more symmetric description of the set $\mathcal{M}_r(K)$. Let $H = K^{r+2}$ be the multiplicative group of sequences

$$c = (c_0, c_1, ..., c_{r+1}), \qquad c_j \in K.$$

It follows from (2.5.4) that one can define an action of H on $\tilde{\mathcal{M}}_r(K)$ by

$$(c\mathbf{M})_J = \left(c_0 \prod_{j \in J} c_j \right) \cdot M_J. \tag{2.7.4}$$

We denote by $\tilde{\mathcal{M}}/H$ the set of orbits under this action. The following proposition is a straightforward consequence of the definitions and Theorem 2.7.1.

2.7.5. Proposition. **1.** *The set $\mathcal{M}_r(K)$ is a set of representatives of H-orbits in $\tilde{\mathcal{M}}_r(K)$.*

2. *The map $\mathbf{M} \mapsto \mathbf{t}(\mathbf{M})$ is constant on H-orbits, and induces a bijection*

$$\tilde{\mathcal{M}}_r(K)/H \to \mathcal{L}_r(K).$$

Thus one can naturally identify each of the varieties $\mathcal{L}$ and $\mathcal{M}$ with the orbit space $\tilde{\mathcal{M}}/H$.

2.8. *Transition from* $\mathbf{n}^0$

In this section, we will use the Chamber Ansatz for "reversing the direction" in Theorem 2.4.6, i.e., we will compute the transition map from $\mathbf{n}^0$ to an arbitrary normal ordering $\mathbf{n}$ of Π, where $\mathbf{n}^0$ is given by (2.4.13).

For a family $t = (t_{ij}) \in P^{\Pi}$ and a subset $J \subset [1, r+1]$, define $\tilde{M}_J(t)$ by the formula (2.6.8):

$$\tilde{M}_J(t) = \sum_A \prod_{1 \leqslant p < q \leqslant l} t_{a_{pq},\, a_{pq}+q-p}, \tag{2.8.1}$$

where the sum is over all J-tableaux A (see Example 2.6.6).

2.8.1. Proposition. *For any normal ordering $\mathbf{n}$ of Π, the transition map from $\mathbf{n}^0$ to $\mathbf{n}$ is given by*

$$(R^{\mathbf{n}}_{\mathbf{n}^0}(t))_{ij} = \frac{\tilde{M}_L(t)\, \tilde{M}_{Lij}(t)}{\tilde{M}_{Li}(t)\, \tilde{M}_{Lj}(t)} \tag{2.8.2}$$

where $L = L^{\mathbf{n}}(i, j)$ is given by (2.5.1).

Proof. Let $\tilde{\mathbf{M}}(t) = (\tilde{M}_J(t))$. By Proposition 2.6.7, the family $\tilde{\mathbf{M}}(t)$ belongs to $\tilde{\mathcal{M}}_r(K)$ for any $t \in K^{\Pi}$. By Proposition 2.5.1, applying the

Chamber Ansatz to this family produces an element $\mathbf{t} = \mathbf{t}(\tilde{\mathbf{M}}(t))$ of the Lusztig variety $\mathscr{L}_r(K)$. Thus the proposition can be reformulated as follows:

$$\begin{aligned} &\mathbf{t} = \mathbf{t}(\tilde{\mathbf{M}}(t)) \text{ is the unique element of } \mathscr{L}_r(K) \text{ such that} \\ &\quad t_{ij}^{\mathbf{n}^0} = t_{ij} \text{ for all } i \text{ and } j. \end{aligned} \tag{2.8.3}$$

In other words, we only need to establish (2.8.2) for $\mathbf{n} = \mathbf{n}^0$. Using (2.5.1) or (2.5.2), we obtain

$$L^{\mathbf{n}^0}(i, j) = [i+1, j-1]. \tag{2.8.4}$$

Thus, the equality we need to prove takes the form

$$t_{ij} = \frac{\tilde{M}_{[i+1, j-1]}(t)\, \tilde{M}_{[i, j]}(t)}{\tilde{M}_{[i, j-1]}(t)\, \tilde{M}_{[i+1, j]}(t)}. \tag{2.8.5}$$

To check (2.8.5), we apply the definition of the $\tilde{M}_J(t)$ to the case when J is an interval: $J = [i+1, j]$. Then there is only one J-tableau $A = (a_{pq})$, given by

$$a_{pq} = i + p, \qquad p = 1, ..., j-i,$$

and we obtain

$$\tilde{M}_{[i+1, j]}(t) = \prod_{i < a < b \leqslant j} t_{ab}. \tag{2.8.6}$$

To prove (2.8.5), it remains to substitute the expressions given by (2.8.6) into its right-hand side, and perform the cancellation. This completes the proof of Proposition 2.8.1. ∎

It will be convenient for us to simplify (2.8.2) by extracting from each $\tilde{M}_J(t)$ the greatest common divisor of all its monomials. To describe this g.c.d., we need some useful terminology and notation. Any subset $J \subset [1, r+1]$ can be written as a disjoint union of intervals

$$\begin{gathered} J = [a_1+1, b_1] \cup [a_2+1, b_2] \cup \cdots \cup [a_s+1, b_s] \\ (0 \leqslant a_1 < b_1 < a_2 < b_2 < \cdots < a_s < b_s \leqslant r+1); \end{gathered} \tag{2.8.7}$$

the intervals $[a_k+1, b_k]$ will be called the *components* of J. Let $l_k = b_k - a_k$ be the size of the kth component of J, and let $l = l_1 + \cdots + l_s$ be the size of J. Let us subdivide the interval $[1, l]$ into disjoint intervals $I_1, ..., I_s$ of sizes $l_1, ..., l_s$, numbered from left to right. We set

$$E(J) = \bigcup_{1 \leqslant u < v \leqslant s} (I_u \times I_v) \subset [1, l] \times [1, l], \tag{2.8.8}$$

and define the polynomial $Q_J(t)$ by

$$Q_J(t) = \sum_A \prod_{(p,\, q) \in E(J)} t_{a_{pq},\, a_{pq}+q-p}, \tag{2.8.9}$$

where the sum is over all J-tableaux $A = (a_{pq})$. We will call $E(J)$ the *essential set* for J.

To explain this terminology, consider any J-tableau $A = (a_{pq})_{1 \leqslant p < q \leqslant l}$. Conditions (2.6.7) imply that, whenever indices $p < q$ belong to the same interval I_k, we have

$$a_{pq} = a_k + p - (l_1 + l_2 + \cdots + l_{k-1}). \tag{2.8.10}$$

Thus, if a pair (p, q) does not belong to the essential set $E(J)$, then a_{pq} has the same value (2.8.10) for every J-tableau A. Splitting each monomial in $\tilde{M}_J(t)$ into its essential and non-essential parts, we can rewrite (2.8.1) as follows:

$$\tilde{M}_J(t) = \left(\prod_{a<b} t_{ab}^{e(J;\, a,\, b)} \right) \cdot Q_J(t), \tag{2.8.11}$$

where

$$e(J;\, a,\, b) = \begin{cases} 1, & \text{if } a \text{ and } b \text{ lie in the same component of } J; \\ 0, & \text{otherwise.} \end{cases} \tag{2.8.12}$$

We are now in a position to reformulate Proposition 2.8.1.

2.8.2. THEOREM. *For any normal ordering* $\mathbf{n}$ *of* Π, *the transition map from* $\mathbf{n}^0$ *to* $\mathbf{n}$ *is given by*

$$(R_{\mathbf{n}^0}^{\mathbf{n}}(t))_{ij} = \left(\prod_{[i,\, j] \subset [a,\, b]} t_{ab}^{e(Lij;\, a,\, b)} \right) \frac{Q_L(t)\, Q_{Lij}(t)}{Q_{Li}(t)\, Q_{Lj}(t)} \tag{2.8.13}$$

where $L = L^{\mathbf{n}}(i, j)$, *and the* $Q_J(t)$ *are given by* (2.8.9).

Proof. Substitute the expressions given by (2.8.11) into (2.8.2) and compare the resulting expression with (2.8.13). To prove our theorem, we only need to check that the exponent

$$e(Lij;\, a,\, b) + e(L;\, a,\, b) - e(Li;\, a,\, b) - e(Lj;\, a,\, b)$$

equals $e(Lij;\, a,\, b)$ if $[i, j] \subset [a, b]$, and vanishes otherwise. This is straightforward. ∎

2.8.3. Example. If $J=[a+1, b]$ is an interval, then the essential set $E(J)$ is empty, and there exists exactly one J-tableau given by (2.8.10). Thus

$$Q_{[a+1, b]}=1. \tag{2.8.14}$$

Now consider a set $J=[a+1, b] \cup [c+1, d]$ with two components. Then $E(J)$ is a "rectangle" $[1, b-a] \times [b-a+1, b-a+d-c]$. The shift of indices $(p, q) \mapsto (p, q-b+a)$ turns $E(J)$ into $[1, b-a] \times [1, d-c]$, while the shift $a_{pq} \mapsto a_{pq}-a$ converts any J-tableau into a semi-standard Young tableau with entries in $[1, c-a]$. Performing both shifts, we can rewrite Q_J as follows:

$$Q_{[a+1, b] \cup [c+1, d]}(t)=\sum_{\tau} \prod_{(p, q) \in [1, b-a] \times [1, d-c]} t_{\tau(p, q)+a,\, \tau(p, q)+b+q-p}, \tag{2.8.15}$$

where the sum is over all Young tableaux τ: $[1, b-a] \times [1, d-c] \rightarrow [1, c-a]$.

For instance, if $J=\{1, 3, 4\}$ (cf. example (2.6.9)), then the Young tableaux contributing to (2.8.15) are $\boxed{1 \mid 1}$, $\boxed{1 \mid 2}$, and $\boxed{2 \mid 2}$, yielding

$$Q_J(t)=t_{12} t_{13}+t_{12} t_{24}+t_{23} t_{24}$$

(cf. (2.6.10)).

As explained in the Introduction, Theorems 2.4.6 and 2.8.1 allow us to compute any transition map $R_{\mathbf{n}}^{\mathbf{n}'}$ as the composition $R_{\mathbf{n}^0}^{\mathbf{n}'} \circ R_{\mathbf{n}}^{\mathbf{n}^0}$. It is desirable, however, to find a direct formula for $R_{\mathbf{n}}^{\mathbf{n}'}$. Some partial results in this direction will be obtained in the next chapters.

2.9. *Polynomials $T_J^{\mathbf{n}}$ and Z_a*

In this section, we develop a general framework that will be used in subsequent computations of transition maps.

According to Theorem 2.7.1, the Chamber Ansatz identifies the Lusztig variety $\mathscr{L}_r(K)$ with the set $\mathscr{M}_r(K)$ of tuples (M_J) satisfying the 3-Term Relations (2.5.4) and the normalization conditions (2.7.1). If we use this identification, each component M_J becomes a function on $\mathscr{L}_r(K)$ with values in K. For any normal ordering $\mathbf{n}$ of Π, we will denote by $M_J^{\mathbf{n}}$ the corresponding function $K^{\Pi} \rightarrow K$, which describes M_J in terms of the coordinates $t=(t_{ij})$ associated with $\mathbf{n}$ (see Section 2.3). For example, if J is a chamber set for $\mathbf{n}$, then, in accordance with (2.7.2),

$$M_J^{\mathbf{n}}(t)=\left(\prod_{i \notin J,\, j \in J,\, i<j} t_{ij}\right)^{-1}. \tag{2.9.1}$$

In particular,

$$M^{\mathbf{n}}_{[a+1,\, r+1]}(t) = \left(\prod_{i \leqslant a < j} t_{ij} \right)^{-1}, \qquad a = 1, ..., r \tag{2.9.2}$$

for any $\mathbf{n}$ whatsoever.

Using this notation, we can reformulate Theorem 2.7.1 as a statement about transition maps, as follows.

2.9.1. THEOREM. *For any two normal orderings $\mathbf{n}$ and $\mathbf{n}'$ of Π, the transition map from $\mathbf{n}$ to $\mathbf{n}'$ is given by*

$$(R^{\mathbf{n}'}_{\mathbf{n}}(t))_{ij} = \frac{M^{\mathbf{n}}_{L'}(t)\, M^{\mathbf{n}}_{L'ij}(t)}{M^{\mathbf{n}}_{L'i}(t)\, M^{\mathbf{n}}_{L'j}(t)} \tag{2.9.3}$$

where $L' = L^{\mathbf{n}'}(i, j)$.

In view of this theorem, in order to obtain a direct formula for $R^{\mathbf{n}'}_{\mathbf{n}}$, it would be enough to compute the functions $M^{\mathbf{n}}_J$ for all chamber sets J for $\mathbf{n}'$.

One inconvenience of this approach is that the functions $M^{\mathbf{n}}_J$ are not polynomials in the t_{ij} (see, e.g., (2.9.1)). We will now introduce another system of coordinates on $\mathcal{M}_r(K)$ which will not have this drawback. Namely, for $a = 1, ..., r$ and $J \subset [1, r+1]$, we define the functions Z_a and T_J on $\mathcal{M}_r(K)$ by

$$Z_a = \frac{1}{M_{[a+1,\, r+1]}}, \qquad T_J = \frac{M_J}{\prod_{a \notin J,\, a+1 \in J} M_{[a+1,\, r+1]}}. \tag{2.9.4}$$

The collection of all Z_a and T_J determines all the components M_J as follows:

$$M_J = \frac{T_J}{\prod_{a \notin J,\, a+1 \in J} Z_a}, \tag{2.9.5}$$

which shows that the systems of functions (M_J) and (Z_a, T_J) are related to each other by an invertible monomial transformation. Thus, the Z_a and the T_J, taken together, can be used as coordinates on $\mathcal{M}_r(K)$. We will now restate the definition of $\mathcal{M}_r(K)$ in terms of these coordinates.

2.9.2. PROPOSITION. *The 3-Term Relations for the components M_J of an element $\mathbf{M} \in \mathcal{M}_r(K)$ translate into the following relations for the Z_a and T_J:*

$$T_{Lik} T_{Lj} = Z_i^{\delta_{i+1,\, j}} T_{Lij} T_{Lk} + Z_j^{\delta_{j+1,\, k}} T_{Ljk} T_{Li} \tag{2.9.6}$$

$$T_{[1,\, a]} = T_{[a+1,\, r+1]} = 1 \tag{2.9.7}$$

for any $a \in [0, r+1]$, any $1 \leqslant i < j < k \leqslant r+1$, and any $L \subset [1, r+1]$ such that $L \cap \{i, j, k\} = \phi$.

Rewriting the Chamber Ansatz bijection $\mathcal{M}_r(K) \to \mathcal{L}_r(K)$ given by (2.5.3), in terms of the Z_a and T_J yields

$$t_{ij}^{\mathbf{n}} = Z_i^{\delta_{i+1,j}} \frac{T_L T_{Lij}}{T_{Li} T_{Lj}}, \tag{2.9.8}$$

where the set $L = L^{\mathbf{n}}(i, j)$ is given by (2.5.1) or (2.5.2). Using the inverse bijection $\mathcal{L}_r(K) \to \mathcal{M}_r(K)$, we can regard the Z_a and T_J as functions $\mathcal{L}_r(K) \to K$. This, for every normal ordering $\mathbf{n}$ of Π, gives rise to well-defined functions $T_J^{\mathbf{n}}$ and $Z_a^{\mathbf{n}}$ on K^{Π} with values in K. In view of (2.9.2), the functions $Z_a^{\mathbf{n}}$ do not depend on $\mathbf{n}$. With some abuse of notation, we will simply write $Z_a(t)$ for $Z_a^{\mathbf{n}}(t)$. Thus

$$Z_a(t) = \prod_{i \leqslant a < j} t_{ij}. \tag{2.9.9}$$

We hope it will always be clear from the context whether Z_a is regarded as a function on the Lusztig variety, or as a monomial given by (2.9.9). Comparing (2.9.9) with (2.4.11), we see that Z_a can also be written as a "minor:"

$$Z_a = \Delta^{[a+1, r+1]}. \tag{2.9.10}$$

With all this notation, formula (2.9.8) is equivalent to the following formula for the transition map between any two normal orderings:

$$(R_{\mathbf{n}}^{\mathbf{n}'}(t))_{ij} = Z_i(t)^{\delta_{i+1,j}} \frac{T_{L'}^{\mathbf{n}}(t)\, T_{L'ij}^{\mathbf{n}}(t)}{T_{L'i}^{\mathbf{n}}(t)\, T_{L'j}^{\mathbf{n}}(t)}, \tag{2.9.11}$$

where $L' = L^{\mathbf{n}'}(i, j)$ (cf. (2.9.3)).

An advantage of (2.9.11) over (2.9.3) will become clear in the next chapter when we will show that all $T_J^{\mathbf{n}}$ are *polynomials* in the t_{ij} with nonnegative integer coefficients. Unfortunately, the problem of finding explicit combinatorial formulas for all these polynomials remains open. In the next section, we present some partial results in this direction.

2.10. *Formulas for* $T_J^{\mathbf{n}}$

We start with the case when J is a chamber set for $\mathbf{n}$. In view of (2.9.1) and (2.9.2), in this case $T_J^{\mathbf{n}}$ is a monomial which does not depend on $\mathbf{n}$. To describe this monomial, we will write J as the union of its components (see (2.8.7)). Let $s(J)$ be the number of these components, and let $\bar{J}$ denote the complement $[1, r+1] - J$. For any pair $(i, j) \in \Pi$, we set

$$c(J; i, j) = \max(s(J \cap [i, j]), s(\bar{J} \cap [i, j])) - 1. \tag{2.10.1}$$

The following proposition is an easy consequence of (2.9.1), (2.9.2) and (2.9.4).

2.10.1. Proposition. *If J is a chamber set for* $\mathbf{n}$, *then*

$$T^{\mathbf{n}}_J = \prod_{i<j} t_{ij}^{c(J;\, i,\, j)}. \tag{2.10.2}$$

Our next goal is to compute the polynomial $T^{\mathbf{n}}_{[a+1,\, b]}$ for arbitrary $\mathbf{n}$. Thus we are looking at the case when J has only one component.

2.10.2. Proposition. *As a function on the Lusztig variety, $T_{[a+1,\, b]}$ coincides with the "minor" $\Delta^{[b+1,\, r+1]}_{[b+1-a,\, r+1-a]}$ (see Section 2.4). Hence for any normal ordering $\mathbf{n}$ of Π, the function $T^{\mathbf{n}}_{[a+1,\, b]}: K^{\Pi} \to K$ is a polynomial in the t_{ij} which is given by* (2.4.10), *with $I = [b+1-a, r+1-a]$ and $J = [b+1, r+1]$.*

Proof. It is enough to verify the case $\mathbf{n} = \mathbf{n}^0$. In this case, $[a+1, b]$ is a chamber set (see (2.8.4)). For this set, (2.10.1) becomes

$$c([a+1, b];\, i,\, j) = \begin{cases} 1, & \text{if} \quad i \leqslant a < b < j; \\ 0, & \text{otherwise.} \end{cases} \tag{2.10.3}$$

Then, by (2.10.2), (2.10.3), and (2.4.16), we have

$$T^{\mathbf{n}^0}_{[a+1,\, b]} = \prod_{i<j} t_{ij}^{c([a+1,\, b];\, i,\, j)} = \prod_{i \leqslant a < b < j} t_{ij} = \left(\Delta^{[b+1,\, r+1]}_{[b+1-a,\, r+1-a]}\right)^{\mathbf{n}^0},$$

as desired. ∎

Using the above proposition, we can check directly that formula (2.4.14) for the transition map $R^{\mathbf{n}^0}_{\mathbf{n}}$ is indeed a special case of the general formula (2.9.11). Note that the factor $Z_i(t)$ in (2.9.11) only appears when $i+1 = j$, i.e., exactly when $L^{\mathbf{n}^0}(i, j) = \phi$. This explains why, in the special case $\mathbf{n}' = \mathbf{n}^0$, the right-hand side of (2.9.11) contains at most four factors $\neq 1$.

The formulas for the transition maps $R^{\mathbf{n}}_{\mathbf{n}^0}$ given in Proposition 2.8.1 and Theorem 2.8.2 can be also seen to be a special case of (2.9.11). To demonstrate this, we will give explicit formulas for the polynomials $T^{\mathbf{n}^0}_J$ for all $J \subset [1, r+1]$.

2.10.3. Theorem. *For any subset $J \subset [1, r+1]$, the polynomial $T^{\mathbf{n}^0}_J$ is given by*

$$T^{\mathbf{n}^0}_J(t) = \left(\prod_{i<j} t_{ij}^{d(J;\, i,\, j)}\right) \cdot Q_J(t) \tag{2.10.4}$$

where the polynomial $Q_J(t)$ is given by (2.8.9), *and $d(J; i, j)$ denotes the number of components $[a_k+1, b_k]$ of J which are contained in $[i+1, j-1]$, i.e., such that $i \leqslant a_k < b_k < j$.*

Proof. Fix elements $t_{ij} \in K$, and let $\tilde{M}_J$ be defined by (2.8.1). By Proposition 2.7.5, there is a unique element $\mathbf{M} = (M_J) \in \mathcal{M}_r(K)$ in the H-orbit of $\tilde{M} = (\tilde{M}_J)$. In view of (2.8.3), $M_J = M_J^{\mathbf{n}^0}(t)$ for all J. By definition (2.7.4),

$$\tilde{M}_J = \left(c_0 \prod_{j \in J} c_j \right) \cdot M_J \tag{2.10.5}$$

for some $c_0, c_1, ..., c_{r+1} \in K$. To find the coefficients c_j, we use (2.10.5) for $J = [1, b]$, $b = 0, 1, ..., r+1$. By (2.8.6),

$$\tilde{M}_{[1, b]} = \prod_{1 \leqslant i < j \leqslant b} t_{ij};$$

on the other hand, $M_{[1, b]} = 1$ by (2.7.1). Substituting these values into (2.10.5), we find

$$c_b = \frac{\tilde{M}_{[1, b]}}{\tilde{M}_{[1, b-1]}} = t_{1b} t_{2b} \cdots t_{b-1, b}.$$

Now, using (2.10.5) for an arbitrary J, we obtain

$$M_J^{\mathbf{n}^0}(t) = \frac{\tilde{M}_J}{\prod_{i<1} t_{ij}^{\delta_J(j)}} \tag{2.10.6}$$

where δ_J is the indicator function of J.

It remains to pass from $M_J^{\mathbf{n}^0}$ to $T_J^{\mathbf{n}^0}$. Combining (2.9.4) with (2.10.6) and (2.8.11), we get

$$T_J^{\mathbf{n}^0}(t) = \left(\prod_{i<j} t_{ij}^{\#\{k:\, i \leqslant a_k < j\} - \delta_J(j) + e(J;\, i,\, j)} \right) \cdot Q_J(t).$$

To complete the proof of our theorem, we need to show that

$$d(J; i, j) = \#\{k: i \leqslant a_k < j\} - \delta_J(j) + e(J; i, j),$$

which is straightforward. ▮

2.11. *Symmetries of the Lusztig Variety*

We conclude this chapter by exploring some symmetry properties of the Lusztig variety.

For a normal ordering

$$\mathbf{n} = ((i_1, j_1), ..., (i_m, j_m))$$

of Π, let $\bar{\mathbf{n}}$ and $\mathbf{n}^*$ denote the normal orderings

$$\begin{aligned} \bar{\mathbf{n}} &= ((i_m, j_m), ..., (i_1, j_1)), \\ \mathbf{n}^* &= ((r+2-j_m, r+2-i_m), ..., (r+2-j_1, r+2-i_1)). \end{aligned} \tag{2.11.1}$$

If $\mathbf{n} = \mathbf{n}(\mathbf{h})$ corresponds to a reduced word $\mathbf{h} = (h_1, ..., h_m) \in R(w_0)$ as described in Section 2.3, then $\bar{\mathbf{n}} = \mathbf{n}(\bar{\mathbf{h}})$ and $\mathbf{n}^* = \mathbf{n}(\mathbf{h}^*)$, where $\bar{\mathbf{h}}$ and $\mathbf{h}^*$ are defined by

$$\begin{aligned} \bar{\mathbf{h}} &= (r+1-h_m, ..., r+1-h_1), \\ \mathbf{h}^* &= (h_m, ..., h_1). \end{aligned} \tag{2.11.2}$$

In the language of pseudo-line arrangements, the transformations $\mathbf{n} \mapsto \bar{\mathbf{n}}$ and $\mathbf{n} \mapsto \mathbf{n}^*$ correspond to 180° rotation and reflection in a vertical mirror, respectively.

Let $t \mapsto t^*$ be an involutive transformation of K^Π defined by

$$(t^*)_{ij} = t_{r+2-j,\, r+2-i}. \tag{2.11.3}$$

For an element $\mathbf{t} = (t^{\mathbf{n}})$ of the Lusztig variety, let

$$(\tau(\mathbf{t}))^{\mathbf{n}} := t^{\bar{\mathbf{n}}}, \qquad (\iota(\mathbf{t}))^{\mathbf{n}} := (t^{\mathbf{n}^*})^*. \tag{2.11.4}$$

It is straightforward to verify that τ and ι are well-defined transformations of the Lusztig variety $\mathscr{L}_r(K)$, or any subvariety $\mathscr{L}_r(P)$ of it. A formal check shows that τ and ι are involutions which commute with each other. Therefore, they define an action of Klein's four-group $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathscr{L}_r(P)$. In the case when P is the tropical semiring (see Example 2.2.5), we obtain an action of the four-group on the canonical basis. This action was introduced and studied in [7].

The action of $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$ on $\mathscr{L}_r(P)$ induces, in a usual way, an action of the four-group on the set of functions $f\colon \mathscr{L}_r(P) \to P$. Namely, set

$$f^\tau(\mathbf{t}) := f(\tau(\mathbf{t})), \qquad f^\iota(\mathbf{t}) = f(\iota(\mathbf{t})). \tag{2.11.5}$$

Let us compute how τ and ι act on functions Z_a and T_J defined in (2.9.4). For a subset $J \subset [1, r+1]$, let

$$\bar{J} = [1, r+1] - J, \qquad J^* = \{r+2-j\colon j \in J\}. \tag{2.11.6}$$

2.11.1. Proposition. *For $a = 1, ..., r$ and $J \subset [1, r+1]$,*

$$Z_a^\tau = Z_a, \qquad Z_a^\iota = Z_{r+1-a}, \qquad T_J^\tau = T_{\bar{J}}, \qquad T_J^\iota = T_{J^*}. \tag{2.11.7}$$

Proof. First let us prove the formulas for Z^{τ}_a and T^{τ}_J. In view of Proposition 2.9.2, it suffices to check the following two statements.

(i) The transformation $Z_a \mapsto Z_a$, $T_J \mapsto T_{\bar{J}}$ preserves the relations (2.9.6)–(2.9.7).

(ii) Replacing $\mathbf{t}$ by $\tau(\mathbf{t})$ and (Z_a, T_J) by $(Z_a, T_{\bar{J}})$ preserves the relation (2.9.8).

The proof of (i) is straightforward. As for (ii), it follows easily from observing that

$$L^{\bar{\mathbf{n}}}(i, j) = \overline{L^{\mathbf{n}}(i, j) \cup \{i, j\}} \tag{2.11.8}$$

(cf. (2.5.1) or (2.5.2)). The formulas for Z^{ι}_a and T^{ι}_J are proved in the same way, with (2.11.8) replaced by

$$L^{\mathbf{n}^*}(r+2-j, r+2-i) = (L^{\mathbf{n}}(i, j))^*. \quad \blacksquare \tag{2.11.9}$$

Let us now restate Proposition 2.11.1 in terms of the functions $T^{\mathbf{n}}_J$.

2.11.2. Corollary. *For any subset $J \subset [1, r+1]$ and any normal ordering $\mathbf{n}$ of Π,*

$$T^{\bar{\mathbf{n}}}_{\bar{J}}(t) = T^{\mathbf{n}^*}_{J^*}(t^*) = T^{\mathbf{n}}_J(t). \tag{2.11.10}$$

As an application, we obtain the following symmetry properties of the transition maps.

2.11.3. Corollary. *For any two normal orderings $\mathbf{n}$ and $\mathbf{n}'$ of Π,*

$$R^{\mathbf{n}'}_{\mathbf{n}}(t) = R^{\bar{\mathbf{n}}'}_{\bar{\mathbf{n}}}(t) = (R^{\mathbf{n}'^*}_{\mathbf{n}^*}(t^*))^*. \tag{2.11.11}$$

To prove this corollary, express the transition maps in (2.11.11) via (2.9.11), and apply (2.11.8)–(2.11.10). It is also possible to prove it directly, by decomposing the transition from $\mathbf{n}$ to $\mathbf{n}'$ into 2- and 3-moves.

Formulas (2.11.11) can be used to establish the following symmetry property of the transition maps $R^{\mathbf{n}}_{\mathbf{n}^0}$, where $\mathbf{n}^0$ is given by (2.4.13).

2.11.4. Proposition. *For any normal ordering $\mathbf{n}$ of Π,*

$$R^{\mathbf{n}^*}_{\mathbf{n}^0}(t) = (R^{\mathbf{n}}_{\mathbf{n}^0}(t^*))^*. \tag{2.11.12}$$

Proof. We will use the following lemma [7, Lemma 4.2].

2.11.5. LEMMA. *The reduced word* $(\mathbf{h}^0)^*$ (*resp., the normal ordering* $(\mathbf{n}^0)^*$) *can be obtained from* $\mathbf{h}^0$ (*resp., from* $\mathbf{n}^0$) *by a sequence of* 2-*moves.*

According to this lemma, the transition map $R_{\mathbf{n}^0}^{(\mathbf{n}^0)^*}$ is the identity transformation, hence

$$R_{\mathbf{n}^0}^{\mathbf{n}} = R_{(\mathbf{n}^0)^*}^{\mathbf{n}} \tag{2.11.13}$$

for any $\mathbf{n}$. Combining (2.11.13) with (2.11.11) yields (2.11.12). ▮

3. TOTAL POSITIVITY CRITERIA

In this chapter, we are going to apply the results and constructions of Chapter 2 to the case of the ground semifield $\mathbb{R}_{>0}$. Recall that in this case the Lusztig variety can be canonically identified with the variety $N_{>0}$ of totally positive matrices (see Theorem 2.2.3). Under this identification, the components $t_k^{\mathbf{h}}$ (or, in the notation of Section 2.3, the $t_{ij}^{\mathbf{n}}$) are the coefficients in various factorizations of a totally positive matrix x into a product of elementary Jacobi matrices (see (1.1) or (2.2.1)). In Chapter 2, we used the Chamber Ansatz to express these coefficients in terms of the functions M_J on the Lusztig variety (see Section 2.5) or, alternatively, in terms of the functions Z_a and T_J (see Section 2.9). Now we will compute all these functions directly in terms of the matrix entries x_{ij} of x. This computation will provide a family of related total positivity criteria.

3.1. *Factorization of Unitriangular Matrices*

Let $x \in N_{>0}$, and let $\mathbf{t} = (t_{ij}^{\mathbf{n}})$ be the corresponding element of the Lusztig variety $\mathcal{L}_r(\mathbb{R}_{>0})$. By Theorem 2.7.1, the $t_{ij}^{\mathbf{n}}$ are related via the Chamber Ansatz (2.5.3) to a unique family of positive numbers $(M_J = M_J(\mathbf{t}))$ satisfying the 3-Term Relations (2.5.4) and the normalization conditions (2.7.1). Let $y \in N$ be the matrix related to x as in Lemma 1.3, that is,

$$x = [w_0 y^T]_+, \qquad w_0 y^T w_0^{-1} = [x w_0^{-1}]_+, \tag{3.1.1}$$

where y^T is the transpose of y, and $[g]_+$ denotes the last factor in the Gaussian LDU-decomposition of a matrix g. Let us also recall the notation $\Delta^J(y)$ for the flag minor of a matrix y, i.e., the minor that occupies several first rows and whose column set is J.

3.1.1. THEOREM. *Let* $\mathbf{t} \in \mathcal{L}_r(\mathbb{R}_{>0})$, *let* $x \in N_{>0}$ *be the matrix corresponding to* $\mathbf{t}$ *via* (2.2.1), *and let* $y \in N$ *be related to* x *by* (3.1.1). *Then* $M_J(\mathbf{t}) = \Delta^J(y)$ *for any* $J \subset [1, r+1]$.

In other terms, Theorem 3.1.1 asserts that one can compute the coefficients $t_{ij}^{\mathbf{n}}$ participating in the factorization of a totally positive matrix x into elementary Jacobi matrices, by the formula

$$t_{ij}^{\mathbf{n}} = \frac{\Delta^{L}(y)\,\Delta^{Lij}(y)}{\Delta^{Li}(y)\,\Delta^{Lj}(y)};$$

the same formula is therefore valid for a generic matrix $x \in N$. This proves Theorem 1.4 from the introduction.

Proof. We start with establishing some identities that relate the minors of x and y. All of them will be derived from the following statement.

3.1.2. Lemma. *Let $u = [g]_+$ for some matrix $g \in GL_{r+1}$. Then, for any column set $J \subset [1, r+1]$, the corresponding flag minor of u is given by*

$$\Delta^{J}(u) = \frac{\Delta^{J}(g)}{\Delta^{[1,|J|]}(g)}. \tag{3.1.2}$$

Proof of Lemma 3.1.2. Applying the Binet–Cauchy formula to the LDU-decomposition $g = v^T \cdot d \cdot u$, we obtain

$$\Delta^{j}(g) = \Delta^{[1,|J|]}(d)\,\Delta^{J}(u). \tag{3.1.3}$$

In particular,

$$\Delta^{[1,|J|]}(g) = \Delta^{[1,|J|]}(d). \tag{3.1.4}$$

Combining (3.1.3) with (3.1.4) yields (3.1.2). ∎

3.1.3. Lemma. *The following identities hold for the matrices $x, y \in N$ related by* (3.1.1):

$$\Delta^{J}(x) = \frac{\Delta_J^{[r+2-|J|,\,r+1]}(y)}{\Delta^{[r+2-|J|,\,r+1]}(y)} \qquad (J \subset [1, r+1]); \tag{3.1.5}$$

$$\Delta^{[d+1,\,d+b-a] \cup [b+1,\,r+1]}(x) = \frac{\Delta^{[1,d] \cup [a+1,b]}(y)}{\Delta^{[a+1,\,r+1]}(y)} \qquad (0 \leqslant d \leqslant a \leqslant b \leqslant r+1); \tag{3.1.6}$$

$$\Delta^{[a+1,\,r+1]}(x)\,\Delta^{[a+1,\,r+1]}(y)=1 \qquad (0\leqslant a\leqslant r+1); \tag{3.1.7}$$

$$\Delta^{[1,\,d]\cup[a+1,\,b]}(y)=\frac{\Delta^{[d+1,\,d+b-a]\cup[b+1,\,r+1]}(x)}{\Delta^{[a+1,\,r+1]}(x)} \qquad (0\leqslant d\leqslant a\leqslant b\leqslant r+1); \tag{3.1.8}$$

$$\Delta^{[a+1,\,b]}(y)=\frac{\Delta^{[1,\,b-a]\cup[b+1,\,r+1]}(x)}{\Delta^{[a+1,\,r+1]}(x)} \qquad (0\leqslant a\leqslant b\leqslant r+1). \tag{3.1.9}$$

Proof of Lemma 3.1.3. To prove (3.1.5), apply Lemma 3.1.2 for $g=w_0\,y^T$ and $u=x$. To prove (3.1.6), apply (3.1.5) for $J=[d+1,d+b-a]\cup[b+1,r+1]$ and note that, since $y\in N$, then

$$\Delta^{[a+1,\,r+1]}_{[d+1,\,b-a+d]\cup[b+1,\,r+1]}(y)=\Delta^{[a+1,\,b]}_{[d+1,\,b-a+d]}(y)=\Delta^{[1,\,d]\cup[a+1,\,b]}(y).$$

To prove (3.1.7), apply (3.1.6) for $d=0$ and $b=a$. Combining (3.1.6) and (3.1.7), we obtain (3.1.8). Finally, (3.1.9) follows from (3.1.8) by setting $d=0$. ∎

We are in a position now to finish the proof of Theorem 3.1.1. We start with an observation that the flag minors $\Delta^J(y)$ satisfy the 3-Term Relations (2.5.4) (see Proposition 2.6.2); since $y\in N$, they also satisfy the normalization condition (2.7.1). Using Corollary 2.7.4, we conclude that it is enough to prove the claim $M_J(\mathbf{t})=\Delta^J(y)$ for all chamber sets J in any given arrangement Arr($\mathbf{n}$). Taking $\mathbf{n}=\mathbf{n}^0$, we recall from (2.8.4) that the chamber sets for $\mathbf{n}^0$ are the intervals $[a+1,b]\subset[1,r+1]$, so we only need to prove that $M_{[a+1,\,b]}(\mathbf{t})=\Delta^{[a+1,\,b]}(y)$. This is done by combining (2.9.5), Proposition 2.10.2, (2.9.10) and (3.1.9):

$$M_{[a+1,\,b]}(\mathbf{t})=\frac{T_{[a+1,\,b]}(\mathbf{t})}{Z_a(\mathbf{t})}=\frac{\Delta^{[b+1,\,r+1]}_{[b+1-a,\,r+1-a]}(x)}{\Delta^{[a+1,\,r+1]}(x)}=\Delta^{[a+1,\,b]}(y). \quad ∎$$

3.2. *Generalizations of Fekete's Criterion*

3.2.1. THEOREM. *Let $x\in N$, and let* $\mathbf{n}$ *be any normal ordering of Π. Then the following are equivalent*:

(i) *the matrix x is totally positive*;

(ii) *all flag minors $\Delta^J(x)$ of x are positive*;

(iii) *the flag minors $\Delta^{L(C)}(x)$ are positive for all chambers C in the pseudo-line arrangement* Arr($\mathbf{n}$).

Obviously, (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii), so the essential part of Theorem 3.2.1 is the implication (iii) $\Rightarrow$ (i). This can be stated as a criterion for total positivity: in order to test whether $x \in N$ is totally positive, it is enough to check positivity of the flag minors which correspond to chamber sets for $\mathbf{n}$. Note that, for a chamber adjacent to the right boundary, the corresponding flag minor has the form $\Delta^{[1,b]}$ and thus equals 1; therefore, the above criterion involves exactly $m = \dim N$ minors.

Theorem 3.2.1 provides a family of total positivity criteria, one for each $\mathbf{n}$ (or, equivalently, for each commutation class of reduced words for w_0). In particular, in the case $\mathbf{n} = \mathbf{n}^0$ of the minimal reduced word (see (2.4.12)–(2.4.13)), the chamber sets are intervals, and we obtain the following classical result that is essentially due to Fekete (cf. [1, Corollary 2.6]).

3.2.2. Theorem (Fekete's Criterion). *A matrix $x \in N$ is totally positive if and only if $\Delta^{[a+1,b]}(x) > 0$ for all $0 < a < b \leqslant r+1$.*

Theorem 3.2.1 can be deduced from Fekete's criterion combined with the following proposition.

3.2.3. Proposition. *Every flag minor Δ^J considered as a polynomial function on N, can be written as a subtraction-free rational expression in the minors $\Delta^{L(C)}$ corresponding to the chamber sets for a given pseudo-line arrangement.*

Proof of Proposition 3.2.3. By Proposition 2.6.2, the flag minors of any matrix $x \in N$ satisfy the 3-Term Relations; they also obviously satisfy the normalization conditions (2.7.1). The proposition now follows from Corollary 2.7.4. ∎

Proof of Theorem 3.2.1. In view of Proposition 3.2.3, the conditions (ii) and (iii) are equivalent. Using Fekete's criterion 3.2.2, we see that (ii) $\Rightarrow$ (i), as desired.

We would like to give another proof of Theorem 3.2.1, relying upon the results in Section 3.1. In particular, this will provide us with an independent proof of Fekete's criterion.

Let $x \mapsto y = y(x)$ be the birational automorphism of the group N given by (3.1.1). Combining Theorem 1.4 with Corollary 2.7.4 applied to the normal ordering $\mathbf{n}^0$, we obtain the following result.

3.2.4. Lemma. *For a matrix $x \in N$, the following conditions are equivalent:*

(i) $x \in N_{>0}$;

(ii) $\Delta^{[a+1,b]}(y(x)) > 0$ *for all* $0 < a < b \leqslant r+1$.

Now our second proof of Theorem 3.2.1 can be completed as follows. Proposition 3.2.3 shows that (iii) $\Rightarrow$ (ii). In view of Lemma 3.2.4, it remains to prove that the positivity of the flag minors of x implies $\Delta^{[a+1,b]}(y(x)) > 0$. But this is immediate from (3.1.9). ▮

As a byproduct of the above proof, we obtain the following result that we find of independent interest.

3.2.5. THEOREM. *The birational automorphism* $x \mapsto y$ *of the group* N *given by* (3.1.1) *restricts to a bijection* $N_{>0} \to N_{>0}$.

Proof. By Lemma 3.2.4, a matrix $x \in N$ is totally positive if and only if $y = y(x)$ satisfies Fekete's criterion. ▮

As mentioned in the Introduction, Theorem 3.2.1 can be used to obtain a set of total positivity criteria for an arbitrary square matrix; see Corollary 1.9.

3.3. *Polynomials* $T_J(x)$

We now return to the study of the functions T_J on the Lusztig variety, which were introduced in Section 2.9. Identifying, as above, $\mathscr{L}_r(\mathbb{R}_{>0})$ with $N_{>0}$, we can regard the T_J as functions on $N_{>0}$, or as rational functions on the entire group N. Our first task is to prove that all T_J are *polynomials* in the matrix entries of $x \in N$. This will be achieved by obtaining explicit polynomial expressions for the T_J. (Note that the companion functions Z_a are also polynomials, namely, the "principal" flag minors given by (2.9.10).)

To describe the polynomials $T_J(x)$, we will need to recall some notation introduced in Section 2.8 (see (2.8.7) and below). A subset $J \subset [1, r+1]$ can be written as the union of its components:

$$J = [a_1+1, b_1] \cup [a_2+1, b_2] \cup \cdots \cup [a_s+1, b_s]$$
$$(0 \leqslant a_1 < b_1 < a_2 < b_2 < \cdots < a_s < b_s \leqslant r+1). \tag{3.3.1}$$

The size of the kth component of J is $l_k = b_k - a_k$, and $l = l_1 + \cdots + l_s$ is the cardinality of J. We subdivide the interval $[1, l]$ into disjoint intervals $I_1, ..., I_s$ of sizes $l_1, ..., l_s$, numbered from left to right. With this notation at hand, let us denote by $S(J)$ the set of permutations σ of $[1, l]$ which have the following property:

$$\sigma \text{ increases on each interval } I_k \text{ and } \sigma(I_k) \cap [b_k+1, r+1] = \phi$$
$$\text{for } k = 1, ..., s. \tag{3.3.2}$$

3.3.1. THEOREM. *For any subset* $J \subset [1, r+1]$ *written in the form* (3.3.1), *the function* T_J *on* N *is a polynomial given by*

$$T_J(x) = \sum_{\sigma \in S(J)} \operatorname{sgn}(\sigma) \prod_{k=1}^{s} \Delta^{\sigma(I_k) \cup [b_k+1, r+1]}(x). \tag{3.3.3}$$

Proof. Using (2.9.4), (2.9.10), Theorem 3.1.1 and (3.1.7), we obtain

$$T_J(x) = M_J(x)\, Z_{a_1}(x) \cdots Z_{a_s}(x) = \frac{\Delta^J(y)}{\Delta^{[a_1+1, r+1]}(y) \cdots \Delta^{[a_s+1, r+1]}(y)}, \tag{3.3.4}$$

where $y \in N$ is related to x as in (3.1.1). Partitioning the column set J in the minor $\Delta^J(y)$ into the components $[a_k+1, b_k]$ and taking the corresponding Laplace expansion, we obtain

$$\Delta^J(y) = \sum_{\sigma \in S(J)} \operatorname{sgn}(\sigma) \prod_{k=1}^{s} \Delta_{\sigma(I_k)}^{[a_k+1, b_k]}(y). \tag{3.3.5}$$

Since $y \in N$, we have

$$\Delta_{\sigma(I_k)}^{[a_k+1, b_k]}(y) = \Delta_{\sigma(I_k) \cup [b_k+1, r+1]}^{[a_k+1, r+1]}(y).$$

Therefore, in view of (3.1.5),

$$\Delta_{\sigma(I_k)}^{[a_k+1, b_k]}(y) = \Delta^{\sigma(I_k) \cup [b_k+1, r+1]}(x)\, \Delta^{[a_k+1, r+1]}(y). \tag{3.3.6}$$

Substituting (3.3.6) into (3.3.5) and then the resulting expression for $\Delta^J(y)$ into (3.3.4), we see that all the factors $\Delta^{[a_k+1, r+1]}(y)$ cancel out, yielding (3.3.3). ▮

Formula (3.3.3) expresses each $T_J(x)$ as a polynomial in the flag minors of x. For any particular choice of a normal ordering $\mathbf{n}$ and a subset $J \subset [1, r+1]$, we can use (2.4.10) to express the minors participating in (3.3.3), in terms of the corresponding variables t_{ij}. This will result in an explicit formula for $T_J^{\mathbf{n}}(t)$. In particular, this argument shows that all the $T_J^{\mathbf{n}}$ are *polynomials* in the variables t_{ij}. We will later prove (see Theorem 3.7.4) that, in fact, these polynomials have *nonnegative* integer coefficients. They therefore represent $T_J^{\mathbf{n}}$ for an arbitrary ground semifield P (cf. Proposition 2.1.7). Since (3.3.3) involves an alternating sum, some work is needed to establish this nonnegativity.

It is clear from (2.9.10) that the $Z_a(x)$ are irreducible polynomials in the matrix entries of $x \in N$. We will show later (see Proposition 3.6.4) that the same is true for the functions $T_J(x)$. Thus formulas (2.9.8) express every component $t_{ij}^{\mathbf{n}}$ as a ratio of products of irreducible polynomials in the variables x_{ij}.

In view of (2.9.7), there are exactly $\binom{r}{2}$ nontrivial polynomials T_J appearing in (2.9.8) for any given normal ordering $\mathbf{n}$. Namely, J runs over chamber sets for $\mathrm{Arr}(\mathbf{n})$ which correspond to the *bounded* chambers. Besides these polynomials, the formulas (2.9.8) involve $Z_1(x), ..., Z_r(x)$. The total number of irreducible factors appearing in (2.9.8), for a given $\mathbf{n}$, is therefore equal to $m = \binom{r+1}{2}$ (cf. statement 2 of Corollary 1.6).

Let us now illustrate Theorem 3.3.1 by some examples. First, consider the case when $J = [1, d] \cup [a+1, b]$ for some $0 \leqslant d < a < b \leqslant r+1$. In this case, $S(J)$ only contains the identity permutation, and (3.3.3) gives

$$T_{[1, d] \cup [a+1, b]}(x) = \Delta^{[d+1, d+b-a] \cup [b+1, r+1]}(x) \tag{3.3.7}$$

(we could also derive it from (3.1.6)). Note that (3.3.7) specializes to Proposition 2.10.2 when $d = 0$.

For $r = 2$, every subset $J \subset [1, 3]$ can be written in the form $J = [1, d] \cup [a+1, b]$. Hence each $T_J(x)$ is a flag minor of x given by (3.3.7). The only nontrivial (i.e., not equal to 1 by virtue of (2.9.7)) polynomials are:

$$(r=2) \qquad T_{13}(x) = x_{12} \qquad T_2(x) = x_{23}$$

For $r = 3$, there is only one subset $J \subset [1, 4]$ for which (3.3.7) does not apply, namely $J = \{2, 4\}$. For this J, formula (3.3.3) gives

$$T_{24}(x) = \Delta^{134}(x)\, \Delta^2(x) - \Delta^{234}(x)\, \Delta^1(x).$$

This is no longer a flag minor of x, but an easy calculation shows that it still is a minor:

$$T_{24}(x) = \Delta^{34}_{13}(x) \qquad (r=3). \tag{3.3.8}$$

The full list of nontrivial polynomials $T_J(x)$, for $r = 3$, is thus obtained:

$$(r=3) \qquad \begin{aligned} T_{14}(x) &= \Delta^2(x) = x_{12} \\ T_{124}(x) &= \Delta^3(x) = x_{13} \\ T_3(x) &= \Delta^{14}(x) = x_{24} \\ T_{23}(x) &= \Delta^{124}(x) = x_{34} \\ T_{134}(x) &= \Delta^{23}(x) = x_{12}x_{23} - x_{13} \\ T_{13}(x) &= \Delta^{24}(x) = x_{12}x_{24} - x_{14} \\ T_2(x) &= \Delta^{134}(x) = x_{23}x_{34} - x_{24} \\ T_{24}(x) &= \Delta^{34}_{13}(x) = x_{13}x_{34} - x_{14} \end{aligned}$$

For $r=4$, calculations show that there are exactly two subsets J for which $T_J(x)$ is not a minor of x:

$$(r=4) \qquad \begin{aligned} T_{24}(x) &= \Delta^{1345}(x)\,\Delta^{25}(x) - \Delta^{2345}(x)\,\Delta^{15}(x) \\ T_{135}(x) &= \Delta^{245}(x)\,\Delta^{3}(x) - \Delta^{345}(x)\,\Delta^{2}(x) \end{aligned} \qquad (3.3.9)$$

Later (see Proposition 3.6.5), we will characterize all subsets $J \subset [1, r+1]$, for which $T_J(x)$ is a minor of x; the fraction of such subsets appears to be equal to

$$\frac{r+1+\binom{r+1}{3}}{2^r},$$

which equals 1 for $1 \leqslant r \leqslant 3$, but decreases exponentially as r grows.

3.3.2. Example. Let us present a complete solution of Problem 1.1 for $r=4$ and $\mathbf{h}=(1, 3, 2, 4, 1, 3, 2, 4, 1, 3)$ (this reduced word and its generalizations play an important role in [28, 29, 9, 10]). The corresponding pseudo-line arrangement and its chamber sets are shown in Fig. 11. The normal ordering $\mathbf{n}$ is

$$\mathbf{n} = ((45), (23), (25), (13), (24), (15), (14), (35), (12), (34)). \qquad (3.3.10)$$

By (2.9.8), the coefficients $t_{ij} = t_{ij}^{\mathbf{n}}$ are expressed through the $Z_a = Z_a(x)$ and $T_J = T_J(x)$ as follows:

$$t_{45} = \frac{Z_4}{T_4}, \qquad t_{23} = \frac{Z_2}{T_{245}}, \qquad t_{25} = \frac{T_4 T_{245}}{T_{24}}, \qquad t_{13} = \frac{T_{245}}{T_{1245}}, \qquad t_{24} = \frac{T_{24}}{T_2 T_4},$$
$$t_{15} = \frac{T_{24} T_{1245}}{T_{245} T_{124}}, \qquad t_{14} = \frac{T_2 T_{124}}{T_{24}}, \qquad t_{35} = \frac{T_{124}}{T_{1245}}, \qquad t_{12} = \frac{Z_1}{T_2}, \qquad t_{34} = \frac{Z_3}{T_{124}}. \qquad (3.3.11)$$

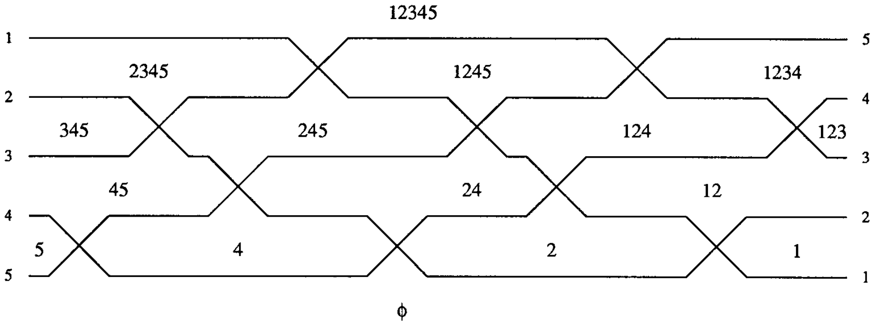


Fig. 11. Pseudo-line arrangement and chamber sets for $\mathbf{h} = 1324132413$.

Using (2.9.10) and Theorem 3.3.1, we obtain the following expressions for the polynomials Z_a and T_J that appear in (3.3.11):

$$Z_1 = \Delta^{[2,5]}(x), \qquad Z_2 = \Delta^{[3,5]}(x),$$

$$Z_3 = \Delta^{[4,5]}(x), \qquad Z_4 = \Delta^{5}(x) = x_{15};$$

$$T_2 = \Delta^{1345}(x), \qquad T_4 = \Delta^{15}(x),$$

$$T_{124} = \Delta^{35}(x), \qquad T_{1245} = \Delta^{34}(x),$$

$$T_{245}(x) = \Delta^{345}_{124}(x), \qquad T_{24} = \Delta^{1345}(x)\, \Delta^{25}(x) - \Delta^{2345}(x)\, \Delta^{15}(x).$$

We conclude this section by providing a family of total positivity criteria in terms of the polynomials $(Z_a(x), T_J(x))$. The following proposition can be easily derived from Theorems 3.2.1 and 3.2.5.

3.3.3. Proposition. *Any pseudo-line arrangement* Arr(**n**) *gives rise to the following criterion: a matrix $x \in N$ is totally positive if and only if $Z_a(x) > 0$ for $a = 1, ..., r$ and $T_J(x) > 0$ for all chamber sets corresponding to the bounded chambers in* Arr(**n**).

3.3.4. *Remark.* Note that the polynomials $Z_1(x), ..., Z_r(x)$ appear in all the criteria given by the last proposition. This phenomenon is closely related to the following recent result [34]: the boundary of $N_{>0}$ in N is the intersection of $N_{\geqslant 0}$ with the union of hypersurfaces $\{Z_a = 0\}$, $a = 1, ..., r$. (Here $N_{\geqslant 0}$ denotes the set of "totally nonnegative" matrices $x \in N$, i.e., those whose all minors are nonnegative.)

3.3.5. *Remark.* Proposition 3.3.3 has a somewhat surprising conclusion that the positivity of all minors of x is controlled by the positivity of the polynomials $Z_a(x)$ and $T_J(x)$, most of which are not minors themselves. The explanation of this phenomenon is provided by the fact that every minor of x is a subtraction-free rational expression in the $Z_a(x)$ and $T_J(x)$. (This follows immediately from (2.4.10) and (2.9.8).)

3.4. *The Action of the Four-Group on $N_{>0}$*

In this section, we consider the automorphisms τ and ι introduced in Section 2.11, for the special case of the Lusztig variety $\mathcal{L}_r(\mathbb{R}_{>0})$. Identifying $\mathcal{L}_r(\mathbb{R}_{>0})$ with $N_{>0}$, we will use the same symbols τ and ι to denote the corresponding transformations of $N_{>0}$, as well as their extensions to N.

The definition (2.11.4) (see also (2.11.2)), when applied to the case under consideration, means that the images of a matrix

$$x = (1 + t_1 e_{h_1}) \cdots (1 + t_m e_{h_m})$$

under τ and ι are

$$\begin{aligned}\tau(x) &= (1 + t_m e_{r+1-h_m}) \cdots (1 + t_1 e_{r+1-h_1}),\\ \iota(x) &= (1 + t_m e_{h_m}) \cdots (1 + t_1 e_{h_1}).\end{aligned} \tag{3.4.1}$$

One can compute $\tau(x)$ and $\eta(x)$ directly from the matrix x, as follows.

3.4.12. Proposition. *The transformations τ and ι of $N_{>0}$ are the restrictions of the involative anti-automorphisms of the group N given by*

$$\begin{aligned}\tau(x) &= w_0 x^T w_0^{-1},\\ \iota(x) &= d_0 x^{-1} d_0^{-1},\end{aligned} \tag{3.4.2}$$

where d_0 is the diagonal matrix with diagonal entries $1, -1, 1, -1, ..., (-1)^r$.

Proof. Both (3.4.1) and (3.4.2) define birational anti-automorphisms of the group N. Thus, it is enough to check that

$$\begin{aligned}1 + te_{r+1-a} &= w_0(1 + te_a)^T w_0^{-1},\\ 1 + te_a &= d_0(1 + te_a)^{-1} d_0^{-1}\end{aligned} \tag{3.4.3}$$

for $a = 1, ..., r$. This is straightforward. ∎

It follows from Proposition 3.4.1 that the anti-automorphisms defined by (3.4.2) preserve the set of totally positive matrices. This fact can be proved in a more direct way, by utilizing the identities.

$$\begin{aligned}\Delta_I^J(\tau(x)) &= \Delta_{J^*}^{I^*}(x),\\ \Delta_I^J(\iota(x)) &= \Delta_{\bar{J}}^{\bar{I}}(x),\end{aligned} \tag{3.4.4}$$

where $\bar{J}$ and J^* are defined by (2.11.6). (The first identity in (3.4.4) is obvious, the second follows from the well known expression for the minors of the inverse matrix.)

The action of τ and ι on the polynomials Z_a and T_J was computed in Proposition 2.11.1. Combining this result with (3.3.7) and (3.4.4), we discover another case where $T_J(x)$ is a minor. Namely, for all $0 \leqslant d < a < b \leqslant r+1$, we have

$$\begin{aligned}T_{[d+1,\,a] \cup [b+1,\,r+1]} &= (T_{[1,\,d] \cup [a+1,\,b]})^\tau = (\Delta^{[d+1,\,d+b-a] \cup [b+1,\,r+1]})^\tau\\ &= \Delta^{[a+1,\,r+1]}_{[1,\,r+1-b] \cup [r+2-d-b+a,\,r+1-d]}.\end{aligned} \tag{3.4.5}$$

Proposition 2.1.7 implies that the identities (3.4.4), in the form

$$(\Delta_I^J)^\tau = \Delta_{J^*}^{I^*}, \qquad (\Delta_I^J)^\iota = \Delta_{\bar{J}}^{\bar{I}},$$

as well as their corollary (3.4.5), remain valid for an arbitrary ground semiring P. We will later show that (3.3.7) and (3.4.5) are the only two cases where T_J coincides with some minor Δ_I^K (see Proposition 3.6.5).

3.5. *The Multi-filtration in the Coordinate Ring of N*

We have already mentioned three properties of the polynomials T_J whose proof was postponed:

(i) Every $T_J(x)$ is an irreducible polynomial in the matrix entries of x.

(ii) The polynomial $T_J(x)$ is a minor of x if and only if $J = [1, d] \cup [a+1, b]$ or $J = [d+1, a] \cup [b+1, r+1]$ for some $0 \leqslant d < a < b \leqslant r+1$.

(iii) For every J and every normal ordering $\mathbf{n}$ of Π, the polynomial $T_J^{\mathbf{n}}(t)$ has nonnegative integer coefficients.

In this section, we introduce some additional structures in the coordinate ring

$$A = \mathbb{Q}[N] = \mathbb{Q}[x_{12}, x_{13}, ..., x_{r,r+1}],$$

following, for the most part, the paper [7], and specializing its pertinent results at $q = 1$. These structures will later be employed to prove (i)–(iii). Although everything can be carried out in the case of a maximal unipotent subgroup N of any semisimple group G, we shall only present the type A_r case where $G = SL_{r+1}$.

Let $\alpha_1, ..., \alpha_r$ be the simple roots, and $\omega_1, ..., \omega_r$ the fundamental weights of type A_r, in the standard notation [5]:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1},$$

$$\omega_i = \sum_{a=1}^{i} \varepsilon_a - \frac{i}{r+1} \sum_{a=1}^{r+1} \varepsilon_a,$$

where $\varepsilon_1, ..., \varepsilon_{r+1}$ is the standard basis in $\mathbb{R}^{r+1}$. The transition matrix between simple roots and fundamental weights is the *Cartan matrix*. For the type A_r,

$$\alpha_i = -\omega_{i-1} + 2\omega_i - \omega_{i+1}$$

(from now on, we use the convention $\omega_0 = \omega_{r+1} = 0$). Let Q_+ be the additive semigroup generated by $\alpha_1, ..., \alpha_r$, and P_+ the additive semigroup generated by $\omega_1, ..., \omega_r$ (thus the elements of P_+ are the highest weights of irreducible finite-dimensional representations of SL_{r+1}). We will now introduce a family of subspaces $A(\lambda, \mu, \nu) \subset A = \mathbb{Q}[N]$ labelled by the triples of elements of P_+.

First, we note that A has a natural Q_+-grading

$$A = \bigoplus_{\gamma \in Q_+} A(\gamma), \tag{3.5.1}$$

where each variable x_{ij} has (multi-)degree

$$\deg(x_{ij}) = \varepsilon_i - \varepsilon_j = \alpha_i + \alpha_{i+1} + \cdots + \alpha_{j-1}. \tag{3.5.2}$$

The dimensions of homogeneous components are given by

$$\dim(A(\gamma)) = p(\gamma) \tag{3.5.3}$$

where $p(\gamma)$ is the *Kostant partition function*, i.e., the number of ways to express γ as a sum of positive roots.

Second, we define the right and left *infinitesimal translation operators* E_i and E_i^*, acting in A by

$$\begin{aligned} (E_i f)(x) &= \frac{d}{dt}\bigg|_{t=0} f(x \cdot (1 + te_i)), \\ (E_i^* f)(x) &= \frac{d}{dt}\bigg|_{t=0} f((1 + te_i) \cdot x), \end{aligned} \tag{3.5.4}$$

for $i = 1, ..., r$. In the coordinate form, these are given by

$$\begin{aligned} E_j &= \sum_{i=1}^{j} x_{ij} \frac{\partial}{\partial x_{i,\, j+1}}, \\ E_i^* &= \sum_{j=i+1}^{r+1} x_{i+1,\, j} \frac{\partial}{\partial x_{ij}}. \end{aligned} \tag{3.5.5}$$

It is easy to check that, for all $\gamma \in Q_+$,

$$E_i, E_i^* \colon A(\gamma) \to A(\gamma - \alpha_i). \tag{3.5.6}$$

To illustrate the above concepts, we observe that

$$\Delta_I^J \in A\left(\sum_{i \in I} \varepsilon_i - \sum_{j \in J} \varepsilon_j\right). \tag{3.5.7}$$

A direct calculation shows that

$$E_j(\Delta_I^J) = \begin{cases} \Delta_0^{J \cup \{j\} - \{j+1\}}, & \text{if} \quad j+1 \in J, \quad j \notin J; \\ 0, & \text{otherwise;} \end{cases} \tag{3.5.8}$$

$$E_i^*(\Delta_I^J) = \begin{cases} \Delta_{I \cup \{i+1\} - \{i\}}^J, & \text{if} \quad i \in I, \quad i+1 \notin I; \\ 0, & \text{otherwise.} \end{cases} \tag{3.5.9}$$

Note that, in the notation of (2.4.6), formula (3.5.8) can be rewritten as

$$E_j(\Delta_I^J) = \Delta_I^{u_j(J)}. \tag{3.5.10}$$

It follows that the restrictions of the E_j onto the linear span of the minors provide a representation of the nil–Temperley–Lieb algebra.

Now let λ, μ, $\nu \in P_+$, and assume that

$$\lambda = l_1\omega_1 + \cdots + l_r\omega_r,$$

$$\nu = n_1\omega_1 + \cdots + n_r\omega_r.$$

We then define the vector subspace $A(\lambda, \mu, \nu) \subset A$ by

$$A(\lambda, \mu, \nu) = \{ f \in A(\lambda - \mu^* + \nu) \colon (E_i^*)^{l_i+1} f = E_i^{n_i+1} f = 0 \quad \text{for} \quad i = 1, ..., r\}, \tag{3.5.11}$$

where $\mu^* = -w_0\mu$, i.e., $\mu \mapsto \mu^*$ is a linear map such that

$$\omega_i^* = \omega_{r+1-i}. \tag{3.5.12}$$

This definition readily implies that the subspaces $A(\lambda, \mu, \nu)$ form a *multiplicative multi-filtration* in A, in the following sense:

$$A(\lambda, \mu, \nu) \cdot A(\lambda', \mu', \nu') \subset A(\lambda + \lambda', \mu + \mu', \nu + \nu'). \tag{3.5.13}$$

Also, for any $\varphi, \psi \in P_+$, we have

$$A(\lambda, \mu, \nu) \subset A(\lambda + \varphi, \mu + \varphi^* + \psi^*, \nu + \psi). \tag{3.5.14}$$

Motivated by (3.5.14), we introduce a partial order on the semigroup P_+^3 by setting

$$(\lambda, \mu, \nu) \leqslant (\lambda + \varphi, \mu + \varphi^* + \psi^*, \nu + \psi).$$

The following proposition follows immediately from the definitions.

3.5.1. PROPOSITION. **1.** *For every nonzero homogeneous polynomial $f \in A$, there is a unique minimal triple (λ, μ, ν) such that $f \in A(\lambda, \mu, \nu)$. Namely, the components l_i of λ and n_i of ν are given by*

$$\begin{aligned} l_i &= l_i(f) = \max\{l \colon (E_i^*)^l f \neq 0\},\\ n_i &= n_i(f) = \max\{n \colon E_i^n f \neq 0\}. \end{aligned} \tag{3.5.15}$$

2. *If nonzero homogeneous polynomials f and f' have minimal triples (λ, μ, ν) and (λ', μ', ν'), respectively, then the minimal triple for the product ff' is $(\lambda+\lambda', \mu+\mu', \nu+\nu')$.*

Let us determine the minimal triple (λ, μ, ν) for a minor Δ_I^J. For a subset J written in the form (3.3.1), we define the weights $\alpha(J), \beta(J) \in P_+$ by

$$\begin{aligned} \alpha(J) &= \omega_{a_1} + \omega_{a_2} + \cdots + \omega_{a_s}, \\ \beta(J) &= \omega_{b_1} + \omega_{b_2} + \cdots + \omega_{b_s}. \end{aligned} \tag{3.5.16}$$

The following proposition is a straightforward consequence of (3.5.7)–(3.5.9).

3.5.2. Proposition. *Let (I, J) be a reduced admissible pair of subsets of $[1, r+1]$ (see Section 2.4). Then*

$$\Delta_I^J \in A(\beta(I), \alpha(I)^* + \beta(J)^*, \alpha(J)), \tag{3.5.17}$$

and the triple $(\beta(I), \alpha(I)^* + \beta(J)^*, \alpha(J))$ is minimal for Δ_I^J. In particular, for any subset J of cardinality $|J|$, we have

$$\Delta^J \in A(\omega_{|J|}, \beta(J)^*, \alpha(J)), \tag{3.5.18}$$

and the triple $(\omega_{|J|}, \beta(J)^*, \alpha(J))$ is minimal for Δ^J unless $J = [1, b]$ for some b.

Note that the case $J = [1, b]$ in the last proposition is indeed an exception, since $\Delta^{[1,b]} = 1 \in A(0, 0, 0)$.

To establish more properties of the subspaces $A(\lambda, \mu, \nu)$, we recall their well-known representation-theoretic interpretation (cf., e.g., [7, 32]). For $\lambda \in P_+$, let

$$V_\lambda = \sum_{\mu, \nu} A(\lambda, \mu, \nu) = \bigcap_i \operatorname{Ker}(E_i^*)^{l_i+1}. \tag{3.5.19}$$

Then there is a unique representation of the group $G = SL_{r+1}$ in V_λ such that the generators e_i of Lie (N) act as the operators E_i given by (3.5.4). This representation is irreducible with highest weight λ (in this realization, the highest vector in V_λ is the function $1 \in A$). Now for any $\mu, \nu \in P_+$, the space $A(\lambda, \mu, \nu)$ regarded as a subspace in V_λ, consists of all vectors $v \in V_\lambda$ of weight $\mu^* - \nu$ such that $e_i^{n_i+1} v = 0$ for $i = 1, ..., r$.

Let us consider this interpretation in the case when $\lambda = w_l$ is a fundamental weight. Then V_λ is isomorphic to $\bigwedge^l \mathbb{C}^{r+1}$, so all the weight subspaces of V_λ are one-dimensional, and correspond to the subsets $J \subset [1, r+1]$ of size l. Using (3.5.18), we obtain the following result.

3.5.3. PROPOSITION. *The space $A(\omega_l, \mu, \nu)$ is zero unless $\mu = \beta(J)^*$ and $\nu = \alpha(J)$ for some subset $J \subset [1, r+1]$ of size l. For any J, the space $A(\omega_{|J|}, \beta(J)^*, \alpha(J))$, is one-dimensional and is spanned by the flag minor Δ^J.*

In a special case of Proposition 3.5.3, we see that the polynomial $Z_a = \Delta^{[a+1, r+1]}$ spans the one-dimensional subspace $A(\omega_{r+1-a}, 0, \omega_a)$. We will need a little more general statement. Assuming, as before, that $\lambda = l_1\omega_1 + \cdots + l_r\omega_r \in P_+$, let us define

$$Z^\lambda = \prod_{i=1}^{r} Z_i^{l_i}. \tag{3.5.20}$$

3.5.4. PROPOSITION. *For every $\lambda \in P_+$, the space $A(\lambda, 0, \lambda^*)$ is one-dimensional and is spanned by Z^{λ^*}. If $\nu \neq \lambda^*$, then $A(\lambda, 0, \nu) = 0$.*

Proof. The above representation-theoretic interpretation of $A(\lambda, \mu, \nu)$ implies that the subspace $A(\lambda, 0, \nu)$ consists of the lowest vectors of weight $-\nu$ in V_λ. It is well known, however, that V_λ has a unique lowest vector (up to scalar multiples), and its weight is $-\lambda^*$. Therefore, $A(\lambda, 0, \nu) = 0$ unless $\nu = \lambda^*$, and dim $A(\lambda, , 0, \lambda^*) = 1$. It is also clear that Z^{λ^*} belongs to $A(\lambda, 0, \lambda^*)$, and hence generates this subspace. ∎

3.6. *The $S_3 \times \mathbb{Z}/2\mathbb{Z}$ Symmetry*

The filtration $(A(\lambda, \mu, \nu))$ exhibits a remarkable symmetry which is a consequence of the following classical result. There is a canonical (up to a scalar multiple) isomorphism of vector spaces between $A(\lambda, \mu, \nu)$ and the space of G-invariants in the tensor product $V_\lambda \otimes V_\mu \otimes V_\nu$:

$$A(\lambda, \mu, \nu) \cong (V_\lambda \otimes V_\mu \otimes V_\nu)^G \tag{3.6.1}$$

(in the case under consideration, $G = SL_{r+1}$). In particular, the dimension of $A(\lambda, \mu, \nu)$ is given by the Littlewood–Richardson rule.

As a consequence of (3.6.1), the space $A(\lambda, \mu, \nu)$ is isomorphic to $A(\lambda', \mu', \nu')$ whenever the triple (λ', μ', ν') is obtained from (λ, μ, ν) by a permutation. Furthermore, since the dual representation $(V_\lambda)^*$ is isomorphic to V_{λ^*} (as before, λ^* is defined via (3.5.12)), we have an isomorphism

$$A(\lambda, \mu, \nu) \cong A(\lambda^*, \mu^*, \nu^*).$$

All these isomorphisms can be chosen in a coherent way, as follows. Consider the group of transformations of the set of triples $(\lambda, \mu, \nu) \in P_+^3$ generated by all permutations of (λ, μ, ν), together with the transformation $(\lambda, \mu, \nu) \mapsto (\lambda^*, \mu^*, \nu^*)$. We will denote this group simply by $S_3 \times \mathbb{Z}/2\mathbb{Z}$. It is not hard to see that one can associate with each $\sigma \in S_3 \times \mathbb{Z}/2\mathbb{Z}$ and each triple $(\lambda, \mu, \nu) \in P_+^3$ an isomorphism

$$\sigma: A(\lambda, \mu, \nu) \to A(\sigma(\lambda, \mu, \nu)) \tag{3.6.2}$$

(denoted by the same symbol σ) in such a way that the isomorphism

$$\sigma\sigma': A(\lambda, \mu, \nu) \to A(\sigma\sigma'(\lambda, \mu, \nu))$$

is equal to the composition

$$A(\lambda, \mu, \nu) \xrightarrow{\sigma'} A(\sigma'(\lambda, \mu, \nu)) \xrightarrow{\sigma} A(\sigma\sigma'(\lambda, \mu, \nu)).$$

These symmetries were discussed in [32] and, in the quantum group setting, in [7]. Let us give explicit formulas for the action of the generators of $S_3 \times \mathbb{Z}/2\mathbb{Z}$.

Once again, let us look at the involutive anti-automorphisms τ and ι of the group N, given by (3.4.2). As in Sections 2.11 and 3.4, we denote by $f \mapsto f^\tau$ and $f \mapsto f^\iota$ the involutive automorphisms of $A = \mathbb{Q}[N]$ given by

$$f^\tau(x) = f(\tau(x)), \qquad f^\iota(x) = f(\iota(x)). \tag{3.6.3}$$

3.6.1. Proposition. *For every $\lambda, \mu, \nu \in P_+$ the restriction of $f \mapsto f^\tau$ to $A(\lambda, \mu, \nu)$ is an isomorphism $A(\lambda, \mu, \nu) \to A(\nu^*, \mu^*, \lambda^*)$ while the restriction of $f \mapsto f^\iota$ to $A(\lambda, \mu, \nu)$ is an isomorphism $A(\lambda, \mu, \nu) \to A(\nu, \mu, \lambda)$.*

Proof. We will prove the statement for τ, the proof for ι being totally similar. Since τ is an automorphism of A sending each x_{ij} to $x_{r+2-j,\, r+2-i}$, it follows from (3.5.2) that τ maps each homogeneous component $A(\gamma)$ to $A(\gamma^*)$. On the other hand, combining the definition (3.5.4) of the operators E_i and E_i^* with (3.4.3), and with the fact that τ is an anti-automorphism of N, we conclude that

$$E_i(f^\tau) = (E^*_{r+1-i} f)^\tau, \qquad E^*_i(f^\tau) = (E_{r+1-i} f)^\tau \qquad (f \in A,\ i = 1, ..., r). \tag{3.6.4}$$

Using the definition (3.5.11), we see that sends each subspace $A(\lambda, \mu, \nu)$ to $A(\nu^*, \mu^*, \lambda^*)$. Since τ is an involution, the map $\tau: A(\lambda, \mu, \nu) \to A(\nu^*, \mu^*, \lambda^*)$ is an isomorphism, as desired. ∎

According to this proposition, the isomorphism (3.6.2) corresponding to an element $\sigma: (\lambda, \mu, \nu) \mapsto (\nu^*, \mu^*, \lambda^*)$ (resp. $\sigma: (\lambda, \mu, \nu) \mapsto (\nu, \mu, \lambda)$) of the group $S_3 \times \mathbb{Z}/2\mathbb{Z}$, can be chosen as the restriction of the map $f \mapsto f^\tau$ (resp. $f \mapsto f^\iota$). The elements $(\lambda, \mu, \nu) \mapsto (\nu^*, \mu^*, \lambda^*)$ and $(\lambda, \mu, \nu) \mapsto (\nu, \mu, \lambda)$ generate a subgroup isomorphic to $\mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}$. Thus, to define the action of the whole group $S_3 \times \mathbb{Z}/2\mathbb{Z}$, we need one more generator. We choose this generator to be the transformation $(\lambda, \mu, \nu) \mapsto (\lambda, \nu, \mu)$, so we need to construct the corresponding isomorphism $A(\lambda, \mu, \nu) \mapsto A(\lambda, \nu, \mu)$ for any triple (λ, μ, ν). This isomorphism will be of special importance for us.

Let $x \mapsto \eta(x)$ be a birational automorphism of N given by

$$\eta(x) = [x w_0^{-1}]_+ \tag{3.6.5}$$

(cf. (3.1.1)). An easy check shows that η is an involution. Let $f \mapsto f^\eta$ be the corresponding involutive automorphism of the field of rational functions $\mathbb{Q}(N)$:

$$f^\eta(x) = f(\eta(x)). \tag{3.6.6}$$

For $f \in \mathbb{Q}(N)$ and $\lambda \in P_+$, we set

$$f^{\eta[\lambda]}(x) = Z^{\lambda^*}(x)\, f(\eta(x)) \tag{3.6.7}$$

(cf. (3.5.20)). The following proposition was given in [32].

3.6.2. Proposition. *For every $\lambda, \mu, \nu \in P_+$, the restriction of $f \mapsto f^{\eta[\lambda]}$ to $A(\lambda, \mu, \nu)$ is an isomorphism $A(\lambda, \mu, \nu) \to A(\lambda, \nu, \mu)$.*

In accordance with this proposition, we choose the restriction of $f \mapsto f^{\eta[\lambda]}$ as an isomorphism (3.6.2) corresponding to the transformation σ: $(\lambda, \mu, \nu) \mapsto (\lambda, \nu, \mu)$. It is straightforward to check that the isomorphisms given by the last two propositions satisfy the relations between the corresponding generators of $S_3 \times \mathbb{Z}/2\mathbb{Z}$. Thus, we can unambigiously define the isomorphism (3.6.2) for all $\sigma \in S_3 \times \mathbb{Z}/2\mathbb{Z}$.

As an application of Propositions 3.6.1 and 3.6.2, let us determine the minimal triple (λ, μ, ν) for an arbitrary polynomial T_J, $J \subset [1, r+1]$. In view of (2.9.7), we may assume that J is not of the form $[1, a]$ or $[a+1, r+1]$.

3.6.3. Proposition. *Let J be not of the form $[1, a]$ or $[a+1, r+1]$. Then the minimal triple for T_J is $(\alpha(J)^*, \omega_{r+1-|J|}, \beta(J))$. The space $A(\alpha(J)^*, \omega_{r+1-|J|}, \beta(J))$ is one-dimensional and is spanned by T_J.*

Proof. Comparing (3.4.2) and (3.6.5) with (3.1.1), we see that the matrix y in (3.1.1) is given by $y = \tau(\eta(x))$. In view of (3.3.4) and (3.5.20), the formula (3.6.7) can be written as

$$T_J = ((\Delta^J)^\tau)^{\eta[\alpha(J)^*]}. \tag{3.6.8}$$

In other words, T_J is obtained from Δ^J by the following composition of isomorphisms of the type (3.6.2):

$$A(\omega_{|J|}, \beta(J)^*, \alpha(J)) \xrightarrow{\tau} A(\alpha(J)^*, \beta(J), \omega_{r+1-|J|})$$

$$\xrightarrow{\eta[\alpha(J)^*]} A(\alpha(J)^*, \omega_{r+1-|J|}, \beta(J)).$$

By Proposition 3.5.3, T_J spans $A(\alpha(J)^*, \omega_{r+1-|J|}, \beta(J))$.

It remains to show that the triple $(\alpha(J)^*, \omega_{r+1-|J|}, \beta(J))$ is minimal for T_J. Since the middle component is a fundamental weight, the only other possibilities for the minimal triple of T_J are $(\alpha(J)^* - \omega_{|J|}, 0, \beta(J))$ or $(\alpha(J)^*, 0, \beta(J) - \omega_{|J|})$. By Proposition 3.5.4, in the first case we would have had $\alpha(J) = \beta(J) + \omega_{r+1-|J|}$, while in the second case we would have had $\beta(J) = \alpha(J) + \omega_{|J|}$. But each of these equalities is impossible unless J is of the form $[1, a]$ or $[a+1, r+1]$, since both weights $\alpha(J)$ and $\beta(J)$ are nonzero and do not have common fundamental weights in their expansions (see (3.5.16)). ∎

The last proposition supplies us with the tools needed to prove statements (i) and (ii) from the first paragraph of Section 3.5.

3.6.4. Proposition. *If J is not of the form $[1, a]$ or $[a+1, r+1]$, then T_J is a non-constant irreducible polynomial in the matrix entries x_{ij}.*

Proof. Suppose $T_J = ff'$ where f and f' are non-constant polynomials. Since T_J is homogeneous, so are f and f'. By Propositions 3.5.1.2 and 3.6.3, the sum of minimal triples for f and f' is equal to $(\alpha(J)^*, \omega_{r+1-|J|}, \beta(J))$. Hence one of these two minimal triples has the middle component 0. By Proposition 3.5.4, this triple has the form $(v^*, 0, v)$ for some nonzero $v \in P_+$. It follows that both $\alpha(J) - v$ and $\beta(J) - v$ belong to P_+. But this is impossible by the argument already used in the proof of Proposition 3.6.3: $\alpha(J)$ and $\beta(J)$ do not have common fundamental weights in their expansions.

3.6.5. Proposition. *The polynomial $T_J(x)$ is a minor of x if and only if $J = [1, d] \cup [a+1, b]$ or $J = [d+1, a] \cup [b+1, r+1]$ for some $0 \leqslant d < a < b \leqslant r+1$.*

Proof. The "if" part follows from (3.3.7) and (3.4.5). To prove the "only if" part, let us introduce some terminology. For a subset $J \subset [1, r+1]$, we define the *complexity* $c(J)$ by

$$c(J) = c(J; 1, r+1) = \max(s(J), s(\bar{J})) - 1, \tag{3.6.9}$$

where $s(J)$ is the number of connected components of J (cf. (2.10.1)). Thus, the sets of complexity 0 are exactly intervals of the form $[1, a]$ or $[a+1, r+1]$. The sets of complexity 1 are the intervals $[a+1, b]$ with $1 \leqslant a < b \leqslant r$, and also the sets with two components, at least one of which is an interval of the form $[1, a]$ or $[a+1, r+1]$. Using this terminology, we can reformulate the "only if" part as follows: if T_J is equal to some minor Δ_I^K then $c(J) \leqslant 1$.

For a weight $\lambda = l_1\omega_1 + \cdots + l_r\omega_r \in P_+$, let us denote $C(\lambda) := l_1 + \cdots + l_r$. The following properties are obvious:

$$C(\lambda+\mu) = C(\lambda) + C(\mu), \qquad C(\lambda^*) = C(\lambda). \tag{3.6.10}$$

An easy inspection shows that, for any $J \subset [1, r+1]$, one has

$$c(J) = \min(C(\alpha(J)), C(\beta(J))) \tag{3.6.11}$$

and

$$|C(\alpha(J)) - C(\beta(J))| \leqslant 1. \tag{3.6.12}$$

Now suppose that $T_J = \Delta_I^K$ for some reduced admissible pair (I, K). Comparing the minimal triples for T_J and Δ_I^K given by Propositions 3.6.3 and 3.5.2, we obtain:

$$\alpha(J)^* = \beta(I), \qquad \omega_{|J|} = \alpha(I) + \beta(K), \qquad \beta(J) = \alpha(K). \tag{3.6.13}$$

It follows that $C(\alpha(I)) + C(\beta(K)) = C(\omega_{|J|}) = 1$, hence $\min(C(\alpha(I)), C(\beta(K))) = 0$. Therefore, we have

$$\begin{aligned} c(J) &= \min(C(\alpha(J)), C(\beta(J))) \\ &= \min(C(\beta(I)), C(\alpha(K))) \\ &\leqslant \min(C(\alpha(I)), C(\beta(K))) + 1 \\ &= 1, \end{aligned}$$

as required. ▮

We have already shown in Proposition 3.4.1 that the anti-automorphisms τ and ι of N preserve the set $N_{>0}$ of totally positive matrices. In conclusion of this section, let us show that the same is true for the birational map η given by (3.6.5).

3.6.6. PROPOSITION. *The birational map η restricts to a bijection $N_{>0} \to N_{>0}$.*

Proof. We have already observed in the proof of Proposition 3.6.3 above, that the composition $\tau \circ \eta$ coincides with the transformation $x \mapsto y$ given by (3.1.1). In other words, we have $\eta(x) = \tau(y)$. The fact that η preserves $N_{>0}$ follows now from Proposition 3.4.1 combined with Theorem 3.2.5. ▮

3.7. *Dual Canonical Basis and Positivity Theorem*

In this section we will finally prove that polynomials $T_J^{\mathfrak{n}}$ have nonnegative coefficients. The proof will be based on the observation that all the T_J belong to the *dual canonical basis* B^* in the coordinate ring $A = \mathbb{Q}[N]$. Let us recall the definition of B^*. The algebra A is easily seen to be graded dual to the universal enveloping algebra $U(\mathrm{Lie}(N))$. By definition, the basis B^* in A is dual to the basis obtained via the specialization $q = 1$ from Lusztig's canonical basis B in the q-deformation U_+ of $U(\mathrm{Lie}(N))$. The q-analogue of B^* was studied in [7, 8]. Let us collect together the properties of B^* that we will need in what follows. We will use the notation $E^{(k)}$ for the *divided power* $E^k/k!$.

3.7.1. Proposition. **1.** *The dual canonical basis B^* contains the minors Δ_I^J for all reduced admissible pairs (I, J); in particular, $1 \in B^*$.*

2. *For every triple of weights (λ, μ, ν), the set $B^* \cap A(\lambda, \mu, \nu)$ is a basis of $A(\lambda, \mu, \nu)$.*

3. *For every $f \in B^* \cap A(\lambda, \mu, \nu)$, the polynomials f^τ, f^ι and $f^{\eta[\lambda]}$ (cf. Propositions 3.6.1 and 3.6.2) also lie in B^*. Therefore, B^* is preserved by all isomorphisms of type (3.6.2).*

4. *Every divided power $E_i^{(k)}$, $k \geqslant 0$ sends any element $f \in B^*$ to a linear combination of elements of B^* with nonnegative integer coefficients.*

All these statements are obtained by specializing at $q = 1$ the corresponding results in [7, 8]: see Proposition 1.3 and Theorem 1.4 in [8], Propositions 6.1 and 7.1 in [7]. As mentioned in [7, 8], statements 2–4 are consequences of more general results due to G. Lusztig and M. Kashiwara.

3.7.2. Corollary. *All polynomials T_J belong to B^*.*

Proof. Follows from (3.6.8) and parts 1 and 3 of Proposition 3.7.1. ▮

Using Corollary 3.7.2 and Proposition 3.6.3, we obtain the following characterization of the polynomials T_J.

3.7.3. Corollary. *For every J not of the form $[1, a]$ or $[a+1, r+1]$, the polynomial T_J is the unique element of B^* in $A(\alpha(J)^*, \omega_{r+1-|J|}, \beta(J))$.*

Now let us turn to the nonnegativity property. Instead of the $T_J^{\mathbf{n}}$, we will work with the polynomials $T_J^{\mathbf{h}}$ given by (1.16) (recall that $T_J^{\mathbf{h}}$ only differs from $T_J^{\mathbf{n}}$ by renaming the variables, as described in Section 2.3). More generally, for any $f \in A = \mathbb{Q}[N]$ and any reduced word $\mathbf{h} = (h_1, ..., h_m) \in R(w_0)$, let us denote by $f^{\mathbf{h}}(t_1, ..., t_m)$ the polynomial

$$f^{\mathbf{h}}(t_1, ..., t_m) = f((1 + t_1 e_{h_1}) \cdots (1 + t_m e_{h_m})). \tag{3.7.1}$$

3.7.4. THEOREM. *For every element f of B^*, the corresponding polynomial $f^{\mathbf{h}}(t_1, ..., t_m)$ has nonnegative integer coefficients In particular, all polynomials $T_J^{\mathbf{h}}$ and hence all the $T_J^{\mathbf{n}}$ have nonnegative integer coefficients.*

Proof. We will deduce our statement from a general formula that expresses the coefficients of $f^{\mathbf{h}}$ in terms of the divided powers $E_i^{(a)}$ of the infinitesimal right translation operators E_i defined in (3.5.4). Note that, in view of (3.5.6), the operator $E_{h_1}^{(a_1)} \cdots E_{h_m}^{(a_m)}$ sends a homogeneous component $A(\gamma)$ to $A(\gamma - \sum_{k=1}^m a_k \alpha_{h_k})$. In particular, if the a_k satisfy

$$\sum_{k=1}^{m} a_k \alpha_{h_k} = \gamma, \tag{3.7.2}$$

then, for each $f \in A(\gamma)$, the element $E_{h_1}^{(a_1)} \cdots E_{h_m}^{(a_m)}(f)$ lies in $A(0)$, i.e., is a rational constant.

3.7.5. LEMMA. *For every homogeneous polynomial $f \in A(\gamma)$, the polynomial $f^{\mathbf{h}}$ is given by*

$$f^{\mathbf{h}}(t_1, ..., t_m) = \sum_{a_1, ..., a_m} E_{h_1}^{(a_1)} \cdots E_{h_m}^{(a_m)}(f) \cdot t_1^{a_1} \cdots t_m^{a_m}, \tag{3.7.3}$$

where the sum is over all sequences $(a_1, ..., a_m)$ of nonnegative integers satisfying (3.7.2).

This lemma is an immediate consequence of the following observation: if we regard e_i as an element of the Lie algebra $\mathrm{Lie}(N)$, and denote by $\exp\colon \mathrm{Lie}(N) \to N$ the usual exponential map, then $\exp(te_i) = 1 + te_i \in N$ for any scalar t.

Theorem 3.7.4 now follows from Lemma 3.7.5 combined with Proposition 3.7.1, parts 1 and 4. ∎

3.7.6. *Remark.* In the case when f is a minor Δ_I^J, formula (3.7.3) combined with (3.5.10) provides another proof of (2.4.8). Let us also reiterate that, in view of Proposition 2.1.7, the polynomials $T_J^{\mathbf{n}}$ for an arbitrary ground semiring P are the same as in the special case $P = \mathbb{R}_{>0}$ considered above.

4. PIECEWISE-LINEAR MINIMIZATION FORMULAS

In this chapter, we will apply the formula (2.9.11) for the transition maps to the case when the ground semifield K is the tropical semifield $(\mathbb{Z}, \min, +)$ of Example 2.1.2. Recall that the Lusztig variety $\mathcal{L}_r(P)$ associated to the tropical semiring $P = (\mathbb{Z}_+, \min, +)$ is identified with the canonical basis B, so the transition maps relate different parametrizations of B.

4.1. *Polynomials over the Tropical Semifield*

In view of Theorem 3.7.4 and Remark 3.7.6, the functions $T^{\mathbf{n}}_J$ appearing in (2.9.11) are polynomials in the variables t_{ij}, with nonnegative integer coefficients. We will begin by providing a simple general description of the structure of such polynomials over the tropical semifield. Roughly speaking, the evaluation of a (Laurent) polynomial f over the tropical semifield only depends on the Newton polytope of f.

To be more precise, let $f \in \mathbb{Z}_+[z_1^{\pm 1}, ..., z_m^{\pm 1}]$ be a Laurent polynomial in the variables $z_1, ..., z_m$, with nonnegative integer coefficients. Let us write f as a sum of monomials:

$$f = \sum_{\xi} c_\xi z^\xi = \sum_{\xi_1, ..., \xi_m} c_{\xi_1, ..., \xi_m} z_1^{\xi_1} \cdots z_m^{\xi_m}. \tag{4.1.1}$$

The exponent vectors $\xi = (\xi_1, ..., \xi_m) \in \mathbb{Z}^m$ appearing in (4.1.1) can be viewed as lattice points in the Euclidean space $\mathbb{R}^m$. The *Newton polytope* of f is defined by

$$N(f) = \mathrm{Conv}\{\xi : c_\xi \neq 0\}, \tag{4.1.2}$$

that is, $N(f)$ is the convex hull of the exponent vectors of all monomials occurring in f. Thus, vertices of the Newton polytope $N(f)$ correspond to the *extremal* monomials of f. We denote the set of vertices of $N(f)$ by $\mathrm{Ver}(f)$.

4.1.1. Proposition. *If the variables $z_1, ..., z_m$ take values in the tropical semifield, then the evaluation of any Laurent polynomial $f \in \mathbb{Z}_+[z_1^{\pm 1}, ..., z_m^{\pm 1}]$ can be written as*

$$f(z_1, ..., z_m) = \min\{\xi_1 z_1 + \cdots + \xi_m z_m : (\xi_1, ..., \xi_m) \in \mathrm{Ver}(f)\}, \tag{4.1.3}$$

where, in the right-hand side, we use the ordinary addition and multiplication.

Proof. Since addition in the tropical semifield $\mathbb{Z}$ is idempotent, replacing all nonzero coefficients c_ξ of f by 1 will not affect the evaluation of f over $\mathbb{Z}$. Thus, we have

$$f(z_1, ..., z_m) = \min\{\xi_1 z_1 + \cdots + \xi_m z_m : c_{\xi_1, ..., \xi_m} \neq 0\}. \tag{4.1.4}$$

It remains to notice that each linear form $\xi_1 z_1 + \cdots + \xi_m z_m$ in (4.1.4) is a convex linear combination of the forms corresponding to the vertices of $N(f)$. ∎

4.1.2. *Remark.* The Newton polytope of the product of two polynomials is equal to the Minkowski sum of their Newton polytopes. One can introduce a semiring structure in the set S of lattice polytopes in $\mathbb{R}^m$ by letting multiplication in S be the Minkowski addition, and letting the sum of two polytopes in S be the convex hull of their union. By virtue of Proposition 4.1.1, the semiring S is isomorphic to the semiring T of those piecewise-linear functions in m variables which can be written as minima of linear forms having integer coefficients. In T, the semiring operations are the "tropical" ones, i.e., addition is minimum and multiplication is the usual addition.

In view of Proposition 4.1.1 and (2.9.9), we can rewrite (2.9.11) as follows.

4.1.3. THEOREM. *The transition maps over the tropical semifield are given by*

$$(R_{\mathbf{n}}^{\mathbf{n}'}(t))_{ij} = \delta_{i+1,j} \sum_{a \leqslant i < b} t_{ab} + \min\left\{\sum \xi_{ab} t_{ab} \colon (\xi_{ab}) \in \mathrm{Ver}(T_{L'}^{\mathbf{n}} T_{L'ij}^{\mathbf{n}})\right\}$$
$$- \min\left\{\sum \xi_{ab} t_{ab} \colon (\xi_{ab}) \in \mathrm{Ver}(T_{L'i}^{\mathbf{n}} T_{L'j}^{\mathbf{n}})\right\}, \qquad (4.15)$$

where $L' = L^{\mathbf{n}'}(i, j)$ (*see* (2.5.1)).

In the rest of this chapter we will discuss various applications of this theorem.

4.2. *Multisegment Duality*

In Theorem 2.8.2, we gave an explicit formula for the transition map $R_{\mathbf{n}^0}^{\mathbf{n}}$ from the normal ordering $\mathbf{n}^0$ to an arbitrary normal ordering $\mathbf{n}$, where $\mathbf{n}^0$ corresponds to the lexicographically minimal reduced word $\mathbf{h}^0 = (1, 2, 1, 3, 2, 1, ..., r, r-1, ..., 1)$ (see (2.4.13)). Translating this result into the "tropical language" yields the following formula.

4.2.1. THEOREM. *The transition map* $R_{\mathbf{n}^0}^{\mathbf{n}}$ *over the tropical semifield is given by*

$$R_{\mathbf{n}^0}^{\mathbf{n}}(t))_{ij} = \sum_{[i,j] \subset [a,b]} e(Lij; a; b)\, t_{ab}$$
$$+ \min\left\{\sum_{(p,q) \in E(L)} t_{a_{pq}, a_{pq}+q-p} \colon (a_{pq}) \in \mathrm{Tab}(L)\right\}$$

$$+\min\left\{\sum_{(p,q)\in E(Lij)} t_{a_{pq},\,a_{pq}+q-p} : (a_{pq})\in \mathrm{Tab}(Lij)\right\}$$

$$-\min\left\{\sum_{(p,q)\in E(Li)} t_{a_{pq},\,a_{pq}+q-p} : (a_{pq})\in \mathrm{Tab}(Li)\right\}$$

$$-\min\left\{\sum_{(p,q)\in E(Lj)} t_{a_{pq},\,a_{pq}+q-p} : (a_{pq})\in \mathrm{Tab}(Lj)\right\}, \qquad (4.2.1)$$

where Tab(J) *stands for the set of all* J*-tableaux* (*see Example* 2.6.6), $E(J)$ *denotes the essential set of* J (*see* (2.8.8)) *and* $L = L^{\mathbf{n}}(i, j)$.

Let us apply this theorem in the special case when $\mathbf{n}$ is the normal ordering $\mathbf{n}^1 = \mathbf{n}(\mathbf{h}^1)$ corresponding to the lexicographically *maximal* reduced word

$$\mathbf{h}^1 = (r, r-1, r, r-2, r-1, r, ..., 1, 2, ..., r). \qquad (4.2.2)$$

Since $\mathbf{h}^1 = \overline{(\mathbf{h}^0)^*}$ (see (2.11.2)), we have

$$\mathbf{n}^1 = \overline{(\mathbf{n}^0)^*} = ((1,2), (1,3), (2,3), ..., (1, r+1), (2, r+1), ..., (r, r+1)). \qquad (4.2.3)$$

4.2.2. Theorem. *The transition map* $R_{\mathbf{n}^0}^{\mathbf{n}^1}$ *over the tropical semifield is given by*

$$\begin{aligned}(R_{\mathbf{n}^0}^{\mathbf{n}^1}(t))_{ij} &= \min\left\{\sum_{(p,q)\in[1,i-1]\times[1,r+1-j]} t_{\tau(p,q),\,\tau(p,q)+i-1-p+q} : \tau\in GYT_{i-1,j}\right\}\\ &\quad+\min\left\{\sum_{(p,q)\in[1,i]\times[1,r+2-j]} t_{\tau(p,q),\,\tau(p,q)+i-p+q} : \tau\in GYT_{i,j-1}\right\}\\ &\quad-\min\left\{\sum_{(p,q)\in[1,i]\times[1,r+1-j]} t_{\tau(p,q),\,\tau(p,q)+i-p+q} : \tau\in GYT_{i,j}\right\}\\ &\quad-\min\left\{\sum_{(p,q)\in[1,i-1]\times[1,r+2-j]} t_{\tau(p,q),\,\tau(p,q)+i-1-p+q} : \tau\in GYT_{i-1,j-1}\right\},\end{aligned} \qquad (4.2.4)$$

where $GYT_{i,j}$ *stands for the set of all* (*generalized, or semi-standard*) *Young tableaux* τ: $[1,i]\times[1,r+1-j]\to[1,j]$ (*see Example* 2.8.3).

Proof. Combining (2.1.8) and (2.11.9) with the description of the chamber sets for $\mathbf{n}^0$ given by (2.8.4), we obtain:

$$L^{\mathbf{n}}(i, j) = \overline{[i, j]} = [1, i-1] \cup [j+1, r+1]. \tag{4.2.5}$$

Remembering (2.8.12), it is easy to see that in our case the first summand in (4.2.1) is equal to

$$\delta_{i+1, j} \sum_{[i, j] \subset [a, b]} t_{ab}. \tag{4.2.6}$$

As for the other four summands, we transform them as in Example 2.8.3, which leads to (4.2.4) (note that if $j = i+1$, then the summand in (4.2.1) that corresponds to Lij disappears and gets replaced by the expression (4.2.6)). ∎

4.2.3. *Remark.* Formula (4.2.4) practically coincides with the expression for the so-called *multisegment duality involution* given in [24, (1.4), (1.9)]. The only difference is that in [24], positive roots of type A_r were not labeled by the elements of our index set $\Pi_r = \{(i, j)\colon 1 \leqslant i < j \leqslant r+1\}$; rather, the set $\{(i, j)\colon 1 \leqslant i \leqslant j \leqslant r\}$ was used which can be obtained from Π_r by the shift $(i, j) \mapsto (i, j-1)$. Thus, the transition map $R_{\mathbf{n}^0}^{\mathbf{n}^1}$ can be identified with the multisegment duality. This fact, already mentioned in [7, 24], is a special case of a general result [23, Theorem 2.0] that relates the transition maps to the geometry of representations of quivers. (We will discuss this connection in more detail in Section 4.4.) Combining this general result with Theorem 4.2.2 yields a new proof of the main theorem in [24].

A fundamental property of the multisegment duality is that it is an involution (which is not obvious at all from (4.2.4). We now show that this fact is an easy consequence of the general symmetry properties of the transition maps given in Section 2.11.

4.2.4. Proposition. *The transformation $R_{\mathbf{n}^0}^{\mathbf{n}^1}\colon P^{\Pi} \to P^{\Pi}$ is an involution, and commutes with the involution $t \to t^*$ given by* (2.11.3).

Proof. We start with an observation that, in view of (2.11.11), (4.2.3) and (2.11.13), for every $\mathbf{n}$ we have

$$R_{\mathbf{n}^1}^{\mathbf{n}} = R_{\overline{(\mathbf{n}^0)^*}}^{\mathbf{n}} = R_{(\mathbf{n}^0)^*}^{\bar{\mathbf{n}}} = R_{\mathbf{n}^0}^{\bar{\mathbf{n}}}. \tag{4.2.7}$$

Setting $\mathbf{n} = (\mathbf{n}^0)^* = \overline{\mathbf{n}^1}$ in (4.2.7), we obtain

$$R_{\mathbf{n}^1}^{\mathbf{n}^0} = R_{\mathbf{n}^1}^{\mathbf{n}} = R_{\mathbf{n}^0}^{\bar{\mathbf{n}}} = R_{\mathbf{n}^0}^{\mathbf{n}^1},$$

which proves that $R_{\mathbf{n}^0}^{\mathbf{n}^1}$ is an involution. The fact that $R_{\mathbf{n}^0}^{\mathbf{n}^1}$ commutes with $t \to t^*$ follows by setting $\mathbf{n} = \mathbf{n}^1$ in (2.11.12). ∎

4.2.5. *Remark.* In view of (4.2.7), an explicit formula for $R^{\mathbf{n}}_{\mathbf{n}^1}$ for an arbitrary normal ordering $\mathbf{n}$ can be obtained from (4.2.1), if we replace $\mathbf{n}$ by $\bar{\mathbf{n}}$.

4.3. *Nested Normal Orderings*

In this section, we consider a family of special reduced words and normal orderings that we call *nested.* This family was introduced and studied in [7]; it contains the orderings $\mathbf{n}^0$ and $\mathbf{n}^1$ and, in a certain sense, interpolates between them.

For every interval $[c, d] \subset [1, r+1]$ with $c<d$, let $S_{[c,d]}$ denote the group of all permutations of $[c, d]$, and let $w_0[c, d]$ be the maximal element of $S_{[c,d]}$; in particular, the whole group S_{r+1} and its maximal element w_0 receive the notation $S_{[1,r+1]}$ and $w_0[1, r+1]$. An easy check shows that

$$\begin{aligned} w_0[c, d] &= w_0[c, d-1]\, s_{d-1}s_{d-2}\cdots s_c \\ &= w_0[c+1, d]\, s_c s_{c+1}\cdots s_{d-1}. \end{aligned} \tag{4.3.1}$$

Using (4.3.1), we give the following recursive definition: a *nested reduced word* for $w_0[c, d]$ is either a nested reduced word for $w_0[c, d-1]$ followed by $d-1, d-2, ..., c$, or a nested reduced word for $w_0[c+1, d]$ followed by $c, c+1, ..., d-1$; the only reduced word (c) for $w_0[c, c+1] = s_c$ is nested by definition.

A normal ordering of Π corresponding to a nested reduced word will also be called nested. Nested normal orderings can be defined recursively as follows. Let $\Pi^{[c,d]}$ be the set of pairs (i, j) with $c \leqslant i < j \leqslant d$. Then a nested normal ordering of $\Pi^{[c,d]}$ is either a nested normal ordering of $\Pi^{[c+1,d]}$ followed by (c, d), $(c, d-1)$, ..., $(c, c+1)$, or a nested normal ordering of $\Pi^{[c,d-1]}$ followed by (c, d), $(c+1, d)$, ..., $(d-1, d)$. Unraveling this definition, we see that nested normal orderings of Π are in a bijective correspondence with nested sequences, or flags, of intervals $[c_1, d_1] \subset [c_2, d_2] \subset \cdots \subset [c_r, d_r] = [1, r+1]$, where $d_k - c_k = k$ for $k = 1, ..., r$. We will encode these flags by their *sign vectors* $\varepsilon = (\varepsilon_2, ..., \varepsilon_r)$, where each ε_k is either $+$ or $-$; we set

$$\varepsilon_k = \begin{cases} +, & \text{if} \quad [c_k, d_k] = [c_{k-1}, d_{k-1}+1] \\ -, & \text{if} \quad [c_k, d_k] = [c_{k-1}-1, d_{k-1}]. \end{cases} \tag{4.3.2}$$

The nested reduced word and normal ordering with the sign vector ε will be denoted by $\mathbf{h}(\varepsilon)$ and $\mathbf{n}(\varepsilon)$, respectively. The ordering $\mathbf{n}(\varepsilon)$ is thus the concatenation $\mathbf{n}(\varepsilon) = (\mathbf{n}^{(1)}, ..., \mathbf{n}^{(r)})$, where

$$\mathbf{n}^{(k)} = \begin{cases} (c_k, d_k), (c_k+1, d_k), ..., (d_k-1, d_k) & \text{if} \quad \varepsilon_k = + \\ (c_k, d_k), (c_k, d_k-1), ..., (c_k, c_k+1) & \text{if} \quad \varepsilon_k = -. \end{cases} \tag{4.3.3}$$

For a given r, there are 2^{r-1} different nested normal orderings. For instance, if $r=2$, then both reduced words for w_0 and the corresponding normal orderings are nested:

$$\mathbf{h}(-)=(1,2,1), \qquad \mathbf{n}(-)=(2,3),(1,3),(1,2);$$
$$\mathbf{h}(+)=(2,1,2), \qquad \mathbf{n}(+)=(1,2),(1,3),(2,3).$$

For $r=3$, the nested reduced words and normal orderings are:

$$\mathbf{h}(-,-)=(1,2,1,3,2,1),$$
$$\mathbf{n}(-,-)=(3,4),(2,4),(2,3),(1,4),(1,3),(1,2);$$
$$\mathbf{h}(-,+)=(2,3,2,1,2,3),$$
$$\mathbf{n}(-,+)=(2,3),(1,3),(1,2),(1,4),(2,4),(3,4);$$
$$\mathbf{h}(+,-)=(2,1,2,3,2,1),$$
$$\mathbf{n}(+,-)=(2,3),(2,4),(3,4),(1,4),(1,3),(1,2);$$
$$\mathbf{h}(+,+)=(3,2,3,1,2,3),$$
$$\mathbf{n}(+,+)=(1,2),(1,3),(2,3),(1,4),(2,4),(3,4).$$

Note that, for arbitrary r, the lexicographically minimal and maximal reduced words and the corresponding normal orderings are nested:

$$\begin{aligned} \mathbf{n}^0 &= \mathbf{n}(-,-,\ldots,-), \\ \mathbf{n}^1 &= \mathbf{n}(+,+,\ldots,+). \end{aligned} \tag{4.3.4}$$

We will say that a pair $(i,j)\in\Pi$ is *ε-positive* if it lies in a segment $\mathbf{n}^{(k)}$ of the normal ordering $\mathbf{n}(\varepsilon)$ (see (4.3.3)), and $\varepsilon_k=+$. In other words, (i,j) is ε-positive if

$$c_k \leqslant i < j = d_k, \qquad \varepsilon_k = +; \tag{4.3.5}$$

similarly, $(i,j)\in\Pi$ is *ε-negative* if

$$c_k = i < j \leqslant d_k, \qquad \varepsilon_k = -. \tag{4.3.6}$$

The following proposition gives a description of all chamber sets for $\mathbf{n}(\varepsilon)$; its proof is straightforward.

4.3.1. Proposition. *For any sign vector ε and any $(i,j)\in\Pi$, we have*

$$L^{\mathbf{n}(\varepsilon)}(i,j)=\begin{cases}[c_k, i-1]\cup[j+1, r+1], & \text{if } (i,j) \text{ is } \varepsilon\text{-positive;}\\ [i+1, j-1]\cup[d_k+1, r+1], & \text{if } (i,j) \text{ is } \varepsilon\text{-negative.}\end{cases} \tag{4.3.7}$$

In particular, taking $\varepsilon = (-, -, ..., -)$ or $\varepsilon = (+, +, ..., +)$ we recover the descriptions of the chamber sets of $\mathbf{n}^0$ and $\mathbf{n}^1$ given by (2.8.4) and (4.2.5):

$$L^{\mathbf{n}^0}(i, j) = [i+1, j-1],$$

$$L^{\mathbf{n}^1}(i, j) = [1, i-1] \cup [j+1, r+1].$$

4.3.2. COROLLARY. *The transition map from an arbitrary normal ordering* $\mathbf{n}$ *to the nested normal ordering* $\mathbf{n}(\varepsilon)$ *is given by the following formulas: if* (i, j) *is* ε*-positive* (*see* (4.3.5)), *then*

$$\begin{aligned}(R_{\mathbf{n}}^{\mathbf{n}(\varepsilon)}(t))_{ij} &\\ &= Z_i(t)^{\delta_{i+1,j}} \frac{(\Delta^{[i, r+1]}_{[1, r+1-j] \cup [r+2-c_k-j+i, r+2-c_k]})^{\mathbf{n}}(t) \times (\Delta^{[i+1, r+1]}_{[1, r+2-j] \cup [r+4-c_k-j+i, r+2-c_k]})^{\mathbf{n}}(t)}{(\Delta^{[i+1, r+1]}_{[1, r+1-j] \cup [r+3-c_k-j+i, r+2-c_k]})^{\mathbf{n}}(t) \times (\Delta^{[i, r+1]}_{[1, r+2-j] \cup [r+3-c_k-j+i, r+2-c_k]})^{\mathbf{n}}(t)};\end{aligned} \tag{4.3.8}$$

if (i, j) *is* ε*-negative* (*see* (4.3.6)), *then*

$$\begin{aligned}(R_{\mathbf{n}}^{\mathbf{n}(\varepsilon)}(t))_{ij} &\\ &= Z_i(t)^{\delta_{i+1,j}} \frac{(\Delta^{[j, r+1]}_{[1, r+1-d_k] \cup [r+1-d_k+j-i, r+1-i]})^{\mathbf{n}}(t) \times (\Delta^{[j+1, r+1]}_{[1, r+1-d_k] \cup [r+3-d_k+j-i, r+2-i]})^{\mathbf{n}}(t)}{(\Delta^{[j, r+1]}_{[1, r+1-d_k] \cup [r+2-d_k+j-i, r+2-i]})^{\mathbf{n}}(t) \times (\Delta^{[j+1, r+1]}_{[1, r+1-d_k] \cup [r+2-d_k+j-i, r+1-i]})^{\mathbf{n}}(t)}.\end{aligned} \tag{4.3.9}$$

Proof. Substitute (4.3.7) into (2.9.11), and use (3.4.5). ▮

In the special case $\varepsilon = (-, -, ..., -)$, (4.3.9) specializes to the formula (2.4.14) for $R_{\mathbf{n}}^{\mathbf{n}^0}$.

Formulas (4.3.8)–(4.3.9) are indeed explicit, since each factor $(\Delta_I^J)^{\mathbf{n}}(t)$ is given by the combinatorial expression (2.4.10), and $Z_i(t)$ is given by (2.9.9). Recall that (2.4.10) is stated in terms of families of vertex-disjoint paths in the graph $\Gamma(\mathbf{n})$. We will now show that, if $\mathbf{n}$ is also a nested ordering, then this combinatorial expression can be translated into a more traditional language of tableaux of a certain kind. While working with partitions, Young diagrams and tableaux, we will use the terminology and notation of [31]. We will identify a partition $\lambda = (\lambda_1 \geqslant \lambda_2 \geqslant \cdots \geqslant \lambda_s \geqslant 0)$ with its diagram

$$\lambda = \{(p, q) \in \mathbb{Z}^2 \colon p \in [1, s], q \in [1, \lambda_i]\}.$$

We will view the first component p of a point (or *box*) $v=(p, q) \in \lambda$ as the row index, and the second component as the column index; thus λ will have rows of length $\lambda_1, ..., \lambda_s$. For $v=(p, q)$, we define

$$d(v)=r+1+p-q. \tag{4.3.10}$$

The values of the function $v \mapsto d(v)$ for the boxes v contained in the rectangular diagram $(r+1)^r$ with r rows and $r+1$ columns are shown below, for $r=4$:

5	4	3	2	1
6	5	4	3	2
7	6	5	4	3
8	7	6	5	4

Now let λ and μ be two partitions such that $\mu \subset \lambda \subset (r+1)^r$. Let $\varepsilon=(\varepsilon_2, ..., \varepsilon_r)$ be any sign vector, and $[c_1, d_1] \subset [c_2, d_2] \subset \cdots \subset [c_r, d_r] = [1, r+1]$ be the flag of intervals related to ε via (4.3.2). By an ε-*tableau of shape* λ/μ we will mean an increasing sequence of partitions $\mu=\lambda^{(0)} \subset \lambda^{(1)} \subset \cdots \subset \lambda^{(r)}=\lambda$ satisfying the following two conditions:

(1) If $\varepsilon_k=+$, then $\lambda^{(k)}-\lambda^{(k-1)}$ is a *horizontal strip*, i.e., contains at most one box in each column; if $\varepsilon_k=-$, then $\lambda^{(k)}-\lambda^{(k-1)}$ is a *vertical strip*, i.e., contains at most one box in each row.

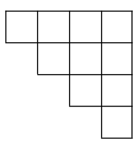
(2) For any $k=1, ..., r$, and any box $v \in \lambda^{(k)}-\lambda^{(k-1)}$, we have

$$c_k \leqslant d(v) < d_k. \tag{4.3.11}$$

An ε-tableau of shape λ/μ can be identified with a function $\tau\colon \lambda-\mu \to [1, r]$ given by

$$\tau(v)=\min\{k\colon v \in \lambda^{(k)}\};$$

equivalently, $\tau(v)=k$ for $v \in \lambda^{(k)}-\lambda^{(k-1)}$. We call k the *entry* in the box v of a tableau τ. Note that (4.3.11) implies that all entries of an ε-tableau are positive integers $\leqslant r$. It follows that the shape $\lambda-\mu$ of such a tableau should be contained inside the staircase



which has r columns and r rows, and whose upper-leftmost box is $(1, 2)$.

For a box v with the entry k in an ε-tableau τ, let us define

$$(i(\tau, v), j(\tau, v)) = \begin{cases} (c_k + d_k - 1 - d(v), d_k), & \text{if} \quad \varepsilon_k = +; \\ (c_k, c_k + d_k - d(v)), & \text{if} \quad \varepsilon_k = -. \end{cases} \tag{4.3.12}$$

The above conditions (1) and (2) imply that for a fixed τ the correspondence $v \mapsto (i(\tau, v), j(\tau, v))$ is an injective mapping $\lambda - \mu \to \Pi$.

4.3.3. Example. Let $\varepsilon = (+, +, ..., +)$. Then $c_k = 1$ and $d_k = k + 1$ for $k = 1, ..., r$, so the definition of an ε-tableau can be restated as follows: this is a (semi-standard) Young tableau τ of shape λ/μ, with entries in $[1, r]$, such that $\tau(v) \geqslant d(v)$ for any box $v \in \lambda - \mu$. Furthermore, (4.3.12) takes the form

$$(i(\tau, v), j(\tau, v)) = (\tau(v) + 1 - d(v), \tau(v) + 1). \tag{4.3.13}$$

For example, for a tableau

$$\begin{array}{|c|c|c|c|} \hline & & 2 & 2 \\ \hline \multicolumn{1}{c|}{} & & 4 & \\ \cline{2-4} \multicolumn{2}{c|}{} & & \\ \cline{3-4} \multicolumn{3}{c|}{} & \\ \cline{4-4} \end{array}$$

of shape $\lambda/\mu = (5, 4)/(3, 3)$, the values $i(\tau, v)$ and $j(\tau, v)$ are shown below:

$$i(\tau, v) \quad \begin{array}{|c|c|c|c|} \hline & & 1 & 1 \\ \hline \multicolumn{1}{c|}{} & & 2 & \\ \cline{2-4} \multicolumn{2}{c|}{} & & \\ \cline{3-4} \multicolumn{3}{c|}{} & \\ \cline{4-4} \end{array} \qquad j(\tau, v) \quad \begin{array}{|c|c|c|c|} \hline & & 3 & 3 \\ \hline \multicolumn{1}{c|}{} & & 5 & \\ \cline{2-4} \multicolumn{2}{c|}{} & & \\ \cline{3-4} \multicolumn{3}{c|}{} & \\ \cline{4-4} \end{array}$$

Now let

$$I = \{i_1 > i_2 > \cdots > i_s\},$$
$$J = \{j_1 > j_2 > \cdots > j_s\}$$

be two subsets in $[1, r+1]$ of the same cardinality, such that $i_k < j_k$ for $k = 1, ..., s$. We associate to (I, J) a pair of partitions (μ, λ) by setting

$$\mu_k = i_k + k - 1, \qquad \lambda_k = j_k + k - 1 \qquad (k = 1, ..., s). \tag{4.3.14}$$

For instance, the pair of sets $I = \{3, 2\}$ and $J = \{5, 3\}$ corresponds to the shape $\lambda/\mu = (5, 4)/(3, 3)$ (see above). Equivalently, one could view λ/μ as a "skew shifted shape" obtained by removing the shifted shape (see, e.g., [31]) with row lengths $i_1, ..., i_s$, from the shifted shape with row lengths $j_1, ..., j_s$.

4.3.4. Proposition. *In the above notation, the polynomial* $(\Delta_I^J)^{\mathbf{n}(\varepsilon)}$, *for any sign vector* ε, *is given by*

$$(\Delta_I^J)^{\mathbf{n}(\varepsilon)}(t) = \sum_{\tau} \prod_{v \in \lambda - \mu} t_{i(\tau, v),\, j(\tau, v)}, \tag{4.3.15}$$

where the sum is over all ε*-tableaux* τ *of shape* λ/μ.

Proof. Direct inspection shows that, for $\mathbf{n} = \mathbf{n}(\varepsilon)$, the families of paths that contribute to (2.4.10), are encoded by ε-tableaux of shape λ/μ, so that the summands in (2.4.10) become the monomials in (4.3.15). This is a modification of the Gessel–Viennot argument [19]. We leave the details to the reader. ∎

4.3.5. Example. Let us compute the polynomial $T^{\mathbf{n}^1}_{[a+1, b]} = (\Delta^{[b+1, r+1]}_{[b+1-a, r+1-a]})^{\mathbf{n}^1}$ (see Proposition 2.10.2). In view of (4.3.4) and (4.3.14), this polynomial is given by (4.3.15) with $\lambda = (r+1)^{r+1-b}$, $\mu = (r+1-a)^{r+1-b}$, and $\varepsilon = (+, +, ..., +)$. Thus, λ/μ is the rectangle $[1, r+1-b] \times [r+2-a, r+1]$. Using the description of ε-tableaux given in Example 4.3.3, we can rewrite (4.3.15) in this case as follows:

$$\begin{aligned}(\Delta^{[b+1, r+1]}_{[b+1-a, r+1-a]})^{\mathbf{n}^1}(t) \\ = \sum_{\tau} \prod_{(p, q) \in [1, r+1-b] \times [r+2-a, r+1]} t_{\tau(p, q) - r - p + q,\, \tau(p, q) + 1},\end{aligned} \tag{4.3.16}$$

where the sum is over all Young tableaux $\tau: [1, r+1-b] \times [r+2-a, r+1] \to [1, r]$ such that $\tau(p, q) > r + p - q$ for all p, q. It is straightforward to show that these tableaux τ are in a bijective correspondence with tableaux $\tau' \in GYT_{a, b}$ (see Theorem 4.2.2). Namely, the correspondence $\tau \mapsto \tau'$ is given by

$$\tau(p, q) - r - p + q = \tau'(p', q'),$$

where (p, q) and (p', q') are related by

$$p' = q - r - 1 + a, \qquad q' = p.$$

Replacing τ by τ' transforms (4.3.16) into the expression for the polynomial $Q_{[1, a] \cup [b+1, r+1]}$ given by (2.8.15). Thus, Proposition 4.3.4 implies that

$$T^{\mathbf{n}^1}_{[a+1, b]} = (\Delta^{[b+1, r+1]}_{[b+1-a, r+1-a]})^{\mathbf{n}^1} = Q_{[1, a] \cup [b+1, r+1]}.$$

This fact can also be derived from (2.10.4) and (2.11.7).

4.3.6. *Remark.* Combining Corollary 4.3.2 and Proposition 4.3.4, we can obtain an explicit formula similar to (4.2.4), for the transition map (over the tropical semifield) between any two nested normal orderings.

4.4. *Quivers and Boundary Pseudo-line Arrangements*

In this section, we will show that our Theorem 4.2.1 can be used to obtain another proof of the piecewise-linear minimization formula of H. Knight in the geometry of quiver representations (see [23, Theorem 6.0]). Let us briefly recall some of the setup in [23]. A *quiver* (of type A_r) is an orientation Ω of the Coxeter–Dynkin graph. Thus, Ω has vertices labelled $1, ..., r$, and each edge $(h-1, h)$, for $h=2, ..., r$, is oriented one way or another. In [23, Section 1], to any two quivers Ω and Ω' there was associated a piecewise-linear bijection

$$\Psi_{\Omega, \Omega'}\colon \mathbb{Z}_+^{\Pi_r} \to \mathbb{Z}_+^{\Pi_r}.$$

We will not reproduce here the precise definition of this bijection; let us only indicate that it relates two different labellings of irreducible components of certain Lagrangian varieties associated with quiver representations. Theorem 2.0 in [23] asserts that $\Psi_{\Omega, \Omega'}$ coincides with the transition map $R_{\mathbf{n}}^{\mathbf{n}'}$ over the tropical semiring, where the normal orderings $\mathbf{n}$ and $\mathbf{n}'$ are related to the quivers Ω and Ω' in the way specified below.

If $h \in [1, r]$ is a sink of Ω, then let $s_h\Omega$ denote the quiver obtained from Ω by reversing the arrows directed to h (thus making h into a source). We will say that a normal ordering $\mathbf{n}$ of Π_r is *adapted* to a quiver Ω if the reduced word $\mathbf{h} = (h_1, ..., h_m) \in R(w_0)$ corresponding to $\mathbf{n}$ has the following property:

$$\text{for } k=1, ..., m, \text{ the vertex } h_k \text{ is a sink of the quiver } s_{h_{k+1}} s_{h_{k+2}} \cdots s_{h_m}\Omega. \tag{4.4.1}$$

(In the terminology of [23, 28], this means that $\mathbf{h}^* = (h_m, ..., h_1)$ is adapted to Ω.) For example, the normal ordering $\mathbf{n} = (23\ 24\ 13\ 14\ 34\ 12)$ is adapted to the quiver

$$\bullet \longleftarrow \bullet \longrightarrow \bullet \tag{4.4.2}$$

of type A_3.

With the help of the symmetry property (2.11.11), Theorem 2.0 from [23] can be reformulated as follows: if normal orderings $\mathbf{n}$ and $\mathbf{n}'$ are adapted to quivers Ω and Ω', respectively, then

$$\Psi_{\Omega, \Omega'} = R_{\mathbf{n}}^{\mathbf{n}'}. \tag{4.4.3}$$

The existence of a reduced word (hence, that of a normal ordering) adapted to an arbitrary quiver was proved in [28, 4.12 (b)]. We will refine upon this result (for type A_r only) by giving an explicit description of a normal ordering adapted to a quiver Ω. As a first step in this direction, let us reformulate the relation "$\mathbf{n}$ is adapted to Ω" in terms of the pseudo-line arrangement $\mathrm{Arr}(\mathbf{n})$.

We will encode a quiver Ω by the set Λ of all indices $h \in [2, r]$ such that the edge $(h-1, h)$ is oriented from h to $h-1$ in Ω. The quiver that corresponds this way to a subset $\Lambda \subset [2, r]$ will be denoted $\Omega(\Lambda)$.

On the other side, let $\mathbf{n}$ be a normal ordering of Π_r. Consider the corresponding graph $\Gamma(\mathbf{n})$ constructed in Section 2.4. Let $\Gamma_0(\mathbf{n})$ be the graph obtained from $\Gamma(\mathbf{n})$ by removing the horizontal parts of all Z-shaped connectors. Alternatively, $\Gamma_0(\mathbf{n})$ can be obtained from (the wiring diagram of) the arrangement $\mathrm{Arr}(\mathbf{n})$ by removing all segments of negative slope. Each connected component of $\Gamma_0(\mathbf{n})$ is a part of a particular pseudo-line in $\mathrm{Arr}(\mathbf{n})$. We will call a connected component of $\Gamma_0(\mathbf{n})$ *non-trivial* if it contains at least one segment of positive slope. The normal ordering $\mathbf{n}$ and the corresponding arrangement $\mathrm{Arr}(\mathbf{n})$ will be called *boundary* if *all* endpoints of nontrivial connected components of $\Gamma_0(\mathbf{n})$ lie on the boundary of the smallest rectangle that contains $\mathrm{Arr}(\mathbf{n})$. For example, Fig. 12 shows the graphs $\Gamma_0(\mathbf{n})$ for the normal orderings (23 24 13 14 34 12) and (23 24 34 14 13 12).

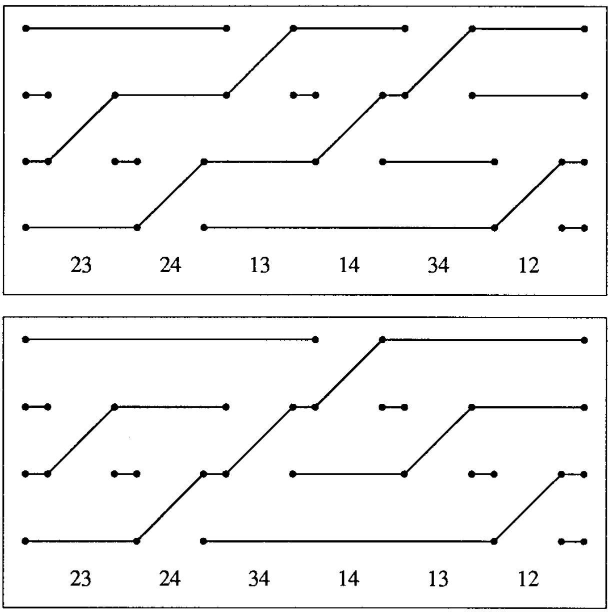


FIG. 12. Two examples of graphs $\Gamma_0(\mathbf{n})$.

(The pseudo-line arrangement corresponding to the first ordering appears in Figs. 3 and 4 of Section 2.) The first of these two orderings is boundary, while the second one is not.

4.4.1. PROPOSITION. *A normal ordering* $\mathbf{n}$ *of* Π_r *is adapted to some quiver* Ω *if and only if* $\mathbf{n}$ *is boundary. If this is the case, then* Ω *is uniquely determined by* $\mathbf{n}$. *Namely,* $\Omega = \Omega(\Lambda)$ *where* Λ *is the set of all indices* $h \in [2, r]$ *such that the right endpoint of* Line_h *in* $\mathrm{Arr}(\mathbf{n})$ *belongs to a non-trivial component of* $\Gamma_0(\mathbf{n})$.

For instance, the ordering $\mathbf{n} = (23\ 24\ 13\ 14\ 34\ 12)$ shown in Figure 12 is adapted to the quiver $\Omega(\{2\})$ of type A_3 (see (4.4.2)).

Proof. Let $\mathbf{n}$ be a boundary normal ordering, and let $\mathbf{h} = (h_1, ..., h_m) \in R(w_0)$ be the corresponding reduced word. For $k = 0, 1, ..., m$, let Λ_k be the set of indices $h \in [2, r]$ such that the segment of the hth horizontal line (counting bottom-up) lying immediately to the right of the kth diagonal connector (counting from the left to the right) belongs to a nontrivial component of $\Gamma_0(\mathbf{n})$. In this notation, the set Λ in our proposition is Λ_m. In view of (4.4.1), the fact that $\mathbf{n}$ is adapted to $\Omega(\Lambda)$ is a consequence of the following lemma.

4.4.2. LEMMA. *For* $k = 1, ..., m$, *the vertex* h_k *is a sink of the quiver* $\Omega(\Lambda_k)$, *and we have*

$$s_{h_k} \Omega(\Lambda_k) = \Omega(\Lambda_{k-1}). \tag{4.4.4}$$

Proof of Lemma 4.4.2. By definition, a vertex $h \in [2, r-1]$ is a sink of a quiver $\Omega(\Lambda)$ if and only if $h \notin \Lambda$, $(h+1) \in \Lambda$; in this case we have

$$s_h \Omega(\Lambda) = \Omega(\Lambda \cup \{h\} \backslash \{h+1\}). \tag{4.4.5}$$

If $h = 1$ or $h = r$, this is modified as follows: the vertex 1 is a sink of $\Omega(\Lambda)$ if and only if $2 \in \Lambda$, and in this case $s_1 \Omega(\Lambda) = \Omega(\lambda \backslash \{2\})$; the vertex r is a sink of $\Omega(\Lambda)$ if and only if $r \notin \Lambda$, and in this case $s_1 \Omega(\Lambda) = \Omega(\Lambda \cup \{r\})$. This description allows to prove the lemma by a straightforward verification. To convince yourself that it can indeed be done, examine Fig. 13 where the vertical arrows describe the quivers $\Omega(\Lambda_k)$. ∎

Lemma 4.4.2 yields the "if" part of Proposition 4.4.1. The "only if" part and the uniqueness of Ω can also be checked from the description of $s_h \Omega(\Lambda)$ given in (4.4.5), and from the above pictorial interpretation. ∎

Now let Λ be an arbitrary subset of $[2, r]$. Using Proposition 4.4.1, we will explicitly construct (an isotopy class of) a boundary pseudo-line arrangement $\mathrm{Arr}(\Lambda)$ adapted to the quiver $\Omega(\Lambda)$. Consider a square on the

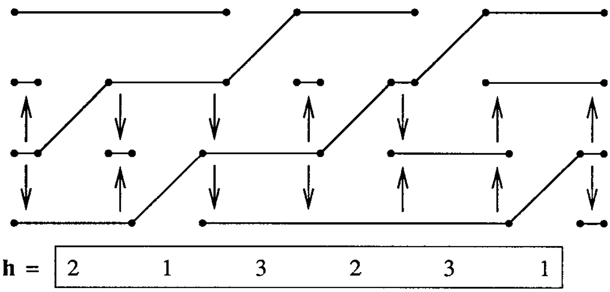


FIG. 13. Quivers appearing in $\Gamma_0(\mathbf{n})$.

plane, with vertical and horizontal sides. The $2(r+1)$ endpoints of our pseudo-lines will divide each vertical side of the square into r equal parts. As in Section 2.3, we number the left endpoints top to bottom, while the right endpoints are numbered bottom-up. Each pseudo-line Line_i will connect the endpoints labelled i. The pseudo-lines Line_1 and Line_{r+1} of $\mathrm{Arr}(\Lambda)$ will be the diagonals of the square. For each $h \in [2, r]$, we define the pseudo-line Line_h to be a union of two line segments of slopes $\pi/4$ and $-\pi/4$, respectively. There are two ways to connect the corresponding points on the left and on the right sides using these slopes, and we choose one way or another depending on whether h belongs to Λ or not. For $h \in \Lambda$, the left segment has negative slope, and the right one has slope $\pi/4$; for $h \in [2, r] \backslash \Lambda$, it goes the other way around. For example, the arrangement $\mathrm{Arr}(2, 4)$ for $r = 5$ is shown in Fig. 14.

4.4.3. PROPOSITION. *A normal ordering* $\mathbf{n}$ *is adapted to the quiver* $\Omega(\Lambda)$ *if and only if the arrangement* $\mathrm{Arr}(\mathbf{n})$ *is isotopic to* $\mathrm{Arr}(\Lambda)$.

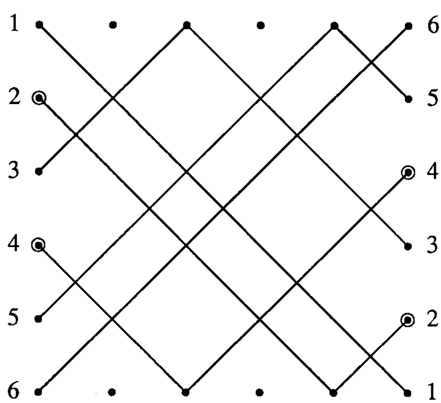


FIG. 14. A boundary pseudo-line arrangement.

Proof. It is easy to see that the nontrivial components of $\Gamma_0(\mathbf{n})$ can be identified with maximal line segments of positive slope in $\mathrm{Arr}(\Lambda)$. Then our statement follows from Proposition 4.4.1 and the pictorial interpretation of Fig. 13. ∎

To produce an actual normal ordering $\mathbf{n}$ using the above procedure, one can list all crossing points in $\mathrm{Arr}(\Lambda)$ from left to right, arranging the crossings lying on the same vertical line in an arbitrary order. For example, in Fig. 14 one can take

$$\mathbf{n}(2, 4) = (23\ 45\ 13\ 46\ 25\ 15\ 26\ 16\ 35\ 24\ 36\ 14\ 56\ 34\ 12).$$

The following two results are simple consequences of Proposition 4.4.3.

4.4.4. Corollary. *Let* $\mathbf{n}^0$ *and* $\mathbf{n}^1$ *be the normal orderings given by* (2.4.13) *and* (4.2.3). *Then* $\mathbf{n}^0$ *is adapted to* $\Omega([2, r])$, *while* $\mathbf{n}^1$ *is adapted to* $\Omega(\phi)$.

4.4.5. Corollary. *If a normal ordering* $\mathbf{n}$ *is adapted to* $\Omega(\Lambda)$ (*cf. Proposition* 4.4.3), *then the chamber sets* $L^{\mathbf{n}}(i, j)$ *for* $\mathbf{n}$ (*see* (2.5.1)) *are given by*

$$L^{\mathbf{n}}(i, j) = ([i+1, j-1] \cap \Lambda) \cup' [1, i-1] \cup' [j+1, r+1], \qquad (4.4.6)$$

where $\cup'$ *means that the set* $[1, i-1]$ (*resp.* $[j+1, r+1]$) *is only taken when* $i \in [2, r] \setminus \Lambda$ (*resp.* $j \in [2, r] \setminus \Lambda$).

Remembering (4.4.3), we see that Corollaries 4.4.4 and 4.4.5 imply the following result.

4.4.6. Corollary. *Let* $\Omega^0 = \Omega([2, r])$ *be the quiver that has each edge* $(h-1, h)$ *oriented from* h *to* $h-1$. *For every quiver* $\Omega = \Omega(\Lambda)$, *the map* $\Psi_{\Omega^0, \Omega}$ *coincides with the transition map* $R^{\mathbf{n}}_{\mathbf{n}^0}$ *given by formula* (4.2.1), *with* $L = L^{\mathbf{n}}(i, j)$ *given by* (4.4.6).

This corollary is essentially a restatement of [23, Theorem 6.0].

5. THE CASE OF AN ARBITRARY PERMUTATION

In this concluding chapter we show that in most of our results, the maximal permutation w_0 can be replaced by an arbitrary element w of the Weyl group S_{r+1}. From now on, we fix an element $w \in S_{r+1}$ of length $l(w) = l$.

5.1. *Generalization of Theorem* 2.2.3

As an obvious generalization of Definition 2.2.1, we will associate to a permutation $w \in S_{r+1}$ the *Lusztig variety* $\mathscr{L}^w(P)$ over a semiring P satisfying condition (2.1.1). An element $\mathbf{t}$ of $\mathscr{L}^w(P)$ is, by definition, a tuple $\mathbf{t} = (t^{\mathbf{h}})_{\mathbf{h} \in R(w)}$ where each $t^{\mathbf{h}} = (t_1^{\mathbf{h}}, ..., t_l^{\mathbf{h}})$ is a "vector" in P^l, and these vectors satisfy the 2-move and 3-move relations (1.13) and (1.14). Thus, the variety $\mathscr{L}^{w_0}(P)$ corresponding to the maximal element w_0 of S_{r+1} coincides with $\mathscr{L}_r(P)$.

Our first result is a description of the Lusztig variety $\mathscr{L}^w(\mathbb{R}_{>0})$ which generalizes Theorem 2.2.3. (We leave aside another important special case where the ground semiring P is the tropical semiring $\mathbb{Z}_+$; it is an intriguing problem to find an interpretation of $\mathscr{L}^w(\mathbb{Z}_+)$ in the spirit of Example 2.2.5.) The first statement in Theorem 2.2.3 generalizes verbatim: for an element $\mathbf{t} = (t_k^{\mathbf{h}}) \in \mathscr{L}^w(\mathbb{R}_{>0})$, the product

$$x(\mathbf{t}) = (1 + t_1^{\mathbf{h}} e_{h_1}) \cdots (1 + t_l^{\mathbf{h}} e_{h_l}) \tag{5.1.1}$$

does not depend on the choice of a reduced word $\mathbf{h} = (h_1 \cdots, h_l) \in R(w)$. Thus, $\mathbf{t} \mapsto x(\mathbf{t})$ is a well-defined map $\mathscr{L}^w(\mathbb{R}_{>0}) \to N$. This map is injective, which follows from the corresponding statement for w_0 and the fact that any reduced word for w is a left factor of a reduced word for w_0. Let us describe the image of this map.

Assume (I, J) is a reduced admissible pair of subsets of $[1, r+1]$, as defined in Section 2.4. Let $w_I^J \in S_{r+1}$ denote the permutation that sends J to I, and whose restrictions onto both J and $\bar{J} = [1, r+1] - J$ are monotone increasing maps. For example, if $r = 4$, $I = \{1, 3\}$, $J = \{2, 5\}$, then $w_I^J = 21453$, in the one-line notation. (The latter means that w_I^J sends 1, 2, 3, 4, 5 to 2, 1, 4, 5, 3, respectively.) It is not hard to see that the correspondence $(I, J) \mapsto w_I^J$ is a bijection between the set of all reduced admissible pairs and the set of all 321-avoiding permutations.

Let $N_{>0}^w \subset N$ denote the set of unipotent upper-triangular matrices z whose minors satisfy the following property:

$$\Delta_I^J(x) > 0, \qquad \text{if} \quad w_I^J \leqslant w;$$

$$\Delta_I^J(x) = 0, \qquad \text{otherwise,}$$

where $\leqslant$ stands for the (strong) Bruhat order (see, e.g., [22, 5.9]). For example, if $r = 3$ and $w = 2431$, in the one-line notation, then there are 10 (out of the total of 14) 321-avoiding permutations $w' \in S_4$ such that $w' \leqslant w$. The remaining four are $3124 = w_{12}^{23}$, $3142 = w_{12}^{24}$, $3412 = w_{12}^{34}$, and $4123 = w_{123}^{234}$. Therefore $N_{>0}^{2431}$ is the set of all unipotent upper-triangular matrices $x = (x_{ij})_{i,j=1}^4$ such that the minors $\Delta_{12}^{23}(x)$, $\Delta_{12}^{24}(x)$, $\Delta_{12}^{34}(x)$, and $\Delta_{123}^{234}(x)$ are equal to 0, and all other minors of x that do not identically vanish on the whole group $N \subset SL_4$ are positive. It is not hard to see that some of these conditions are superfluous, and we can describe $N^{2431}_{>0}$ by the following equations and inequalities:

$$\Delta^{23}_{12}(x) = 0; \qquad \Delta^{34}_{12}(x) = 0; \qquad \Delta^{34}_{23}(x) > 0; \qquad x_{ij} > 0,$$

for all $1 \leqslant i < j \leqslant 4$.

5.1.1. THEOREM. *The map* $\mathbf{t} \mapsto x(\mathbf{t})$ *defined by* (5.1.1) *is a bijection between* $\mathscr{L}^w(\mathbb{R}_{>0})$ *and* $N^w_{>0}$.

Proof. The fact that $x(\mathbf{t}) \in N^w_{>0}$ for all $\mathbf{t} \in \mathscr{L}^w(\mathbb{R}_{>0})$ is a consequence of formulas (2.4.7) and (2.4.8) combined with the subword criterion for the Bruhat order (see, e.g., [22, 5.10]) and the following simple combinatorial lemma whose proof is omitted.

5.1.2. LEMMA. *Let the generators of the nil–Temperley–Lieb algebra* NTL_r *act on the subsets of* $[1, r+1]$ *according to* (2.4.6). *For any reduced admissible pair* (I, J), *the sequences* $(h_1, ..., h_s)$ *such that*

$$u_{h_1} u_{h_2} \cdots u_{h_s}(J) = I$$

are exactly the reduced words for w^J_I. ▮

It remains to prove that the map $\mathbf{t} \mapsto x(\mathbf{t})$ from $\mathscr{L}^w(\mathbb{R}_{>0})$ to $N^w_{>0}$ is surjective. Let us temporarily denote the image of this map by N^w, so we need to show that $N^w = N^w_{>0}$. Let $N_{\geqslant 0}$ be the set of matrices $x \in N$ whose all minors are nonnegative. By definition, all the $N^w_{>0}$ are subsets of $N_{\geqslant 0}$. The desired equality $N^w = N^w_{>0}$ will be an immediate consequence of the following two statements:

(i) the union of all N^w is $N_{\geqslant 0}$;

(ii) the subsets $N^w_{>0}$ are pairwise disjoint.

Let us prove (i). We will need the classical approximation theorem of A. Whitney [35]: $N_{\geqslant 0}$ is the closure of $N_{>0}$ in N, with respect to the usual topology of a real Lie group. For the convenience of the reader, let us prove this fact. Fix a reduced word $\mathbf{h} = (h_1, ..., h_m) \in R(w_0)$, and consider the map

$$(t_1, ..., t_m) \mapsto x_{\mathbf{h}}(t_1, ..., t_m) = (1 + t_1 e_{h_1}) \cdots (1 + t_m e_{h_m}),$$

where all the t_k are nonnegative real numbers; note that so far we always required $t_k > 0$ for all k. Clearly, $t \mapsto x_{\mathbf{h}}(t)$ is a continuous map $\mathbb{R}^m_{\geqslant 0} \to N_{\geqslant 0}$, and $x_{\mathbf{h}}(\mathbb{R}^m_{>0}) = N_{>0}$ by Theorem 2.2.3. Now let $x \in N_{\geqslant 0}$. Then

$$x = x \cdot x_{\mathbf{h}}(0, 0, ..., 0) = \lim_{\lambda \to 0_+} x \cdot x_{\mathbf{h}}(\lambda, \lambda, ..., \lambda),$$

where $x \cdot x_{\mathbf{h}}(\lambda, \lambda, ..., \lambda) \in N_{>0}$, since, generally, $x \in N_{\geqslant 0}$ and $x' \in N_{>0}$ implies $xx' \in N_{>0}$, by the Binet–Cauchy formula. Therefore, x lies in the closure of $N_{>0}$, as desired.

Our next step in proving (i) is to show that, in the notation just introduced, the union of all N^w is equal to the image $x_{\mathbf{h}}(\mathbb{R}^m_{\geqslant 0})$. The inclusion $\bigcup_w N^w \subset x_{\mathbf{h}}(\mathbb{R}^m_{\geqslant 0})$ is clear since any reduced word for w is a subword of $\mathbf{h} \in R(w_0)$. To prove the reverse inclusion, we need to show that any matrix $x_{\mathbf{h}}(t_1, ..., t_m)$ with all $t_k \geqslant 0$ belongs to some set N^w. This follows by removing from the product $x_{\mathbf{h}}(t_1, ..., t_m)$ all factors $(1 + t_k e_{h_k})$ with $t_k = 0$, and simplifying the remaining product with the help of 2- and 3-move relations combined with the obvious relation

$$(1 + se_h)(1 + te_h) = (1 + (s + t)\, e_h).$$

To complete the proof of (i), it remains to show that every matrix $x \in N_{\geqslant 0}$ has the form $x_{\mathbf{h}}(t_1, ..., t_m)$ where all the t_k are nonnegative. We have already shown that x lies in the closure of $N_{>0}$, i.e., is the limit of some sequence of matrices $x_1, x_2, ... \in N_{>0}$. By Theorem 2.2.3, each x_n has the form $x_n = x_{\mathbf{h}}(t_{1n}, ..., t_{mn})$ where all the t_{kn} are positive. We claim that the collection of all numbers t_{kn} is bounded, for a fixed x. This follows from an observation that, for each $n \geqslant 1$, the sum $t_{1n} + \cdots + t_{mn}$ is equal to the sum of all entries of the matrix x_n which lie immediately above the main diagonal. Now the standard compactness argument shows that (replacing, if necessary, the sequence $x_1, x_2, ...$ by a subsequence) we may assume that for each $k = 1, ..., m$, the sequence $t_{k1}, t_{k2}, ...$ converges to some $t_k \geqslant 0$. By continuity, $x = x_{\mathbf{h}}(t_1, t_2, ..., t_m)$, which completes the proof of (i).

Statement (ii) is purely combinatorial: the assertion is that a permutation $w \in S_{r+1}$ is uniquely determined by the family of all 321-avoiding permutations w_I^J which are less than or equal to w in the Bruhat order. This is a corollary of a much stronger recent result due to A. Lascoux and M. P. Schützenberger [26], in which 321-avoiding permutations are replaced by the so-called *bigrassmannian permutations*. In our notation, the bigrassmannian permutations are exactly the w_I^J, where I and J are intervals in $[1, r+1]$. Thus, Theorem 4.4 in [26] can be formulated as follows.

5.1.3. Lemma. *A permutation $w \in S_{r+1}$ is uniquely determined by the family of all triples of integers (a, b, c) such that $0 \leqslant a < b < c \leqslant r+1$ and $w_{[a+1,\, c-b+a]}^{[b+1,\, c]} \leqslant w$.*

Since this lemma implies (ii), Theorem 5.1.1 is proved. ∎

The above proof implies the following description of the variety $N_{\geqslant 0}$ of all unipotent upper-triangular matrices with nonnegative minors.

5.1.4. Theorem. *The variety $N_{\geqslant 0}$ is the disjoint union of its subsets $N^w_{>0}$, for all $w \in S_{r+1}$. Each $N^w_{>0}$ is the set of all matrices $x \in N_{\geqslant 0}$ satisfying the following conditions*:

$$\begin{aligned} &\Delta^{[b+1,\,c]}_{[a+1,\,c-b+a]}(x) > 0, \qquad &&\text{if} \quad 0 \leqslant a < b < c \leqslant r+1 \text{ and } w^{[b+1,\,c]}_{[a+1,\,c-b+a]} \leqslant w;\\ &\Delta^{[b+1,\,c]}_{[a+1,\,c-b+a]}(x) = 0, \qquad &&\text{otherwise.} \end{aligned}$$

Returning to the case of an arbitrary ground semiring P, we can straightforwardly generalize Theorem 2.2.6 in the following way.

5.1.5. Theorem. *For any $\mathbf{h} \in R(w)$, the projection $\mathbf{t} \mapsto t^{\mathbf{h}}$ is a bijection between the Lusztig variety $\mathscr{L}^w(P)$ and P^l.*

As in Section 2.2, this theorem allows us to define the transition maps $R^{\mathbf{h}'}_{\mathbf{h}}$: $P^l \to P^l$ for any two reduced words $\mathbf{h}$ and $\mathbf{h}'$ for an arbitrary permutation w.

5.2. *Extending the Results of Sections* 2.3, 2.4

In this section, we work over an arbitrary ground semiring P satisfying condition (2.1.1). For any $\mathbf{h} \in R(w)$, let us define the map $t \mapsto X_{\mathbf{h}}(t)$ from P^l to the nil–Temperley–Lieb algebra $NTL_r(P)$ (see Section 2.4) by

$$X_{\mathbf{h}}(t) = (1 + t_1 u_{h_1}) \cdots (1 + t_l u_{h_l}). \tag{5.2.1}$$

The arguments from Section 2.4 show that the maps $X_{\mathbf{h}}$ for all $\mathbf{h} \in R(w)$ give rise to a well-defined map $\mathbf{t} \mapsto X(\mathbf{t})$ from the Lusztig variety $\mathscr{L}^w(P)$ to $NTL_r(P)$ (cf. (2.4.5)). The following statement generalizes Corollary 2.4.8.

5.2.1. Proposition. *Each of the maps $X_{\mathbf{h}}$ and hence the map $\mathbf{t} \mapsto X(\mathbf{t})$ from the Lusztig variety $\mathscr{L}^w(P)$ to the nil–Temperley–Lieb algebra $NTL_r(P)$ is injective.*

Proof. This follows from the corresponding statement for w_0 and the fact that any reduced word for w is a left factor of a reduced word for w_0 (the same reasoning was used in Section 5.1 to prove the injectivity of the map $\mathbf{t} \mapsto x(\mathbf{t})$). ∎

As in Section 2.4, we have well-defined "minors" $\Delta^J_I(\mathbf{t}) = \Delta^J_I(X(\mathbf{t}))$ as functions $\mathscr{L}^w(P) \to P \cup \{0\}$. These functions are explicitly given by (2.4.8). Let us describe which of them do not vanish on $\mathscr{L}^w(P)$.

5.2.2. Proposition. *Let $\mathbf{t} \in \mathscr{L}^w(P)$, and let (I, J) be a reduced admissible pair of subsets of $[1, r+1]$. Then $\Delta^J_I(\mathbf{t}) \in P$ if $w^J_I \leqslant w$ (with respect to the Bruhat order), otherwise $\Delta^J_I(\mathbf{t}) = 0$.*

Proof. Follows from (2.4.8) and Lemma 5.1.2. ∎

Now let us adapt the combinatorial constructions of Section 2.3 to the case of an arbitrary permutation w. This will lead to a modified realization of the Lusztig variety $\mathscr{L}^w(P)$, which will involve two-subscript notation and the notion of a normal ordering. With any reduced word $\mathbf{h} \in R(w)$ we associate an arrangement $\mathrm{Arr}(\mathbf{h})$ of labelled pseudo-lines $\mathrm{Line}_1, ..., \mathrm{Line}_{r+1}$, as follows. Choose a reduced word $\tilde{\mathbf{h}} \in R(w_0)$ having $\mathbf{h}$ as its right factor. Consider the arrangement $\mathrm{Arr}(\tilde{\mathbf{h}})$ constructed in Section 2.3. Then $\mathrm{Arr}(\mathbf{h})$ is the part of $\mathrm{Arr}(\tilde{\mathbf{h}})$ lying to the right of the vertical line that separates the l rightmost crossings from the previous ones. Obviously, $\mathrm{Arr}(\mathbf{h})$ does not depend on the choice of $\tilde{\mathbf{h}}$. Scanning the left endpoints of pseudo-lines in $\mathrm{Arr}(\tilde{\mathbf{h}})$ bottom-up yields the inverse permutation w^{-1}, in the one-line notation. See example in Fig. 15.

Let $(i, j) \in \Pi_r$ (see (2.3.1)). It is clear that $\mathrm{Arr}(\mathbf{h})$ contains the crossing $\mathrm{Line}_i \cap \mathrm{Line}_j$ if and only if $w(i) > w(j)$. Thus, the left-to-right ordering of the crossing points in $\mathrm{Arr}(\mathbf{h})$ results in a total ordering on the set of pairs

$$\Pi^w = \{(i, j) \in \Pi_r : w(i) > w(j)\}. \tag{5.2.2}$$

We denote this ordering of Π^w by $\mathbf{n} = \mathbf{n}(\mathbf{h})$ and call it the normal ordering associated to $\mathbf{h}$. By changing the names of the components of a point $\mathbf{t} \in \mathscr{L}^w(P)$ from $t_k^{\mathbf{h}}$ to $t_{ij}^{\mathbf{n}}$, we identify $\mathbf{t}$ with a family of "vectors" $t^{\mathbf{n}} \in P^{\Pi^w}$ satisfying the 2-move and 3-move relations from Section 2.3. For any two

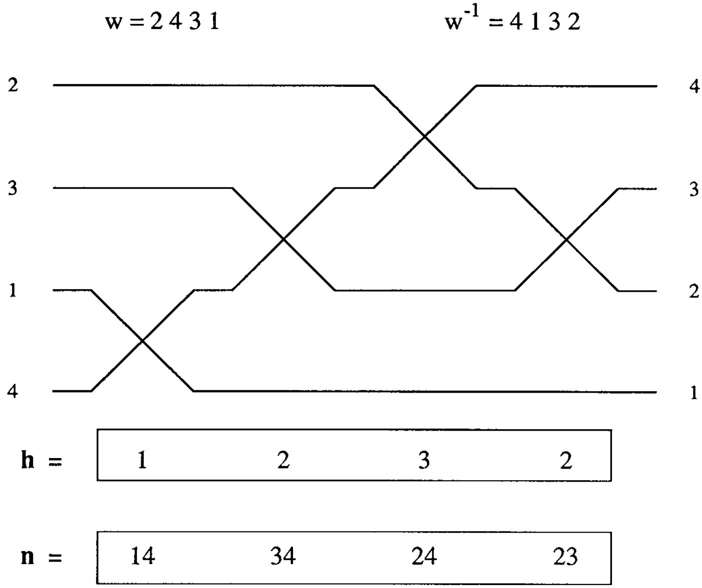


FIG. 15. Arrangement $\mathrm{Arr}(\mathbf{h})$ for $w = 2431$ and $\mathbf{h} = 1232$.

normal orderings $\mathbf{n}$ and $\mathbf{n}'$ of Π^w, there is a well-defined transition map $R_{\mathbf{n}}^{\mathbf{n}'}: P^{\Pi^w} \to P^{\Pi^w}$. For any function f on $\mathscr{L}^w(P)$ and any normal ordering $\mathbf{n}$ of Π^w, there is a function $f^{\mathbf{n}}$ on P^{Π^w} defined via a straightforward modification of the last paragraph of Section 2.3.

In particular, any minor Δ_I^J gives rise to a family of polynomials $(\Delta_I^J)^{\mathbf{n}}$ in the variables t_{ij}, $(i, j) \in \Pi^w$, one for each normal ordering $\mathbf{n}$ of Π^w. Modifying the definition of the graph $\Gamma(\mathbf{n})$ in the natural way, and repeating the argument used in the proof of Theorem 2.4.4, we see that these polynomials are given by (2.4.10). The following proposition is a generalization of Corollary 2.4.5, and is proved in the same way.

5.2.3. Proposition. *For any normal ordering $\mathbf{n}$ of Π^w, any $a \in [1, r+1]$, and $t = (t_{ij})_{(i,j) \in \Pi^w}$, we have*

$$(\Delta^{w^{-1}[1, a-1]})^{\mathbf{n}}(t) = \prod_{w(j) < a \leqslant w(i)} t_{ij}. \tag{5.2.3}$$

In particular, the polynomial $(\Delta^{w^{-1}[1,a]})^{\mathbf{n}}$ does not depend on $\mathbf{n}$.

We will now generalize Theorem 2.4.6. Consider the lexicographically minimal reduced word $\mathbf{h}^0(w) \in R(w)$ and the corresponding normal ordering $\mathbf{n}^0(w) = \mathbf{n}(\mathbf{h}^0(w))$. The ordering $\mathbf{n}^0(w)$ can be described as follows. As in the proof of Lemma 2.7.2, we can view any normal ordering $\mathbf{n}$ of Π^w as the process of sorting the sequence

$$w^{-1}(1), w^{-1}(2), ..., w^{-1}(r+1).$$

In this interpretation, $\mathbf{n}^0(w)$ sorts the sets $w^{-1}([1, a])$, for $a, = 2, ..., r$, successively: after completely sorting the elements of $w^{-1}([1, a])$, we interchange $i = w^{-1}(a+1)$ with all numbers j such that $j > i$ and $w(j) \leqslant a$. For example, if $w = 2431$, then $w^{-1} = 4132$, and $\mathbf{n}^0(w) = (14\ 34\ 24\ 23)$ (cf. Fig. 15).

5.2.4. Theorem. *For any normal ordering $\mathbf{n}$ of Π^w, the transition map from $\mathbf{n}$ to $\mathbf{n}^0(w)$ is given by*

$$(R_{\mathbf{n}}^{\mathbf{n}^0(w)}(t))_{ij} = \frac{(\Delta^{I \cup \{j\}}_{[w(i)-1-|I|,\, w(i)-1]})^{\mathbf{n}}(t)(\Delta^{I}_{[w(i)+1-|I|,\, w(i)]})^{\mathbf{n}}(t)}{(\Delta^{I \cup \{j\}}_{[w(i)-|I|,\, w(i)]})^{\mathbf{n}}(t)(\Delta^{I}_{[w(i)-|I|,\, w(i)-1]})^{\mathbf{n}}(t)}, \tag{5.2.4}$$

where

$$I = I(i, j; w) = \{a \in [1, r+1] \colon a > j,\, w(a) < w(i)\}. \tag{5.2.5}$$

Note that (5.2.4) specializes to (2.4.14) when $w = w_0$.

Proof. We follow the proof of Theorem 2.4.6, with necessary modifications. It is enough to prove the following formula generalizing (2.4.15):

$$t_{ij} = \frac{(\Delta^{I \cup \{j\}}_{[w(i)-1-|I|,\, w(i)-1]})^{\mathbf{n}^0(w)}(t)(\Delta^{I}_{[w(i)+1-|I|,\, w(i)]})^{\mathbf{n}^0(w)}(t)}{(\Delta^{I \cup \{j\}}_{[w(i)-|I|,\, w(i)]})^{\mathbf{n}^0(w)}(t)(\Delta^{I}_{[w(i)-|I|,\, w(i)-1]})^{\mathbf{n}^0(w)}(t)}. \qquad (5.2.6)$$

The minors in the right-hand side of (5.2.6) are given by the following lemma.

5.2.5. Lemma. *Let $I = \{k \in [1, r+1] \colon k > b,\, w(k) < a\}$ for some indices $a, b \in [0, r+1]$. Then*

$$(\Delta^{I}_{[a-|I|,\, a-1]})^{\mathbf{n}^0}(t) = \prod_{i<j,\, j \in I,\, w(i) \geqslant a} t_{ij}. \qquad (5.2.7)$$

This lemma generalizes Lemma 2.4.7 and is proved in the same way. To complete the proof of Theorem 5.2.4, it remains to substitute the expressions given by Lemma 5.2.5 into the right-hand side of (5.2.6), and perform the cancellation. ▮

5.3. *Chamber Sets and Chamber Ansatz for Arbitrary w*

Let $\mathbf{n}$ be a normal ordering of Π^w, and let $\mathrm{Arr}(\mathbf{n})$ be the corresponding pseudo-line arrangement. The definition of chamber sets $L^{\mathbf{n}}(i, j)$ given by (2.5.1), and the Chamber Ansatz substitution (2.5.3) extend to arbitrary w without any changes, provided $(i, j) \in \Pi^w$. However, for a given w, not all subsets $J \subset [1, r+1]$ can appear as chamber sets. We say that a subset $J \subset [1, r+1]$ is a *w-chamber set* if J is a chamber set for some normal ordering of Π^w. The following characterization of w-chamber sets follows easily from the definitions.

5.3.1. Proposition. *A subset $J \subset [1, r+1]$ is a w-chamber set if and only if J has the following property*:

$$\text{if } i < j,\ i \notin J, \text{ and } j \in J, \text{ then } w(i) > w(j), \text{ i.e., } (i, j) \in \Pi^w. \qquad (5.3.1)$$

Clearly, the corresponding version of Proposition 2.5.1 remains valid for any w: in order to define a point of the Lusztig variety $\mathcal{L}^w(P)$ via the Chamber Ansatz, the components M_J must satisfy the 3-Term Relations (2.5.4), where the six participating subsets are assumed to be w-chamber sets. We then define $\mathcal{M}^w(P)$ as the set of tuples $\mathbf{M} = (M_J)$ of elements of P indexed by w-chamber sets J and satisfying the 3-Term Relations (2.5.4) and normalization conditions (2.7.1). Theorem 2.7.1 generalizes as follows (the same proof works).

5.3.2. THEOREM. *The Chamber Ansatz gives rise to a bijection* $\mathbf{M} \mapsto \mathbf{t}(\mathbf{M})$ *between* $\mathcal{M}^w(P)$ *and the Lusztig variety* $\mathcal{L}^w(P)$. *The inverse bijection* $\mathbf{t} = (t_{ij}^{\mathbf{n}}) \mapsto \mathbf{M}(\mathbf{t})$ *from* $\mathcal{L}^w(P)$ *to* $\mathcal{M}^w(P)$ *is given by* (2.7.2).

As in Section 2.7, a consequence of Theorem 5.3.2 is Corollary 2.7.4, for $\mathcal{M} = \mathcal{M}^w(P)$. Another important consequence is Theorem 2.9.1 which obviously extends to the case of arbitrary w (for each w-chamber set J, we view M_J as a function on $\mathcal{L}^w(P)$).

It should be possible to extend to arbitrary $\mathcal{M}^w(P)$ the change of coordinates $(M_J) \to (Z_a, T_J)$ on $\mathcal{M}_r(P)$ defined in Section 2.9. We will not attempt at doing it here, since the meaning of such a change is not as clear as in the case $w = w_0$. Instead, we compute the function M_J in the case when J is a chamber set for the ordering $\mathbf{n}^0(w)$ defined in Section 5.2. The chamber sets of $\mathbf{n}^0(w)$ can be described as follows (cf. (2.8.4)):

$$L^{\mathbf{n}^0(w)}(i, j) = \{k \in [1, r+1]: k < j, w(k) < w(i)\} \tag{5.3.2}$$

for all $(i, j) \in \Pi^w$. Thus, any chamber set for $\mathbf{n}^0(w)$ has the form

$$J = \{k \in [1, r+1]: k \leqslant b, w(k) < a\}$$

for some indices $a, b \in [0, r+1]$.

5.3.3. PROPOSITION. *Let*

$$I = \{k \in [1, r+1]: k > b, w(k) < a\},$$

$$J = \{k \in [1, r+1]: k \leqslant b, w(k) < a\},$$

for some $a, b \in [0, r+1]$. *Then the function* M_J *on the Lusztig variety* $\mathcal{L}^w(P)$ *is given by*

$$M_J = \frac{\Delta^I_{[a-|I|, a-1]}}{\Delta^{w^{-1}[1, a-1]}}. \tag{5.3.3}$$

Proof. Combine (5.2.3) and (5.2.7) with (2.7.2). ∎

In particular, setting $b = r+1$ in (5.3.3), we obtain

$$M_{w^{-1}[1, a-1]} = \frac{1}{\Delta^{w^{-1}[1, a-1]}}, \tag{5.3.4}$$

for $a = 1, ..., r+1$.

5.4. *Generalizations of Theorems* 3.1.1 *and* 3.2.5

We now return to the case of the ground semifield $\mathbb{R}_{>0}$. By Theorem 5.1.1, the Lusztig variety $\mathcal{L}^w(\mathbb{R}_{>0})$ can be identified with the subset $N^w_{>0} \subset N$. Thus, for any w-chamber set J, we can view each term M_J in the Chamber Ansatz as a function $N^w_{>0} \to \mathbb{R}_{>0}$. We will compute these functions explicitly, generalizing Theorem 3.1.1 that gives an answer for the case $w = w_0$.

Our first task is to extend to an arbitrary w the birational automorphism $x \to y$ given by (3.1.1). To do this, we need some geometric information on the set $N^w_{>0}$. Let $B_- \subset G = GL_{r+1}$ be the Borel subgroup of lower-triangular matrices. The unipotent radical N_- of B_- is N^T, the subgroup transpose to N. It is well known (see, e.g., [18, Section 23.4]) that G is the disjoint union of its *Bruhat cells* $B_- w B_-$, for all $w \in S_{r+1}$; here a permutation w is identified with the matrix $(\delta_{i,\, w(j)})$. On the other hand, by Theorem 5.1.4, the subsets $N^w_{>0}$, for $w \in S_{r+1}$, form a disjoint decomposition of the variety $N_{\geqslant 0}$. The following proposition relates these two phenomena.

5.4.1. Proposition. *For any* $w \in S_{r+1}$, *we have*

$$N^w_{>0} = N_{\geqslant 0} \cap B_- w B_- . \tag{5.4.1}$$

Proof. It is enough to show that $N^w_{>0} \subset B_- w B_-$. In view of Theorem 5.1.1, any element $x \in N^w_{>0}$ has the form (5.11), i.e., can be factored as

$$x = (1 + t_1 e_{h_1}) \cdots (1 + t_l e_{h_l}),$$

where $(h_1, ..., h_l) \in R(w)$, and $t_k > 0$ for all k. It is straightforward to check that if $h = 1, ..., r$ and $t \neq 0$ then $(1 + t e_h) \in B_- s_h B_-$. Therefore, we have

$$x \in B_- s_{h_1} B_- s_{h_2} B_- \cdots s_{h_l} B_- = B_- w B_- ,$$

as desired. (This argument follows the proof of Proposition 42.2.4 in [29].) ∎

Now consider the *flag variety* $B_-\backslash G$, and the natural projection π: $G \to B_-\backslash G$. The Bruhat decomposition of G gives rise to the decomposition of $B_-\backslash G$ into the disjoint union of *Schubert cells* $X_w = \pi(B_- w B_-)$, $w \in S_{r+1}$. An explicit isomorphism of X_w with an affine space is constructed as follows. Define the subgroup $N(w)$ of N by

$$\begin{aligned} N(w) &= N \cap w^{-1} N_- w \\ &= \{ y \in N\colon y_{ij} = 0 \text{ whenever } i < j,\ w(i) < w(j) \}. \end{aligned} \tag{5.4.2}$$

It is well known (and easy to check) that there is an isomorphism of algebraic varieties $\pi_w\colon N(w) \to X_w$ given by

$$\pi_w(y) = \pi(w \cdot y^T). \tag{5.4.3}$$

Thus, we can take the matrix entries y_{ij}, for $(i, j) \in \Pi^w$ (see (5.2.2)), as affine coordinates on X_w.

Now everything is ready for a generalization of (3.1.1). Since $N \cap B_- = \{e\}$, the restriction of the projection π to $N^w_{>0}$ is an embedding of $N^w_{>0}$ to X_w. Identifying the Schubert cell X_w, with $N(w)$ via the map π_w given by (5.4.3), we obtain an embedding $\pi_w^{-1} \circ \pi\colon N^w_{>0} \to N(w)$. As in (3.1.1), we will write this embedding simply as $x \mapsto y$. Unraveling the definitions and using the same notation as in Section 3.1, we see that a matrix $x \in N^w_{>0}$ and the corresponding matrix $y \in N(w)$ are related by

$$x = [wy^T]_+, \qquad wy^Tw^{-1} = [xw^{-1}]_+; \tag{5.4.4}$$

note that this formula specializes to (3.1.1) when $w = w_0$. Theorem 3.1.1 generalizes as follows.

5.4.2. Theorem. *For* $\mathbf{t} \in \mathscr{L}^w(\mathbb{R}_{>0})$, *let* $x = x(\mathbf{t}) \in N^w_{>0}$ *be the matrix corresponding to* $\mathbf{t}$ *via* (5.1.1), *and let* $y \in N(w)$ *be related to* x *by* (5.4.4). *Then, for every w-chamber set* $J \subset [1, r+1]$, *we have* $M_J(\mathbf{t}) = \Delta^J(y)$.

Proof. Following the proof of Theorem 3.1.1, we begin by generalizing the identities in Lemma 3.1.3. Assume that matrices $x \in N$ and $y \in N(w)$ are related by (5.4.4). The proof of the following two lemmas is essentially the same as that of Lemma 3.1.3.

5.4.3. Lemma. *For any* $J \subset [1, r+1]$, *we have*

$$\Delta^J(x) = \frac{\Delta_J^{w^{-1}[1, |J|]}(y)}{\Delta^{w^{-1}[1, |J|]}(y)}; \tag{5.4.5}$$

in particular, for $1 \leqslant a \leqslant r+1$, *we have*

$$\Delta^{w^{-1}[1, a-1]}(x)\, \Delta^{w^{-1}[1, a-1]}(y) = 1. \tag{5.4.6}$$

5.4.4. Lemma. *Let* I *and* J *have the same meaning as in Proposition* 5.3.3, *and let* $0 \leqslant d < \min(J)$. *Then*

$$\Delta^{[d+1, d+|J|] \cup I}(x) = \frac{\Delta^J_{[d+1, d+|J|]}(y)}{\Delta^{w^{-1}[1, a-1]}(y)} \tag{5.4.7}$$

and

$$\Delta^J_{[d+1,\,d+|J|]}(y) = \frac{\Delta^{[d+1,\,d+|J|]\,\cup\, I}(x)}{\Delta^{w^{-1}[1,\,a-1]}(x)}. \tag{5.4.8}$$

In particular, setting $d=0$ *in* (5.4.8) *yields*

$$\Delta^J(y) = \frac{\Delta^I_{[a-|I|,\,a-1]}(x)}{\Delta^{w^{-1}[1,\,a-1]}(x)}. \tag{5.4.9}$$

The proof of Theorem 5.4.2 can now be completed as follows. The same argument as in the proof of Theorem 3.1.1 shows that it is enough to check the desired identity $M_J(\mathbf{t}) = \Delta^J(y)$ for all chamber sets in any given arrangement $\mathrm{Arr}(\mathbf{n})$ for w. Take $\mathbf{n} = \mathbf{n}^0(w)$ (see Section 5.2). Then the chamber sets for $\mathbf{n}$ are precisely the sets J that appear in Proposition 5.3.3 and Lemma 5.4.4, so our statement follows by comparing (5.3.3) with (5.4.9). ∎

Our next goal is a generalization of Theorem 3.2.5. Let $N(w)_{>0}$ (not to be confused with $N^w_{>0}$) denote the set of all matrices $y \in N(w)$ such that $\Delta^J(y) > 0$ for all w-chamber sets J.

5.4.5. THEOREM. *The embedding* $x \mapsto y$ *given by* (5.4.4) *is a bijection between* $N^w_{>0}$ *and* $N(w)_{>0}$.

Proof. The fact that the image of $N^w_{>0}$ under the embedding $x \mapsto y$ is contained in $N(w)_{>0}$, follows from Theorems 5.4.2 and 5.3.2. To prove the reverse inclusion, we only need to show that a matrix $y \in N(w)_{>0}$ is uniquely determined by its flag minors $\Delta^J(y)$ corresponding to the w-chamber sets J. This is a consequence of the following algebraic lemma. Let us regard the matrix entries y_{ij} and the minors $\Delta^J(y)$ as elements of the field $\mathbb{Q}(y_{ij})$ of rational functions in the variables y_{ij}, for $(i, j) \in \Pi^w$.

5.4.6. LEMMA. *Every matrix entry* y_{ij} *with* $(i, j) \in \Pi^w$ *can be expressed in* $\mathbb{Q}(y_{ij})$ *as a Laurent polynomial in the elements* $\Delta^J(y)$, *where* J *runs over all* w-*chamber sets.*

Proof. Fix a pair $(i, j) \in \Pi^w$ and consider the set $I = \{k \in [i+1, j-1]\colon w(k) < w(j)\}$. The description (5.3.1) of w-chamber sets implies readily that both $J = [1, i-1] \cup I \cup \{j\}$ and $J' = [1, i] \cup I$ are w-chamber sets. As polynomials on N, the flag minors Δ^J and $\Delta^{J'}$ can be rewritten in the form $\Delta^J = \Delta^{I \cup \{j\}}_{[i,\, i+|I|]}$ and $\Delta^{J'} = \Delta^I_{[i+1,\, i+|I|]}$. Expanding the first of these minors in the last column, we obtain

$$\Delta^J(y) = (-1)^{|I|}\, \Delta^{J'}(y)\, y_{ij} + Q(y),$$

where $Q(y)$ is a polynomial in the variables $y_{i'j'}$ such that $(i', j') \neq (i, j)$ and $[i', j'] \subset [i, j]$. Using induction on $j-i$, we can assume that $Q(y)$ is a Laurent polynomial in the flag minors corresponding to w-chamber sets. Hence, the same is true for

$$y_{ij} = (-1)^{|I|} \frac{\Delta^J(y) - Q(y)}{\Delta^{J'}(y)}.$$

Lemma 5.4.6 and Theorem 5.4.5 are proved. ∎

5.4.7. Example. Let $r = 3$ and $w = 4231$, i.e., w is the transposition of 1 and 4. Then $l(w) = 5$ and $\Pi^w = \{(i, j)\colon 1 \leqslant i < j \leqslant 4, (i, j) \neq (2, 3)\}$. In this case, each of the entries y_{12}, y_{14}, y_{24}, y_{34} is a flag minor corresponding to a w-chamber set, namely

$$y_{12} = \Delta^2, \qquad y_{14} = \Delta^4, \qquad y_{24} = \Delta^{14}, \qquad y_{34} = \Delta^{124}.$$

As for the remaining entry y_{13}, the procedure in the above proof expresses it as $y_{13} = -\Delta^{23}$. In particular, we see that this entry is negative for $y \in N(w)_{>0}$.

The last result can be viewed as a generalization of the total positivity criteria given in Theorem 3.2.1.

5.4.8. Proposition. *Let* $\mathbf{n}$ *be a normal ordering of* Π^w. *A matrix* $y \in N(w)$ *belongs to* $N(w)_{>0}$ *if and only if* $\Delta^J(y) > 0$ *for any chamber set* J *of* $\mathbf{n}$.

Proof. This follows from the generalization of Corollary 2.7.4 mentioned in Section 5.3. ∎

In particular, taking $\mathbf{n} = \mathbf{n}^0(w)$ and using (5.3.2), we obtain the following generalization of the Fekete criterion (Theorem 3.2.2).

5.4.9. Corollary. *A matrix* $y \in N(w)$ *belongs to* $N(w)_{>0}$ *if and only if*

$$\Delta^{[1, j] \cap w^{-1}[1, w(i)-1]}(y) > 0$$

for all $(i, j) \in \Pi^w$.

APPENDIX: CONNECTIONS WITH THE YANG–BAXTER EQUATION

Here we explain the connection between the 2-move and 3-move relations and the ubiquitous *Yang–Baxter equation.*

Denoting $R_i(x) = 1 + xu_i$, we can rewrite (2.4.1) and (2.4.2) as

$$R_i(t_1)\, R_j(t_2) = R_j(t_2)\, R_i(t_1), \qquad |i-j| \geqslant 2 \tag{A.1}$$

and

$$\begin{aligned} &R_i(t_1)\, R_j(t_2)\, R_i(t_3) \\ &\quad = R_j\left(\frac{t_2 t_3}{t_1+t_3}\right) R_i(t_1+t_3)\, R_j\left(\frac{t_1 t_2}{t_1+t_3}\right), \qquad |i-j| = 1, \end{aligned} \tag{A.2}$$

respectively. Let us also remark that

$$R_i(0) = 1 \tag{A.3}$$

for all i. In this language, transition maps for the Lusztig variety establish relationships between different products of the form $R_{i_1}(t_1)\, R_{i_2}(t_2) \cdots$. We will now describe a general setup for the study of the equations (A.1)–(A.3).

Let $\mathscr{A}$ be an associative algebra over a field K of characteristic zero, and let $R_i(x) \in \mathscr{A}[[x]]$, $i = 1, ..., r$, be formal power series with coefficients in $\mathscr{A}$. In this context, the Yang–Baxter equations are (A.1) and (A.3) together with

$$R_i(x)\, R_{i+1}(x+y)\, R_i(y) = R_{i+1}(y)\, R_i(x+y)\, R_{i+1}(x), \tag{A.4}$$

where x and y are formal variables which commute with each other (cf. [2, 12]). We will be interested in those solutions of the Yang–Baxter equations which are "re-scalable" in the following sense: for any scalars $\alpha_1, ..., \alpha_r \in K$, the rescaled functions $\tilde{R}_i(x) = R_i(\alpha_i x)$ also satisfy the same equations (A.1), (A.3), and (A.4). An easy inspection shows that imposing this condition amounts to replacing the Yang–Baxter equation (A.4) by a stronger condition

$$\begin{aligned} &R_i(x)\, R_{i+1}(\beta(x+y))\, R_i(y) \\ &\quad = R_{i+1}(\beta y)\, R_i(x+y)\, R_{i+1}(\beta x), \qquad \beta \in K. \end{aligned} \tag{A.5}$$

A.1. Proposition. *Equations* (A.1)–(A.3) *are equivalent to the "scale-invariant Yang–Baxter equations"* (A.1), (A.3), *and* (A.5).

Proof. Observe that the change of variables

$$t_1 = x, \qquad t_2 = \beta(x+y), \qquad t_3 = y$$

converts (A.5) into (A.2). ∎

The general solution to the system of equations (A.1)–(A.3) is given by the following theorem.

A.2. THEOREM. *Let $\mathcal{A}$ be an associative algebra over a field of characteristic zero. The formal power series $R_i \in \mathcal{A}[[x]]$, $i = 1, ..., r$, satisfy the equations* (A.1)–(A.3) *if and only if they are given by*

$$R_i(x) = \exp(xv_i) \tag{A.6}$$

where $v_1, ..., v_r$ are some elements of $\mathcal{A}$ satisfying the relations

$$v_iv_j - v_jv_i = 0, \qquad |i-j| \geqslant 2, \tag{A.7}$$

$$v_i^2v_j - 2v_iv_jv_i + v_jv_i^2 = 0, \qquad |i-j| = 1. \tag{A.8}$$

Proof. Setting $\beta = 0$ in (A.5) results in

$$R_i(x+y) = R_i(x)\,R_i(y). \tag{A.9}$$

It follows from (A.3) and (A.9) that the R_i are given by (A.6), where $v_1, ..., v_r$ are some elements of the associative algebra $\mathcal{A}$: one can simply take $v_i = R_i'(0)$. (In the terminology of [16], this means that the $R_i(x)$ are *exponential* solutions of the Yang–Baxter equations.) We then see from (A.1) that the elements v_i must satisfy the partial commutativity condition (A.7). Conversely, suppose that the v_i satisfy (A.7), and the R_i are defined by (A.6). Then conditions (A.1) and (A.3) become obvious. It remains to show that (A.5) is equivalent to the relations (A.8). This is a consequence of Theorem 1 in [16] that describes all exponential solutions of the Yang–Baxter equations. ∎

A.3. *Remarks.* **1.** Equations (A.1), (A.3), (A.4), and (A.9) are sometimes called the *colored braid relations*. As shown in [15], solutions of these relations give rise to generalized Schubert polynomials and provide an adequate algebraic formalism for their study.

2. The relations (A.7)–(A.8) are the well known Serre relations for type A_r. By Serre's theorem [33, 6.4], they form a system of defining relations for the universal enveloping algebra $U(\mathrm{Lie}(N))$. Thus, choosing the v_i in (A.6) to be the standard generators of $\mathcal{A} = U(\mathrm{Lie}(N))$ provides the universal solution to relations (A.1)–(A.3). (This solution was found in [15].) Any other solution can be obtained from the universal one via a homomorphism $U(\mathrm{Lie}(N)) \to \mathcal{A}$. An example of this kind is given by the nil–Temperley–Lieb algebra which is the quotient algebra of $U(\mathrm{Lie}(N))$ modulo the additional relations $v_i^2 = 0$. The corresponding solution has the form $R_i(x) = \exp(xu_i) = 1 + u_i$ where the u_i are generators of the algebra NTL, (this returns us to the starting point of this Appendix). Adding to the defining relations of NTL one more relation

$$v_iv_j = 0 \qquad \text{unless} \quad j = i+1,$$

we obtain the associative algebra of nilpotent upper-triangular matrices, with its standard generators e_i. This brings us back to the original Lusztig's solution of the equations (A.1)–(A.3).

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