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LINEAR ALGEBRA AND ITS APPLICATIONS

Circular planar graphs and resistor networks

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Abstract

We consider circular planar graphs and circular planar resistor networks. Associated with each circular planar graph Γ there is a set $\pi(\Gamma) = \{(P;Q)\}$ of pairs of sequences of boundary nodes which are connected through Γ. A graph Γ is called critical if removing any edge breaks at least one of the connections $(P;Q)$ in $\pi(\Gamma)$. We prove that two critical circular planar graphs are Y–Δ equivalent if and only if they have the same connections. If a conductivity γ is assigned to each edge in Γ, there is a linear from boundary voltages to boundary currents, called the network response. This linear map is represented by a matrix Λ_γ. We show that if (Γ,γ) is any circular planar resistor network whose underlying graph Γ is critical, then the values of all the conductors in Γ may be calculated from Λ_γ. Finally, we give an algebraic description of the set of network response matrices that can occur for circular planar resistor networks.

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1. Introduction

This article is a continuation of Refs. [5–7], and was inspired by Refs. [1,2]. Some related results have been announced in Ref. [3].

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A *graph with boundary* is a triple $\Gamma = (V, V_B, E)$, where (V,E) is a finite graph with V = the set of nodes, E = the set of edges, and V_B is a non-empty subset of V called the set of boundary nodes. Γ is allowed to have multiple edges (i.e., more than one edge between two nodes) or loops (i.e., an edge joining a node to itself).

A *circular planar graph* is a graph with boundary which is embedded in a disc D in the plane so that the boundary nodes lie on the circle C which bounds D, and the rest of Γ is in the interior of D. The boundary nodes can be labelled $v_1, \ldots, v_n$ in the (clockwise) circular order around C. A pair of sequences of boundary nodes $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ such that the sequence $(p_1, \ldots, p_k, q_k, \ldots, q_1)$ is in circular order is called a *circular pair.*

A circular pair $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ of boundary nodes is said to be *connected through* Γ if there are k disjoint paths $\alpha_1, \ldots, \alpha_k$ in Γ, such that α_i starts at p_i, ends at q_i and passes through no other boundary nodes. We say that α is a *connection* from P to Q. The notion of a connection between a pair of sequences of boundary nodes appears in Refs. [1,2]. The definition of a well-connected critical graph was given in Ref. [1]. In this paper, we consider graphs which are not necessarily well-connected.

For each circular planar graph Γ, let $\pi(\Gamma)$ be the set of all circular pairs $(P;Q)$ of boundary nodes which are connected through Γ.

There are two ways to remove an edge from a graph.
1. By deleting an edge.
2. By contracting an edge to a single node. (An edge joining two boundary nodes is not allowed to be contracted to a single node.)

We say that removing an edge *breaks the connection* from P to Q if there is a connection from P to Q through Γ, but there is not a connection from P to Q after the edge is removed. A graph Γ is called *critical* if the removal of any edge breaks some connection in $\pi(\Gamma)$.

Theorem 1. *Suppose Γ_1 and Γ_2 are two critical circular planar graphs. Then $\pi(\Gamma_1) = \pi(\Gamma_2)$ if and only if Γ_1 and Γ_2 are Y–Δ equivalent.*

A *conductivity* on a graph Γ is a function γ which assigns to each edge e in E a positive real number $\gamma(e)$. A *resistor network* (Γ,γ) consists of a graph with boundary together with a conductivity function γ.

Suppose (Γ,γ) is a resistor network with n boundary nodes. There is a linear map from boundary functions to boundary functions, constructed as follows. To each function $f = \{f(v_i)\}$ defined at the boundary nodes, there is a unique extension of f to all the nodes of Γ which satisfies Kirchhoff's current law at each interior node. This function then gives a current I where $I(v_i)$ is the current into the network at boundary node v_i. The linear map which sends f to I is called the Dirichlet-to-Neumann map in Refs. [5–7]. This map is represented by an $n \times n$ matrix, $\Lambda_\gamma (= \Lambda(\Gamma, \gamma))$, called the *network response.*

Theorem 2. *Suppose (Γ, γ) is a circular planar resistor network which is critical as a graph. Then the values of the conductors are uniquely determined by, and can be calculated from Λ_γ.*

In this situation we say γ is *recoverable* from Λ_γ.

Notation. Suppose $A = \{a_{s,t}\}$ is a matrix, $P = (p_1, \ldots, p_k)$ is an ordered subset of the rows, and $Q = (q_1, \ldots, q_m)$ is an ordered subset of the columns. Then $A(P;Q)$ denotes the $k \times m$ matrix obtained by taking the entries of A which are in rows $p_1, \ldots, p_k$ and columns $q_1, \ldots, q_m$. Specifically, for each $1 \leqslant i \leqslant k$ and $1 \leqslant j \leqslant m$,

$$A(P;Q)_{i,j} = a_{p_i,q_j}.$$

A pair of sequences of indices $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$ is called a *circular pair* if a cyclic permutation of $(p_1, \ldots, p_k; q_k, \ldots, q_1)$ is in order. If $(P;Q)$ is a circular pair of indices, $A(P;Q)$ is called a *circular minor* of A.

Definition 1.1. For each integer $n \geqslant 2$, let Ω_n be the set of $n \times n$ symmetric matrices M for which the sum of the entries in each row is 0, and which satisfy the following condition.
If $M(P;Q)$ is a $k \times k$ circular minor of M, then $(-1)^k \det M(P;Q) \geqslant 0$.

This condition says that if $M \in \Omega_n$ and $(P;Q)$ is a circular pair of indices, then the matrix $-M(P;Q)$ is *totally non-negative* as in Ref. [9]. This condition implies that if $M \in \Omega_n$, each off-diagonal entry is non-positive and each diagonal entry is non-negative.

Theorem 3. *Suppose M is in Ω_n. Then there is a circular planar graph with a conductivity γ so that $M = \Lambda(\Gamma, \gamma)$.*

Definition 1.2. Suppose Γ is a circular planar graph with n boundary nodes, and $\pi = \pi(\Gamma)$ is the set of circular pairs $(P;Q)$ which are connected through Γ. A subset $\Omega(\pi)$ of Ω_n is defined by the following condition. For each circular pair of indices $(P;Q) = (p_1, \ldots, p_k; q_1, \ldots, p_k)$,
(a) If $(P;Q) \in \pi$, then $(-1)^k \det M(P;Q) > 0$.
(b) If $(P;Q) \notin \pi$, then $\det M(P;Q) = 0$.

Let (Γ, γ) be a critical circular planar resistor network and $\pi(\Gamma) = \pi$. In Section 4, we show that the network response matrix Λ_γ is in $\Omega(\pi)$. In Section 12, we show that if $M \in \Omega(\pi)$, then there is a conductivity γ on Γ so that $M = \Lambda_\gamma$. More generally, we have the following.

Theorem 4. *Suppose Γ is a critical circular planar graph with N edges and $\pi = \pi(\Gamma)$. Then the map which sends γ to Λ_γ is a diffeomorphism of $(R^+)^N$ onto $\Omega(\pi)$.*

Remark 1. Theorems 1–4 show that there is a close relationship between circular planar resistor networks and matrices. The set of network response matrices for all circular planar graphs with n boundary nodes is Ω_n, which is the disjoint union of the sets $\Omega(\pi)$. For each $M \in \Omega_n$, let $\pi = \{(P;Q)\}$ be the set of circular pairs $(P;Q)$ of indices for which $\det M(P;Q) \neq 0$. Associated with this π, there is a circular planar graph Γ with $\pi(\Gamma) = \pi$, and there is a conductivity γ on Γ with $\Lambda(\Gamma, \gamma) = M$. The graph Γ may be chosen to be critical, and then Γ is unique to within $Y-\Delta$ equivalence. If a graph Γ is chosen in this $Y-\Delta$ equivalence class, then the conductivity γ on Γ which gives $M = \Lambda(\Gamma, \gamma)$ is unique.

Remark 2. For each of the sets π, let $N(\pi)$ be the number of edges in a critical graph with $\pi(\Gamma) = \pi$. Suppose Γ be a circular planar graph with N edges. Then Γ is critical if and only if $N = N(\pi(\Gamma))$. If Γ is not critical, then there is a critical graph Γ', with $\pi(\Gamma') = \pi(\Gamma)$. The graph Γ' may be obtained from Γ by removal (by deletion and/or contraction) of $N - N(\pi(\Gamma))$ edges. If γ is a conductivity on Γ, there is a conductivity γ' on Γ' so that $\Lambda(\Gamma', \gamma') = \Lambda(\Gamma, \gamma)$.

This paper is almost entirely self-contained. In addition to matrix algebra, the proofs make use of the medial graphs of Steinitz and Theorem 5.2 of Ref. [7]. In Section 2, Schur complements are used to prove a determinant identity, originally due to Dodgson, that is used extensively in Section 10. For (Λ, γ) a resistor network, the response matrix Λ_γ is constructed in Section 3. The important properties of Λ_γ are established in Section 4. Section 5 describes $Y-\Delta$ and $\Delta-Y$ transformations of planar graphs. The medial graph of a circular planar graph, is defined in Section 6. In Section 7, the methods of Steinitz are used to show that in each $Y-\Delta$ equivalence class of critical circular planar graphs there is a standard representative. In Section 8, we define three ways to adjoin an edge to a graph and we describe the effects of each of these adjunctions on the response matrices. Theorem 2 was proven in Ref. [7] for the standard representative of a well-connected critical circular planar graph. Section 9 of the present paper makes use of Ref. [7] to prove Theorem 2 for an arbitrary critical graph. Section 10 uses Dodgson's identity to prove some facts about the matrices M in Ω_n. In Section 11, we show that removing a boundary edge or boundary spike from a critical graph results in another critical graph. In Section 12, we prove Theorems 3 and 4. In Section 13, we prove Theorem 1.

2. The Schur complement

Suppose M is a square matrix and D be a non-singular square submatrix of M. For convenience, assume that D is the lower right-hand corner of M, so that M has the block structure.

$$M = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

The Schur complement of D in M is the matrix $M/D = A - BD^{-1}C$. The Schur complement satisfies the following determinantal identity.

$$\det M = \det (M/D) \cdot \det D.$$

If E is a non-singular square submatrix of D, then

$$\det M = \det (M/D) \cdot \det (D/E) \cdot \det E.$$

In this situation, the following quotient formula is due to Haynsworth [4].

$$M/D = (M/E)/(D/E).$$

Let $A = \{a_{i,j}\}$ be an $n \times n$ matrix, and $a_{h,k}$ is a non-zero entry. The 1×1 matrix with entry $a_{h,k}$ is denoted by $[a_{h,k}]$. For the Schur complement, $A/[a_{h,k}]$, we have

$$\det A = (-1)^{h+k} a_{h,k} \cdot \det (A/[a_{h,k}]).$$

Suppose A is an $n \times n$ matrix, with $n \geqslant 2$. If i and j are any two indices, $A[i;j]$ will denote the $(n-1) \times (n-1)$ matrix obtained by *deleting* row i and column j. Similarly, if (h,i) and (j,k) are indices, then $A[h,i;j,k]$ will denote the $(n-2) \times (n-2)$ matrix obtained by *deleting* rows h and i and columns j and k. We shall make extensive use of the following identity, due to Dodgson [8].

Lemma 2.1. *For any indices* $[h,i;j,k]$ *with* $1 \leqslant h < i \leqslant n$ *and* $1 \leqslant j < k \leqslant n$,

$$\det A \cdot \det A[h,i;j,k] = \det A[h;j] \cdot \det A[i;k] - \det A[h;k] \cdot \det A[i;j].$$

Proof. By re-ordering the rows and columns, we may assume that the indices are $(h,i) = (1,2)$ and $(j,k) = (1,2)$. Let $B = A[1,2;1,2]$. Then A has the form:

$$A = \begin{bmatrix} a & b & x \\ c & d & y \\ w & z & B \end{bmatrix},$$

where x and y are $1 \times (n-2)$ row vectors, w and z are $(n-2) \times 1$ column vectors. Temporarily assume that B is non-singular. For the Schur complement A/B we have:

$$A/B = \begin{bmatrix} a - xB^{-1}w & b - xB^{-1}z \\ c - yB^{-1}w & d - yB^{-1}z \end{bmatrix},$$

$$\begin{aligned}\det(A/B) &= (a - xB^{-1}y)(d - yB^{-1}z) - (b - xB^{-1}z)(c - yB^{-1}w) \\ &= \det(A[2;2]/B) \cdot \det(A[1;1]/B) - \det(A[1;2]/B) \cdot \det(A[2;1]/B).\end{aligned}$$

Using the determinantal identity for Schur complements, we have

$$\det A \cdot \det B = \det A[2;2] \cdot \det A[1;1] - \det A[1;2] \cdot \det A[2;1].$$

This is a polynomial relation which holds for the n^2 values of the entries of A whenever $\det B \neq 0$. Therefore it is an identity in the coefficients of A. □

3. Resistor networks

In this section we construct the response matrix $A(\Gamma, \gamma)$ for a resistor network (Γ, γ). This is done first when Γ is connected as a graph; the response matrix for a general network is obtained by taking the direct sum of the response matrices of the connected components.

Suppose $(\Gamma, \gamma) = (V, V_B, E, \gamma)$ is a connected resistor network, with d vertices numbered $v_1, \ldots, v_d$. The Kirchhoff matrix $K = K(\Gamma, \gamma)$ is the $d \times d$ matrix K constructed as follows.

1. If $i \neq j$ then $K_{i,j} = -\sum \gamma(e)$, where the sum is taken over all edges e joining v_i to v_j. (If there is no edge joining v_i to v_j, then $K_{i,j} = 0$.)
2. $K_{i,i} = \sum \gamma(e)$, where the sum is taken over all edges e with one endpoint at v_i and the other endpoint not v_i.

The Kirchhoff matrix has the following interpretation. If u is a voltage defined at the nodes of Γ, then $c = Ku$ is the resulting current flow. In coordinates, if $u = \{u(v_i)\}$, then $c_j = \sum_i K_{i,j} u(v_i)$ is the current flowing into the network at node v_j.

If a function f is imposed at the boundary nodes, the function u which satisfies Kirchhoff's current law $c_j = 0$ at each interior node v_j, and which agrees with f at the boundary nodes, is called the *potential* due to f.

Suppose there are N edges numbered $e_1, \ldots, e_N$. A $d \times N$ matrix Q is constructed as follows. If e is an edge joining v_i to v_j with $i < j$, then

$$Q_{i,k} = +\sqrt{\gamma(e)},$$

$$Q_{j,k} = -\sqrt{\gamma(ek)},$$

$$Q_{h,k} = 0, \quad \text{otherwise}.$$

A calculation shows that $K = Q \cdot Q^{\mathrm{T}}$. Thus K is positive semi-definite. Suppose $x = (x_1, \dots, x_d)$. Then $xKx^{\mathrm{T}} = 0$ if and only if $xQ = 0$. Let $e = v_i v_j$ be an edge in Γ. Then $xQ = 0$ implies that

$$x_i \sqrt{\gamma(e)} - x_j \sqrt{\gamma(e)} = 0.$$

Thus $x_i = x_j$. Since Γ is connected as a graph, $xK^{\mathrm{T}}x = 0$ if and only if $x_i = x_j$ for all vertices v_i and v_j.

Lemma 3.1. *Suppose (Γ, γ) is a connected resistor network. Let $P = (p_1, \dots, p_k)$ be a non-empty proper subset of the vertices. Then the matrix $K(P;P)$ is positive definite.*

Proof. Let $A = K(P;P)$, and suppose $y = (y_1, \dots, y_k)$ is a vector with $yAy^{\mathrm{T}} = 0$. Let $x = (x_1, \dots, x_d)$ be the vector with $x_{p_i} = y_i$ for $1 \leqslant i \leqslant k$, and $x_j = 0$ if j is not in P. Then $xKx^{\mathrm{T}} = yAy^{\mathrm{T}} = 0$. Since P is a proper subset of V, at least one of the x_i is 0. Since Γ is connected, all the x_i must be 0. Hence the y_i are 0 also. □

Suppose $(\Gamma, \gamma) = (V, V_B, E, \gamma)$ is a connected resistor network. Let $I = V - V_B$ be the set of interior nodes. By Lemma 3.1, if I is not empty, the matrix $K(I,I)$ is nonsingular.

Theorem 3.2. *Suppose (Γ, γ) is a connected resistor network. Then the network response matrix Λ_γ is the Schur complement*

$$\Lambda_\gamma = K/K(I;I).$$

Proof. If I is the empty set, $K/K(I;I)$ is defined to be K, and $\Lambda_\gamma = K$. Otherwise, I is non-empty. For convenience, assume the nodes are numbered so that $V_B = \{v_1, v_2, \dots, v_n\}$, and $I = \{v_{n+1}, v_{n+2}, \dots, v_d\}$. Let $D = K(I;I)$. The K has a block structure.

$$K = \begin{bmatrix} A & B \\ C & D \end{bmatrix}.$$

Suppose that $f = \{f(v_i); i = 1, \dots, n\}$ is a function imposed at the boundary nodes. Let $g = \{g(v_i); i = n+1, \dots, d\}$ be the resulting potential at the interior nodes. Kirchhoff's current law says that the sum of the currents into each interior node is 0. Thus

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} c \\ 0 \end{bmatrix}.$$

This implies that $(A - BD^{-1}C)f = c$. Therefore the response matrix representing the Dirichlet-to-Neumann map is $\Lambda_\gamma = A - BD^{-1}C$. □

If $A = (a_1, \ldots, a_s)$ and $B = (b_1, \ldots, b_t)$ are two sequences of nodes, $A + B$ stands for the sequence $(a_1, \ldots, a_s, b_1, \ldots, b_t)$.

Lemma 3.3. *Suppose (Γ, γ) is a connected resistor network, and let Λ_γ be its response matrix. Let P and Q be two sequences of boundary nodes of Γ. Then the submatrix $\Lambda_\gamma(P;Q)$ is obtained as the Schur complement*

$$\Lambda_\gamma(P;Q) = K(P+I;Q+I)/K(I;I).$$

Proof. This follows from Theorem 3.2 and the definition of Schur complement. □

Suppose $\Gamma = (V, V_B, E)$ is a connected graph with n boundary nodes. Let p be one of the boundary nodes. Let $\Gamma' = (V', V_B', E')$ be the graph with $V' = V, V_B' = V_B - p$ and $E' = E$. That is Γ' is the same as Γ, except that p is declared to be an interior node. If γ is a conductivity on Γ, we assign the same values to the conductors in Γ'. Let Λ'_γ denote the response matrix for Γ'. By Theorem 3.2,

$$\Lambda'_\gamma = K/K(I+p;I+p).$$

Suppose $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ are two sequences of boundary nodes, and p is a boundary node not in $P \cup Q$.

Lemma 3.4. *In this situation,*
1. $\Lambda'_\gamma(P;Q) = \Lambda_\gamma(P+p;Q+p)/\Lambda_\gamma(p;p)$
2. $\det \Lambda'_\gamma(P;Q) = \det \Lambda_\gamma(P+p;Q+p)/\det \Lambda_\gamma(p;p)$

Proof. The first follows from the Haynsworth quotient formula. The second follows from the determinantal identity for Schur complements. □

4. Connections and determinants

Suppose $\Gamma = (V, V_B, E)$ is a connected graph with boundary. Γ is not assumed to be planar. Let $I = V - V_B$ denote the set of interior nodes. If p and q are two boundary nodes, a *path* from p to q through Γ is a sequence of edges $p, r_1, r_1, r_2, \ldots, r_m, q$ in Γ where the r_j are distinct interior nodes. Suppose $P = (p_1, \ldots, p_k)$ and $Q = (q_1, \ldots, q_k)$ are two disjoint sets of boundary nodes. A *connection* from P to Q through Γ is a set $\alpha = (\alpha_1, \ldots, \alpha_k)$ of disjoint paths through Γ, where for each $1 \leqslant i \leqslant k$, α_i is a path from P_i to $Q_{\tau(i)}$, and τ is an element of the permutation group S_k. Let $\mathscr{C}(P;Q)$ be the set of connections from P to Q. For each $\alpha = (\alpha_1, \ldots, \alpha_k)$ in $\mathscr{C}(P;Q)$, let

τ_α be the permutation of $(q_1, q_2, \dots, q_k)$ which results at the endpoints of the paths $(\alpha_1, \alpha_2, \dots, \alpha_k)$;
E_α be the set of edges in α;
J_α be the set of interior nodes which are not the ends of any edge in α.

Lemma 4.1. *Let (Γ, γ) be a connected resistor network. Let $P = (p_1, p_2, \dots, p_k)$ and $Q = (q_1, q_2, \dots, q_k)$ be two disjoint sequences of boundary nodes. Then*

$$\det \Lambda(P;Q) \cdot \det K(I,I) = (-1)^k \sum_{\tau \in S_k} \operatorname{sgn}(\tau) \{ \sum_{\substack{\alpha \in \mathscr{C}(P;Q) \\ \tau_\alpha = \tau}} \prod_{e \in E_\alpha} \gamma(e) \cdot \det K(J_\alpha; J_\alpha) \}.$$

Proof. Let $v = k + k'$, where k' is the number of interior nodes in Γ. Let the interior nodes be numbered r_i for $i = k+1, \dots, k+k'$. By taking the Schur complement with respect to $K(I,I)$, we have

$$\det \Lambda(P;Q) \cdot \det K(I,I) = \det K(P+I; Q+I).$$

The $v \times v$ matrix $K(P+I;Q+I)$ is denoted $M = \{m_{i,j}\}$. Then

$$\det M = \sum_{\sigma \in S_v} \operatorname{sgn}(\sigma) \prod_{i=1}^{v} m_{i,\sigma(i)}.$$

Here S_v denotes the symmetric group on v symbols. For each $1 \leqslant i \leqslant k$, let n_i be the first index j for which $\sigma^j(i) \leqslant k$. For each $1 \leqslant i \leqslant k$, and $0 \leqslant j \leqslant n_i$, let $a(i,j) = \sigma^j(i)$. Let τ be the permutation of $1, 2, \dots, k$ where $\tau(i) = a(i, n_i)$. Thus each $\sigma \in S_v$ gives a diagram of the following form:

$$\begin{gathered}
1 = a(1,0) \overset{\sigma}{\mapsto} a(1,1) \overset{\sigma}{\mapsto} a(1,2) \overset{\sigma}{\mapsto} \cdots \overset{\sigma}{\mapsto} a(1,n_1) = \tau(1), \\
2 = a(2,0) \overset{\sigma}{\mapsto} a(2,1) \overset{\sigma}{\mapsto} a(2,2) \overset{\sigma}{\mapsto} \cdots \overset{\sigma}{\mapsto} a(2,n_2) = \tau(2), \\
\cdots \\
k = a(k,0) \overset{\sigma}{\mapsto} a(k,1) \overset{\sigma}{\mapsto} a(k,2) \overset{\sigma}{\mapsto} \cdots \overset{\sigma}{\mapsto} a(k,n_k) = \tau(k).
\end{gathered}$$

Let A be the subset of $\{1, 2, \dots, v\}$ consisting of the $a(i,j)$ for $1 \leqslant i \leqslant k$, $0 \leqslant j < n_i$. Let $t = \sum n_i$, which is the cardinality of A. Let B be the set $\{1, 2, \dots, v\} - A$. Then σ may be expressed as a product $\sigma = \phi \cdot \mu$, where ϕ is a permutation of A, and μ is a permutation of B. Let ϕ be expressed as a product of disjoint cycles $\phi = \phi_1 \cdot \phi_2 \dots \phi_s$. Then $\operatorname{sgn}(\sigma) = (-1)^{t-s} \operatorname{sgn}(\mu)$. Then τ will also be expressed as a product of s cycles. $\tau = \psi_1 \cdot \psi_2 \cdots \psi_s$ and $\operatorname{sgn}(\tau) = (-1)^{k-s}$. Thus $\operatorname{sgn}(\sigma) = (-1)^{k+t} \operatorname{sgn}(\tau) \operatorname{sgn}(\mu)$.

The diagram above determines a set $\alpha = (\alpha_1, \dots, \alpha_k)$ of sequences of nodes in Γ, where α_i is the sequence $a(i,0)$, $a(i,1), \dots, a(i,n_i)$. For each $1 \leqslant i \leqslant k$, $a(i,0) = p_i$ and $a(i,n_i) = q_{\tau(i)}$. For each $1 \leqslant i \leqslant k$, and $0 < j < n_i$, $a(i,j)$ is the interior node $r_{a(i,j)}$. The product $\prod_{i=1}^{v} m_{i,\sigma(i)}$ can be non-zero only if $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_k)$ forms a connection through Γ from P to Q. For each

$\alpha \in \mathscr{C}(P;Q)$, let $S(\alpha)$ be the set of $\sigma \in S_v$ for which the connection is α. As σ varies over $S(\alpha)$, μ varies over the permutations of J_α. Then

$$\sum_{\sigma\in S(\alpha)} \mathrm{sgn}(\sigma)\prod_{i=1}^{v} m_{i,\sigma(i)} = \sum_{\sigma\in s(\alpha)} (-1)^{k+t}\,\mathrm{sgn}(\tau)\prod_{e\in E_\alpha}(-\gamma(e))\cdot \mathrm{sgn}(\mu)\cdot \prod_{i\in J_\alpha} m_{i,\mu(i)}$$
$$= (-1)^k\,\mathrm{sgn}(\tau)\cdot \prod_{e\in E_\alpha}\gamma(e)\cdot \det K(J_\alpha;J_\alpha).$$

For each $\tau \in S_k$, take the sum over all α which induce this τ. Then take the sum over all $\tau \in S_k$, and the proof is complete. □

This answers a question raised by Ref. [2]. In particular, it follows from Lemma 4.1 that if $\det \Lambda(P;Q) = 0$, then either

1. There is no connection from P to Q; or
2. There are (at least) two connections α and β from P to Q, with permutations τ_α and τ_β of opposite sign.

The following theorem is very important for our purposes. It was first proved for well-connected circular planar networks in Ref. [7], and for general circular planar networks in Ref. [1]. The proof we give here is based on Lemma 4.1.

Theorem 4.2. *Suppose Γ is a circular planar resistor network and $(P;Q) = (p_1,\ldots,p_k;q_1,\ldots,q_k)$ is a circular pair of sequences of boundary nodes.*
(a)*If $(P;Q)$ are not connected through Γ, then* $\det \Lambda(P;Q) = 0$.
(b)*If $(P;Q)$ are connected through Γ, then* $(-1)^k \det \Lambda(P;Q) > 0$.

Proof. We first consider the case when Γ is connected as a graph. By Lemma 3.1, $K(I,I)$ is positive definite, so $\det K(J,J) > 0$ for all $J \subseteq I$. The sequence $(p_1,\ldots,p_k, q_k,\ldots,q_1)$ is in circular order around the boundary of Γ. If there is a connection from P to Q, it must connect p_i to q_i for $1 \leqslant i \leqslant k$. Thus each τ which appears in Lemma 4.1 is the identity permutation, so all the terms in the sum have the same sign. In the general case, Γ is a disjoint union of connected components Γ_i, and $\Lambda(\Gamma, \gamma)$ is a direct sum of the $\Lambda(\Gamma_i, \gamma_i)$. □

When we say that removal of an edge e from Γ breaks the connection from P to Q, we mean that P and Q are connected through Γ (possibly in many ways), and that P and Q are not connected through the graph Γ' which is the graph Γ with e removed. By Theorem 4.2, this is equivalent to the two assertions that $\det \Lambda(P;Q) \neq 0$, and $\det \Lambda'(P;Q) = 0$.

An edge e between a pair of adjacent boundary nodes is called a *boundary edge*. If r is a boundary node which is joined by an edge to only one other node p which is an interior node, the edge rp is called a *boundary spike*.

Corollary 4.3. *Suppose Γ is a connected circular planar resistor network and $e = pq$ is a boundary edge, such that deleting e breaks the connection between a circular pair $(P; Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$. Then pq is either p_1q_1 or p_kq_k, and*

$$\det \Lambda(P; Q) = -\gamma(e) \cdot \det \Lambda(P - p; Q - q).$$

Proof. The edge pq must be either p_1q_1 or p_kq_k. As the two cases are similar, WLOG assume the former. We consider $\det K(P + I; Q + I)$ as a linear function $F(z)$ of the first column z of $K(P + I; Q + I)$. Let $\xi = \gamma(e)$. Then $z = x + y$, where

$$x = \begin{bmatrix} -\xi \\ 0 \end{bmatrix} \quad \text{and} \quad y = \begin{bmatrix} 0 \\ a \end{bmatrix}.$$

Then $F(z) = F(x) + F(y)$. But $F(y) = 0$, since P and Q are not connected through Γ after p_1q_1 is deleted. Thus

$$\det K(P + I; Q + I) = -\xi \det K(P - p_1 + I; Q - q_1 + I).$$

The result follows by taking the Schur complement with respect to $K(I; I)$, and using Lemma 3.3. □

Corollary 4.4. *Suppose Γ is a connected circular planar resistor network and rp is boundary spike joining a boundary node r to an interior node p. Suppose that contracting rp breaks the connection between a circular pair $(P; Q) = (p_1, \ldots, p_k; q_1, \ldots, q_k)$. Then $r \notin P \cup Q$, and*

$$\det \Lambda(P + r; Q + r) = \gamma(pr) \cdot \det \Lambda(P; Q).$$

Proof. It is clear that $r \notin P \cup Q$. Let $\xi = \gamma(\mathrm{pr})$. Then $K(P + r + I; Q + r + I)$ has a submatrix $K(r, p; r, p)$ which has the form:

$$K(r, p; r, p) = \begin{bmatrix} \xi & -\xi \\ -\xi & w \end{bmatrix}.$$

The remaining entries of $K(P + r + I;\ Q + r + I)$ in the column corresponding to r are 0, and the remaining entries of $K(P + r + I;\ Q + r + I)$ in the row corresponding to r are 0. Thus

$$\begin{aligned} \det K(P + r + I;\ Q + r + I) = {} & \xi \det K(P + I;\ Q + I) \\ & - \xi^2 \det K(P + I - p;\ Q + I - p). \end{aligned}$$

The assertion of the corollary follows upon dividing by $K(I;\ I)$, interpreting each of the terms as the determinant of a Schur complement, and using Lemma 3.3. □

5. Y–Δ transformations

Suppose $\Gamma = (V, V_B, E)$ is a circular planar graph, and s is a trivalent interior node with incident edges sp, sq and sr, as in Fig. 1(A). A *Y–Δ transformation* removes the vertex s, the edges sp, sq, sr and adds three new edges pq, qr, and rp as in Fig. 1(B). Similarly, if pqr is a triangle in Γ as in Fig. 1(B), then a Δ–Y transformation removes the edges pq, qr, and rp, inserts a new vertex s, and adds three new edges ps, qs, and rs, to arrive at Fig. 1(A). All other nodes are fixed during the transformation.

We say that two circular planar graphs Γ_1 and Γ_2 are *Y–Δ equivalent* if Γ_1 can be transformed to Γ_2 by a sequence of Y–Δ or Δ–Y transformations.

Lemma 5.1. *If Γ_1 and Γ_2 are two circular planar graphs which are Y–Δ equivalent, then $\pi(\Gamma_1) = \pi(\Gamma_2)$.*

Proof. Suppose Γ_1 is transformed into Γ_2 where the Y of Fig. 1(A) is replaced by the triangle of Fig. 1(B). Let α and β be disjoint paths in Γ_1 where α passes through p and β passes through edges rs and sq. The corresponding paths in Γ_2 are α and β', where β' is the same as β except that the two edges rs and sq are replaced by the single edge rq. □

Lemma 5.2. *Suppose Γ_1 and Γ_2 are two circular planar graphs which are Y–Δ equivalent. Then Γ_1 is critical if and only if Γ_2 is critical.*

Proof. Suppose Γ_1 is transformed into Γ_2 where the Y of Fig. 1(A) is replaced by the triangle of Fig. 1(B). Assume that Γ_1 is not critical. We need to consider three cases.

(1) Suppose e is an edge in Γ_1 which is not ps, qs, or rs and e can be removed without breaking a connection in $\pi(\Gamma_1)$. Then removal of the same edge in Γ_2 breaks no connection in $\pi(\Gamma_2)$.

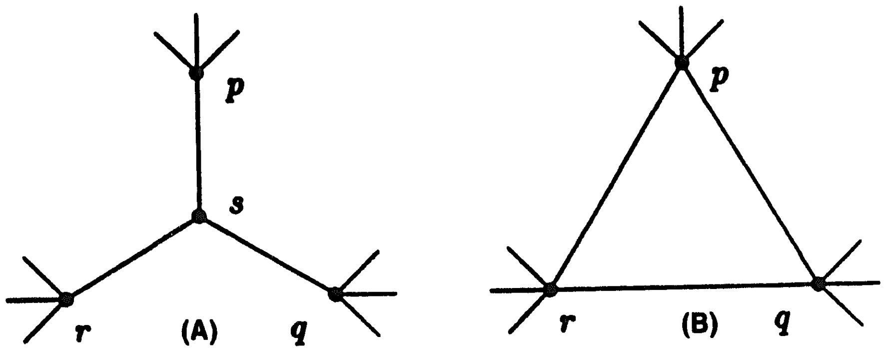

Fig. 1. (A) Y with incident edges sp, sq, sr changed to (B) Δ with edges pq, qr, rp.

(2) Suppose deletion [illegible] breaks no connection in $\pi(\Gamma_1)$. Then deletion of pr breaks no connection in $\pi(\Gamma_2)$.

(3) Suppose contraction of ps breaks no connection in $\pi(\Gamma_1)$. Then deletion of rq breaks no connection in $\pi(\Gamma_2)$.

Assume that Γ_2 is not critical. Again there are three cases.

(4) Suppose e is an edge in Γ_2 which is not pq, qr, or rp and e can be removed without breaking a connection in $\pi(\Gamma_2)$. Then removal of the same edge in Γ_1 breaks no connection in $\pi(\Gamma_1)$.

(5) Suppose deletion of rq breaks no connection in $\pi(\Gamma_2)$. Then contraction of ps breaks no connection in $\pi(\Gamma_1)$.

(6) Suppose contraction of rq breaks no connection in $\pi(\Gamma_2)$. Then contraction of rs breaks no connection in $\pi(\Gamma_1)$. □

Lemma 5.3. *Suppose Γ_1 and Γ_2 are two circular planar graphs which Y–Δ equivalent. If γ_1 is a conductivity on Γ_1 then there is a conductivity γ_2 on Γ_2, with $\Lambda(\Gamma_1,\gamma_1) = \Lambda(\Gamma_2,\gamma_2)$.*

Proof. Suppose Γ_1 is transformed into Γ_2 where the Y of Fig. 1(A) is replaced by the triangle of Fig. 1(B). Suppose $\gamma_1(ps)=a$, $\gamma_1(qs)=b$, $\gamma_1(rs)=c$. The corresponding conductivity γ_2 on Γ_2 is

$$\gamma_2(pq) = \frac{ab}{a+b+c},$$
$$\gamma_2(qr) = \frac{bc}{a+b+c},$$
$$\gamma_2(rp) = \frac{ac}{a+b+c},$$

and $\gamma_2(e)=\gamma_1(e)$ for all other edges.

Suppose Γ_1 is transformed into Γ_2 where the triangle of Fig. 1(B) is replaced by the Y of Fig. 1(A). Suppose $\gamma_1(pq)=a$, $\gamma_1(qr)=b$, $\gamma_1(rp)=c$. The corresponding conductivity γ_2 on Γ_2 is

$$\gamma_2(ps) = \frac{ab+ac+bc}{b},$$
$$\gamma_2(qs) = \frac{ab+ac+bc}{c},$$
$$\gamma_2(rs) = \frac{ab+ac+bc}{a},$$

and $\gamma_2(e)=\gamma_1(e)$ for all other edges. If u is a function defined at the nodes of Γ_1 which satisfies Kirchhoff's current law, the same function (omitting the point s) satisfies Kirchhoff's current law on Γ_2. Hence $\Lambda(\Gamma_1,\gamma_1) = \Lambda(\Gamma_2,\gamma_2)$. □

Lemma 5.2. *Suppose Γ_1 and Γ_2 are two circular planar graphs which are Y–Δ equivalent. If γ_1 is recoverable from $\Lambda(\Gamma_1,\gamma_1)$, then γ_2 is recoverable from $\Lambda(\Gamma_2,\gamma_2)$.*

Proof. This follows from Lemma 5.3. □

6. Medial graphs

Suppose $\Gamma = (V, V_B, E)$ is a circular planar graph with n boundary nodes. Γ is assumed to be embedded in the plane so that the boundary nodes $v_1, v_2, \dots, v_n$ occur in clockwise order around a circle C and the rest of Γ is in the interior of C. The construction of the medial graph $\mathscr{M}(\Gamma)$ is similar to that in (Ref. [10], p. 239). The medial graph $\mathscr{M}(\Gamma)$ depends on the embedding. First, for each edge e of Γ, let m_e be its midpoint. Next, place $2n$ points $t_1, t_2, \dots, t_{2n}$ on C so that

$$t_1 < v_1 < t_2 < t_3 < v_2 < \cdots < t_{2n-1} < v_n < t_{2n} < t_1$$

in the clockwise circular order around C.

(1) The vertices of $\mathscr{M}(\Gamma)$ consist of the points m_e for $e \in E$, and the points t_i for $i = 1,2, \dots ,2n$.

(2) The edges in $\mathscr{M}(\Gamma)$ are as follows. Two vertices m_e and m_f are joined by an edge whenever e and f have a common vertex and e and f are incident to the same face in Γ. There is also one edge for each point t_j as follows. The point t_{2i} is joined by an edge to m_e where e is the edge of the form $e = v_i r$ which comes first after arc $v_i t_{2i}$ in clockwise order around v_i. The point t_{2i-1} is joined by an edge to m_f where f is the edge of the form $f = v_i s$ which comes first after arc $v_i t_{2i-1}$ in counter-clockwise order around v_i.

The vertices of the form m_e of $\mathscr{M}(\Gamma)$ are 4-valent; the vertices of the form t_i are 1-valent. An edge uv of $\mathscr{M}(\Gamma)$ has a *direct extension* vw if the edges uv and vw separate the other two edges incident to the vertex v. A path $u_0 u_1 \dots u_k$ in $\mathscr{M}(\Gamma)$ is called a *geodesic arc* if each edge $u_{i-1} u_i$ has edge $u_i u_{i+1}$ as a direct extension. A geodesic arc $u_0 u_1 \dots u_k$ is called a *geodesic* if either

(1) u_0 and u_k are points on the circle C,

or (2) $u_k = u_0$ and $u_{k-1} u_k$ has $u_0 u_1$ as direct extension.

A subgraph $\mathscr{L}$ of $\mathscr{M}(\Gamma)$ is called a *lens* provided that:

(1) $\mathscr{L}$ consists of a simple closed path $u_0 u_1 \dots u_k v_0 v_1 \dots v_m u_0$ and all the nodes and edges of $\mathscr{M}(\Gamma)$ in the bounded connected component of the complement of $\mathscr{L}$ in the plane.

(2) $u_0 u_1 \dots u_k v_0$ and $v_0 v_1 \dots v_m u_0$ are two geodesic arcs such that no inner edge of $\mathscr{L}$ is incident to u_0 or v_0.

If each geodesic in $\mathscr{M}(\Gamma)$ begins and ends on C, has no self-intersection, and if $\mathscr{M}(\Gamma)$ has no lenses, we say that $\mathscr{M}(\Gamma)$ is *lensless*.

A *triangle* in $\mathscr{M}(\Gamma)$ is a triple $\{f,g,h\}$ of geodesics which intersect to form a triangle with no other intersections within the configuration, as in Fig. 2(A).

Suppose $\{f,g,h\}$ form a triangle as in Fig. 2(A). A *motion* of $\{f,g,h\}$ consists of replacing this configuration with that of Fig. 2(B).

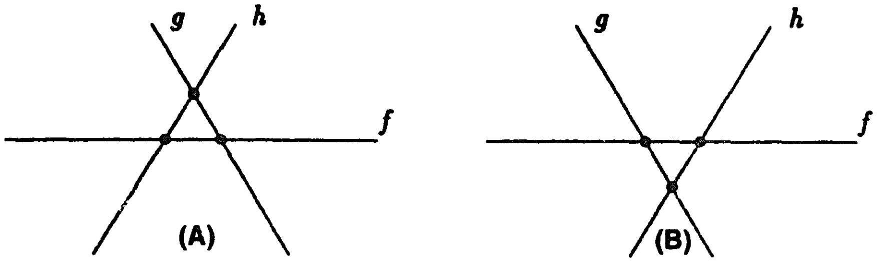

Fig. 2. (A) Triangle changed by motion of $\{f, g, h\}$ to another triangle (B).

Lemma 6.1. *Two circular planar graphs are Y–Δ equivalent if and only if their medial graphs are equivalent under motions.*

Proof. Each Y–Δ transformation of Γ corresponds to a motion on $\mathscr{M}(\Gamma)$. Conversely, a motion on $\mathscr{M}(\Gamma)$ corresponds to a Y–Δ transformation of Γ. □

We shall make extensive use of the following lemma. Our proof is an adaptation of a proof of Steinitz to our situation; see Refs. [10,11].

Lemma 6.2. *Suppose Γ is a circular planar graph, for which $\mathscr{M}(\Gamma)$ is lensless. Suppose g and h intersect at p. Suppose g intersects C at q and h intersect C at r. Assume $\mathscr{F} = \{f_1, \ldots, f_m\}$ is a set of geodesics with the property that for each $1 \leqslant i \leqslant m$, f_i, intersects g between p and q if and only if f_i intersects h between p and r. Then a finite sequence of motions will remove all members of $\mathscr{F}$ from the sector qpr.*

Proof. For each $i = 1, \ldots, m$, let v_i be the point of intersection (if there is one) of f_i with g between p and q. For each f_i which intersects another of the f_j within sector qpr, let D_i be the first point of intersection on f_i after v_i in sector qpr. Let $\mathscr{D} = \{D_i\}$ be the set of points obtained in this way. If $\mathscr{D}$ is empty, let f_i be the geodesic in $\mathscr{F}$ such that v_i is closest to p, and $\{g, h, f_i\}$ form a triangle. A motion will remove f_i from sector qpr. Otherwise, $\mathscr{D}$ is non-empty. Each point $D_i \in \mathscr{D}$ is the point of intersection of two of the geodesics, say f_i and f_j. Let D be a point in $\mathscr{D}$ for which the number of regions within the configuration formed by f_i and f_j and g is a minimum. This minimum must be one, or there would be another geodesic which intersects f_i between v_i and D or which intersects f_j between v_j and D. Then $\{g, f_i, f_j\}$ form a triangle. A motion will reduce the number of regions within sector qpr. After a finite number of motions, no f_i crosses into the sector. □

Lemma 6.3. *Suppose Γ is a circular planar graph, for which $\mathscr{M}(\Gamma)$ has a lens. Then Γ is Y–Δ equivalent to a graph Γ' which has either a pair of edges in series, or a pair of edges in parallel.*

Proof. Suppose g and h are two geodesics which intersect at p_1 and p_2 to form a lens $\mathscr{L}$. w.l.o.g. assume that $\mathscr{L}$ is a lens with the fewest number of regions inside $\mathscr{L}$. Each geodesic f which intersects g between p_1 and p_2 also intersects h between p_1 and p_2, or there would be a lens with fewer regions than $\mathscr{L}$. An argument similar to that of Lemma 6.2 shows that all of these f may be removed from $\mathscr{L}$. Thus Γ is Y–Δ equivalent to a graph Γ' for which $\mathscr{M}(\Gamma')$ has an empty lens. This empty lens corresponds either to a pair of edges in series (if there is a vertex of Γ' within $\mathscr{L}$), or to a pair of edges in parallel (if there is no vertex of Γ' within $\mathscr{L}$). □

Lemma 6.4. *If Γ is a critical circular planar graph, then $\mathscr{M}(\Gamma)$ is lensless.*

Proof. If there were a lens, a closed geodesic or a geodesic with a self-intersection in $\mathscr{M}(\Gamma)$, then Γ would be Y–Δ equivalent to a graph Γ' with a pair of edges in series or in parallel, or with an interior pendant or an interior loop. In each case an edge could be removed from Γ' without breaking any connection, so Γ' would not be critical, and hence also Γ would not be critical. □

In Section 13, we show that if $\mathscr{M}(\Gamma)$ is lensless, then Γ is critical.

7. Standard graphs

Suppose Γ is a circular planar graph with n boundary nodes which is embedded in the plane so that the boundary nodes $v_1, \dots, v_n$ occur in clockwise order on a circle C and the rest of Γ is in the interior of C. Assume the medial graph $\mathscr{M}(\Gamma)$ is lensless. Then $\mathscr{M}(\Gamma)$ has n geodesics each of which intersects C twice. The n geodesics intersect C in $2n$ distinct points. These $2n$ points are labelled $t_1, \dots, t_{2n}$, so that

$$t_1 < v_1 < t_2 < t_3 < v_2 < \cdots < t_{2n-1} < v_n < t_{2n} < t_1$$

in the circular order around C. The geodesics are labelled as follows. Let g_1 be the geodesic which begins at t_1. The remaining geodesics are labelled $g_2, g_3, \dots, g_n$ so that if $i < j$, then the first point of intersection of g_i with C occurs before the first point of intersection of g_j with C in the clockwise order starting from t_1. For each $i = 1, 2, \dots, 2n$, let z_i be the number associated with the geodesic which intersects C at t_i. In this way we obtain a sequence $z = z_1, z_2, \dots, z_{2n}$, called the z-sequence for $\mathscr{M}(\Gamma)$. Each of the numbers from 1 to n occurs in z exactly twice. If $i < j$, and if the occurrences of i and j appear in z in the order

$$\dots i \dots j \dots i \dots j \dots$$

we say that i and j *interlace* in z; otherwise, we say that i and j do not interlace in z.

Suppose $z = z_1, z_2, \ldots, z_{2n}$ is a sequence which contains each of the numbers $1, 2, \ldots, n$ twice. Assume that if $i < j$, the first occurrence of i comes before the first occurence of j. Associated with this sequence, there is a standard arrangement $\mathscr{A}(z)$, of n pseudolines $\{g_i\}$ in the disc, constructed as follows. Place $2n$ points in clockwise order around the circle C and label them $x_1, \ldots, x_n$ and $y_1, \ldots, y_n$ as follows. The points labelled x_i and y_i are to be placed at positions corresponding to the two occurrences of i in the sequence $z_1, \ldots, z_{2n}$, with $x_i < y_i$. We join each x_i to y_i by a geodesic g_i. If i and j interlace in z, then g_i will be made to intersect g_j; the point of intersection is denoted $x(i,j)$, with the convention that $x(j,i)$ denotes the same point as $x(i,j)$.

First, join x_1 to y_1 by pseudoline g_1. After $g_1, \ldots, g_{m-1}$ have been placed within C, the pseudoline g_m joining x_m to y_m is placed as follows. For each $i \leqslant m-1$, if m interlaces i in z, place a point $x(i,m)$ on g_i closer to y_i than any previously placed point on g_i. Now let g_m join x_m to y_m passing through the points $x(i,m)$ which have just been placed. The points y_i which are between x_m and y_m occur in the same order on C as the points $x(i,m)$ occur on g_m.

When all the pseudolines $g_1, \ldots, g_n$ are in place, the arrangement $\mathscr{A}(z)$ has sequence z. The intersection points $x(i,j)$ occur as follows. For each $i \leqslant m-1$, the points $x(i,j)$ which are on g_i appear between x_i and y_i so that:

1. If $i < j < k$, then $x(i,j)$ appears before $x(i,k)$.
2. If $j < i < k$, then $x(i,j)$ appears before $x(i,k)$.
3. If $j < k < i$, then $x(i,j)$ and $x(i,k)$ appear on g_i in the same order as y_j and y_k appear in z.

Let $\{x_1, \ldots, x_n\} \cup \{y_1, \ldots, y_n\} = \{t_1, \ldots, t_{2n}\}$ where $t_1 < \cdots < t_{2n}$ in the clockwise order around the circle C. Place n boundary points $v_1, \ldots, v_n$ on C so that the points

$$t_1 < v_1 < t_2 < t_3 < v_2 < \ldots < t_{2n-1} < v_n < t_{2n}$$

are in clockwise circular order on C. Next color the regions formed by $\mathscr{M}(\Gamma(z))$ inside C in two colors, black and white, with each v_i in a black region. To obtain the standard graph $\Gamma(z)$, for which $\mathscr{M}(\Gamma(z)) = A(z)$, we must assume that each of the black regions contains at most one of the vertices v_i. After a vertex has been placed in each black region, they are joined by edges, with one edge passing through each of the points $x(i,j)$.

Lemma 7.1. *Let Γ be a connected circular planar graph with n boundary nodes. Assume $\mathscr{M}(\Gamma)$ is lensless. Let $z = z_1, z_2, \ldots, z_{2n}$ be the z-sequence associated with Γ, and let $\Gamma(z)$ be the standard graph constructed above. Then Γ is $Y - \Delta$ equivalent to $\Gamma(z)$.*

Proof. We make motions in $\mathcal{M}(\Gamma)$ to transform it to $A(z)$. Geodesic g_i intersects the outer circle C at two points x_i and y_i, with $x_i < y_i$. The points $x_1, x_2, \dots, x_n$ and $y_1, y_2, \dots, y_n$ occur in the order of z around C. If i and j interlace in z, the geodesic g_j intersects g_i. Let $x(i,j) = x(j,i)$ be the point of intersection of g_i with g_j, and let $S(i,j)$ be the sector formed by x_i, $x(i,j)$ and x_j. The location of the points $x(i,j)$ is changed by the motions of $\mathcal{M}(\Gamma)$.

Let k be the first index for which g_k intersects a previous geodesic. Then g_k must intersect g_{k-1}. Consider the geodesics from the set $\{g_{k+1}, g_{k+2}, \dots, g_n\}$ which intersect g_k between $x(k-1,k)$ and x_k. Any such geodesic also intersects g_{k-1} between $x(k-1,k)$ and x_{k-1}. Lemma 6.2 implies that finite sequence of motions will remove $g_{k+1}, \dots, g_n$ from the sector $S(k-1,k)$. This process is repeated to remove all intersections of $g_{k+1}, \dots, g_n$ from the sectors $S(i,k)$ for $i = k-2, \dots, 1$.

We perform a similar process at steps $k+1, \dots, n-1$. After step $(m-1)$, the geodesics are in position so that if $i < j < m$, the geodesics $g_m, g_{m+1}, g_{k+1}, \dots, g_n$ have no intersections within any of the sectors $S(i,j)$. Note that for each $1 \leqslant i < m$, if g_m intersects g_i, then for all $j < m$ the point of intersection $x(i,m)$ is between $x(i,j)$ and y_i on g_i. Also the set of points

$$\{x_m, x(m,1), \dots, x(m,m-1), y_m\}$$

occur in the following order along g_m: if $j < m$ and $k < m$, with $j \neq k$, then $x(m,j)$ and $x(m,k)$ appear in the same order along g_m as y_j and y_k appear in z.

For step (m), Lemma 6.2 implies that we can remove geodesics $g_{m+1}, \dots, g_n$ from the sectors $S(j,m)$ for $1 \leqslant j < m$. These geodesics are removed from the sectors $S(j,m)$ in the same order in which the $x(m,j)$ appear on g_m.

Continue until $m = n-1$, when all intersections are as in $\mathcal{A}(z)$. □

Theorem 7.2. *Suppose Γ_1 and Γ_2 are two connected circular planar graphs, each with n boundary nodes. Assume the medial graphs $\mathcal{M}(\Gamma_1)$ and $\mathcal{M}(\Gamma_2)$ are lensless. Then Γ_1 and Γ_2 are Y–Δ equivalent if and only if $\mathcal{M}(\Gamma_1)$ and $\mathcal{M}(\Gamma_2)$ and have the same z-sequence.*

Proof. A Y–Δ transformation does not change the z-sequence. Conversely, if $\mathcal{M}(\Gamma_1)$ and $\mathcal{M}(\Gamma_2)$ and have the same z-sequence, then Γ_1 and Γ_2 are each be Y–Δ equivalent to the same standard graph $\Gamma(z)$. □

When the sequence z is $1, \dots, n, 1, \dots, n$, the standard arrangement $\mathcal{A}(z)$ is denoted $\mathcal{A}_n$ and the standard graph $\Gamma(z)$ is denoted Σ_n. In $\mathcal{A}_n$, every pseudoline g_i intersects every other pseudoline, and there are $\frac{1}{2}n(n-1)$points of intersection $x(i,j)$. For each $1 \leqslant i \leqslant n$, the points

$$x_i, x(i,1), x(i,2), \dots, x(i,i-1), x(i,i+1), \dots, x(i,n), y_i$$

occur in order along g_i.

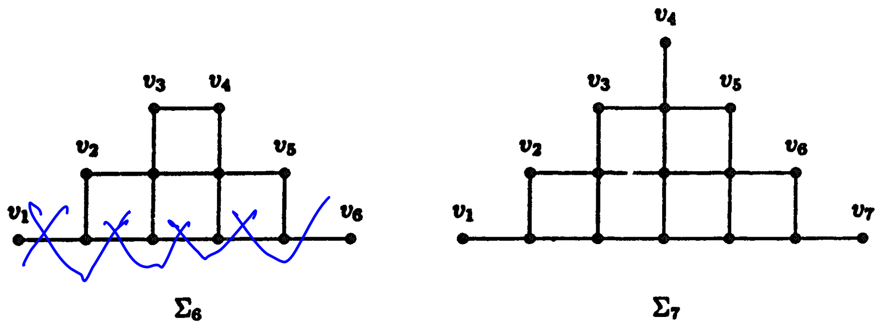

Fig. 3. Graphs Σ_6 and Σ_7.

The standard graph Σ_n has $\frac{1}{2}n(n-1)$ edges. The graphs Σ_6 and Σ_7 are shown in Fig. 3.

As in Ref. [1], a circular planar graph is called *well-connected* if every circular pair $(P;Q)$ is connected through Γ.

Proposition 7.3. *For each integer $n \geqslant 3$, the graph Σ_n is critical and well-connected.*

Proof. The proof is left to the reader. □

Corollary 7.4. *Let $n = 4m + 3$, and let $C(m, 4m + 3)$ be the circular graph of* [7]. *Suppose that Γ is a circular planar graph with n boundary nodes, Assume that $\mathcal{M}(\Gamma)$ is lensless and has z-sequence, $1, \ldots, n, 1, \ldots, n$. Then Γ is Y–Δ equivalent to $C(m, 4m + 3)$. In particular, Σ_n and $C(m, 4m + 3)$ are Y–Δ equivalent.*

Proof. The medial graph $\mathcal{M}(C(m,4m + 3))$ is lensless. The z-sequence is $1, \ldots, n, 1, \ldots, n$. By Lemma 7.2, Γ and $C(m,4m + 3)$ are Y–Δ equivalent. □

8. Adjoining edges

Let (Γ, γ) be a circular planar resistor network with n boundary nodes $v_1, \ldots, v_n$. We will describe three ways to adjoin an edge to Γ, and the effect of each on the matrix $\Lambda(\Gamma, \gamma)$. In this section $\Lambda(\Gamma)$ stands for $\Lambda(\Gamma, \gamma)$, with the conductivity γ implicit from the context.

(1) Let p and q be two adjacent boundary nodes. For convenience of notation, we make a cyclic re-labelling of the boundary nodes, so that $p = v_1$ and $q = v_2$. We add an edge pq so that the new graph is still be a circular planar graph with n boundary nodes. We call this process *adjoining a boundary edge*. If a boundary edge pq is adjoined to Γ, with $\gamma(pq) = \xi$, the resulting resistor network is denoted $\mathcal{T}_\xi(\Gamma)$.

Suppose $M = \{m_{i,j}\}$is an $n \times n$ matrix and ξ is a real number. We define a new matrix $\mathscr{T}_\xi(M)$ as follows.

$$T_\xi(M)_{1,1} = m_{1,1} + \xi,$$
$$T_\xi(M)_{2,2} = m_{2,2} + \xi,$$
$$T_\xi(M)_{1,2} = m_{1,2} - \xi,$$
$$T_\xi(M)_{2,1} = m_{2,1} - \xi,$$
$$T_\xi(M)_{i,j} = m_{i,j} \quad \text{otherwise.}$$

Clearly, $T_{-\xi}T_\xi =$ identity. From the definition of Kirchhoff matrix, we have

$$K(\mathscr{T}_\xi(\Gamma)) = T_\xi(K(\Gamma)).$$

From Theorem 3.2, it follows that

$$\Lambda(T_\xi(\Gamma)) = T_\xi(\Lambda(\Gamma)),$$

$$\Lambda(\Gamma) = T_{-\xi}(\Lambda(\mathscr{T}_\xi(\Gamma)).$$

Suppose given (Γ,γ) and ξ. Then $\Lambda(\Gamma)$ uniquely determines $\Lambda(\mathscr{T}_\xi(\Gamma))$.. Also $\Lambda(\mathscr{T}_\xi(\Gamma))$ uniquely determines $\Lambda(\Gamma)$.

(2) Let p be a boundary node. By a cyclic re-labelling of the boundary nodes, assume that $p = v_1$. We place a new vertex v_0 on the boundary circle C, between v_n and v_1, and adjoin a new edge v_0v_1 to Γ. The new graph is a circular planar graph with $n + 1$ boundary nodes. We call this process *adjoining a boundary spike without interiorizing*. If a boundary spike v_0v_1 is adjoined to Γ, without interiorizing the vertex v_1, and with $\gamma(v_0v_1) = \xi$, the resulting resistor network is denoted $\mathscr{P}_\xi(\Gamma)$.

Suppose $M = \{m_{i,j}\}$ is an $n \times n$ matrix, written in block form

$$M = \begin{bmatrix} m_{1,1} & a \\ b & C \end{bmatrix}.$$

If ξ a real number, let $P_\xi(M)$ be the $(n + 1) \times (n + 1)$ matrix, with indices $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$,

$$P_\xi(M) = \begin{bmatrix} \xi & -\xi & 0 \\ -\xi & m_{1,1} + \xi & a \\ 0 & b & C \end{bmatrix}.$$

Then by Theorem 3.2,

$$\Lambda(\mathscr{P}_\xi(\Gamma) = P_\xi(\Lambda(\Gamma)).$$

Suppose given (Γ, γ) and ξ. Then $\Lambda(\Gamma)$ uniquely determines $\Lambda(\mathscr{P}_\xi(\Gamma))$. Also, $\Lambda(\mathscr{P}_\xi(\Gamma))$ uniquely determines $\Lambda(\Gamma)$

(3) Let p be a boundary node. By cyclic re-labelling of the boundary nodes, assume that $p = v_1$. We adjoin a boundary spike rv_1 to Γ, then declare v_1 to be an interior node, and renumber so that r is the first boundary node. The new graph is a circular planar graph with n boundary nodes. We call this process *adjoining a boundary spike*. If a boundary spike rv_1 is adjoined to Γ, with $\gamma(rv_1) = \xi$, the resulting resistor network is denoted $S_\xi(\Gamma)$.

Suppose $M = \{m_{i,j}\}$ is an $n \times n$ matrix, written in block form

$$M = \begin{bmatrix} m_{1,1} & a \\ b & C \end{bmatrix}.$$

For any real number ξ, the $(n+1) \times (n+1)$ matrix $P_\xi(M)$ has been defined in part (2). The indexing is $0 \leqslant i \leqslant n$ and $0 \leqslant j \leqslant n$. If the (1,1) entry $\delta = m_{1,1} + \xi$ is not 0, we take the Schur complement of $P_\xi(M)$ with respect to this entry, to obtain

$$S_\xi(M) = P_\xi/[m_{1,1} + \xi] = \begin{bmatrix} \xi - \xi^2/\delta & a\xi/\delta \\ b\xi/\delta & C - ba/\delta \end{bmatrix}.$$

A calculation shows that $S_{-\xi} \circ S_\xi =$ identity. From the definition of the Kirchhoff matrix in Section 3,

$$K(\mathscr{S}_\xi(\Gamma)) = K(P_\xi(\Gamma)).$$

Thus $\Lambda(\mathscr{S}_\xi(\Gamma))$ is the Schur complement of $P_\xi(K(\Gamma))$ with respect to the block corresponding to $I \cup \{v_1\}$. From Theorem 3.2 and Lemma 3.4, it follows that

$$\Lambda(\mathscr{S}_\xi(\Gamma)) = S_\xi(\Lambda(\Gamma)),$$

$$\Lambda(\Gamma) = S_{-\xi}(\Lambda(\mathscr{S}_\xi(\Gamma)).$$

Suppose given (Γ, γ) and the positive real number ξ. Then $\Lambda(\Gamma)$ uniquely determines $\Lambda(\mathscr{S}_\xi(\Gamma))$. Also $\Lambda(\mathscr{S}_\xi(\Gamma))$ uniquely determines $\Lambda(\Gamma)$.

Remark 8.1. We have adjoined the boundary edge at v_1v_2 for convenience of notation. The construction $\mathscr{T}_\xi(\Gamma)$ may be made at any pair of boundary nodes p and q which are adjacent in the circular order. The construction $T_\xi(M)$ may be made at any pair of indices of which are adjacent in the circular order. Similarly the constructions $\mathscr{P}_\xi(\Gamma)$ or $\mathscr{S}_\xi(\Gamma)$ may be made at any boundary node, and $P_\xi(M)$ or $S_\xi(M)$ may be made at any index. In each case, the location of the nodes (or indices) where the construction is to be made will be clear from the context.

9. Recovering conductivities

Lemma 9.1. *Suppose Γ is a circular planar graph with n boundary nodes for which the medial graph $\mathscr{M}(\Gamma)$ is lensless. Assume that the z-sequence for the medial graph $\mathscr{M}(\Gamma)$ is not the sequence $1, 2, \ldots, n, 1, 2, \ldots, n$. Then either*

(1) *there is a boundary node where a boundary spike may be adjoined to Γ, so that after the adjunction, the resulting graph Γ' is lensless,*

or (2) *there is a pair of consecutive boundary nodes where a boundary edge may be adjoined,* so that after the adjunction, the resulting graph Γ' *is lensless.*

Proof. Let t be a number in the sequence such that two repetitions of t are closest in the circular order around C. By a cyclic relabelling, we may assume that $t = 1$, so that the z-sequence for $\mathcal{M}(\Gamma)$ is

$$z = 1, 2, \ldots, m, 1, z_{m+2}, \ldots, z_{2n}$$

with $m < n$. Let h be the first index for which z_h is not in the set $\{1, 2, \ldots, m\}$. Then z_{h-1} and z_h are a pair of numbers which do not interlace in z (see Section 7). The corresponding geodesics in $\mathcal{M}(\Gamma)$ do not cross. We now make the single alteration in $\mathcal{M}(\Gamma)$ so that these two geodesics do cross, and the new z-sequence is

$$1, 2 \ldots, m, 1, z_{m+2}, \ldots, z_h, z_{h-1}, \ldots, z_{2n}.$$

The new medial graph is lensless. This change in the medial graph corresponds to adjoining either a boundary edge or a boundary spike to Γ. □

Lemma 9.2. *Suppose Γ is a circular planar graph with n boundary nodes for which the medial graph $\mathcal{M}(\Gamma)$ is lensless. There is a sequence of circular planar graphs $\Gamma = \Gamma_0, \Gamma_1, \Gamma_2, \ldots, \Gamma_k$, where each Γ_{i+1} is obtained from Γ_i by adjoining a boundary edge or a boundary spike, and where Γ_k is Y–Δ equivalent to the standard graph Σ_n.*

Proof. We adjoin boundary edges or boundary spikes until the z-sequence for the medial graph $\mathcal{M}(\Gamma_k)$ is $1, 2, \ldots, n, 1, 2, \ldots, n$. By Corollary 7.4, Γ_k is Y–Δ equivalent to Σ_n. □

Proof of Theorem 2. By taking connected components, we need only consider the case when Γ is connected. First let (Γ, γ) be a resistor network whose underlying graph is the graph $C(m, 4m + 3)$ of Ref. [7]. In Theorem 5.2 of Ref. [7] we showed that for this graph, the conductivity γ may be recovered from Λ_γ. By Corollary 5.4, any resistor network whose underlying graph is Y–Δ equivalent to $C(m, 4m + 3)$ is also recoverable. In particular, any conductivity on Σ_{4m+3} is recoverable.

Next suppose (Γ, γ) is any connected critical circular planar resistor network with n boundary nodes. If n is not of the form $4m + 3$, first adjoin 1, 2, or 3 boundary spikes without interiorizing as in Section 8, to obtain a resistor network which does have 4m + 3 boundary nodes. Combining this with Lemma 9.2, we obtain a sequence of circular planar resistor networks $\Gamma = \Gamma_0$, $\Gamma_1, \Gamma_2, \ldots, \Gamma_k$, where Γ_k *is a graph with* $4m + 3$ *boundary nodes, which is* Y–Δ

equivalent to Σ_{4m+3}. Each Γ_{i+1} is obtained from Γ_i by adjoining a boundary edge, or adjoining a boundary spike (with or without interiorizing). The resistor network Γ_k is recoverable, and hence each of the resistor networks Γ_i for $k \geqslant i \geqslant 0$ is also recoverable. In particular, the resistor network $\Gamma = \Gamma_0$ is recoverable. □

10. Totally non-negative matrices

We continue the notations of Sections 1 and 2. Specifically, let $A = \{a_{i,j}\}$ be a matrix. If $P = (p_1, \ldots, p_k)$ is an ordered subset of the rows, and $Q = (q_1, \ldots, q_m)$ is an ordered subset of the columns, then $A(P;Q)$ is the $k \times m$ submatrix of A with

$$A(P;Q)_{i,j} = a_{p_i,q_j}.$$

$A[P;Q]$ is the matrix obtained by *deleting* the rows for which the index is in P, and *deleting* the columns for which the index is in Q. The empty set is ϕ. Thus $A[\phi;1]$ refers to the matrix A with the first column deleted.

Following Ref. [9], a rectangular matrix A is called *totally non-negative* (TNN) if every square minor has determinant $\geqslant 0$. The following facts about TNN matrices will be needed in Sections 11 and 12.

Lemma 10.1. *Suppose $A = \{a_{i,j}\}$ is an $m \times m$ matrix which is TNN and non-singular. Then any principal minor is non-singular.*

Proof. *Induction on m.* For $m = 1$, there is nothing to prove. Let $m > 1$. The entry $a_{1,1}$ must be > 0, else either the first row or the first column of A would be entirely 0, contradicting the assumption that A is non-singular. By the determinantal formula for Schur complements, the Schur complement $A/[a_{1,1}]$ is non-singular and TNN. Similarly $a_{m,m} > 0$, $A/[a_{m,m}]$ is non-singular and TNN. By the inductive assumption, every principal minor of $A/[a_{1,1}]$ is non-singular. Let $A(P;P)$ be a principal minor of A, where $P = (p_1, \ldots, p_k)$ is an ordered subset of the index set $(1, 2, \ldots, m)$. If $1 \in P, A(P;P)/[a_{1,1}]$ is a principal minor of $A/[a_{1,1}]$ and hence is non-singular. Thus det $A(P;P) \neq 0$, so $A(P;P)$ is non-singular. Similarly if $m \in P$, $A(P;P)$ is non-singular. Otherwise, P contains neither 1 nor m, and $k \leqslant m-2$. Let $Q = (1, p_1, \ldots, p_m)$. The $k+1 \times k+1$ matrix $A(Q;Q)$ is TNN and non-singular. $A(P;P)$ is a principal minor of $A(Q;Q)$, so is non-singular by induction. □

Lemma 10.2. *Suppose that $A = \{a_{i,j}\}$ is an $m \times m$ matrix, and suppose that $a_{s,1} < 0$ for some index s with $1 \leqslant s \leqslant m$. Assume also that*

(i) $A[\phi;1]$ *is TNN.*

(ii) $A(s+1, \ldots, m; 1, \ldots, m)$ *is TNN.*

(iii) $A(1,\ldots,s-1;2,\ldots,m,1)$ *is TNN.*
Then
(1) $(-1)^s \det A \geqslant 0$.
(2) *If it is further assumed that* $\det A[s;1] > 0$, *then* $(-1)^s \det A > 0$.

Proof. *Induction on m.* The assertion of (1) for $m=2$ is immediate. For $m>2$, first consider the case $s=1$, with $a_{1,1}<0$. If all the cofactors of the entries in the first column are 0, then $\det A = 0$. If the only non-zero cofactor of an entry in the first column is $A[1;1]$, then

$$\det A = a_{1,1}\cdot \det A[1;1] < 0.$$

Otherwise, suppose $\det A[t;1] > 0$ with $t>1$. $A[1,t;1,2]$ is a principal minor of $A[t;1]$ which is assumed to be TNN, so $\det A[1,t;1,2]>0$ by Lemma 10.1. Dodgson's identity (Lemma 2.1) gives

$$\det A \cdot \det A[1,t;1,2] = \det A[1;1]\cdot \det A[t;2] - \det A[1;2]\cdot \det A[t;1]. \quad (1)$$

$\det A[1;2]$ and $\det A[t;1]$ are non-negative by assumption (ii). By the inductive assumption $\det A[t;2]\leqslant 0$. Hence $\det A \leqslant 0$.

The case $s=m$ is similar, by considering the matrix $A(1,\ldots,m;2,\ldots,m,1)$. The only negative entry is in the last column. Assumption (iii) is used in place of (ii).

This leaves the case when $1<s<m$. If the only non-zero cofactor of an entry in the first column in $A[s;1]$, then

$$\det A = (-1)^{s+1}\cdot a_{s,1}\cdot \det A[s;1].$$

If another cofactor is non-zero, w.l.o.g., assume $\det A[t;1] > 0$ with $1<s<t\leqslant m$. Then $A[1,t;1,2]$ is a principal minor of $A[t;1]$, so $\det A[1,t;1,2] > 0$ by Lemma 10.1. Dodgson's identity (Lemma 2.1) gives

$$\det A\cdot \det A[1,t;1,2] = \det A[1;1]\cdot \det A[t;2] - \det A[1;2]\cdot \det A[t;1].$$

The factors $\det A[1;1]$ and $\det A[t;1]$ are non-negative. By the inductive assumption, $(-1)^s \det A[t;2]\geqslant 0$ and $(-1)^{s-1}\det A[1;2]\geqslant 0$. In every case, $(-1)^s \det A \geqslant 0$.

The proof of (2) is also by induction on m. For $m=2$, the assertion is immediate. Let $m>2$. If the only non-zero cofactor of an entry in the first column is $A[s;1]$, then

$$(-1)^s \det A = -a_{s,1}\cdot \det A[s;1] > 0.$$

If more than one cofactor is non-zero, w.l.o.g., assume $\det A[s;1] > 0$ and $\det A[t;1] > 0$ with $1<s<t\leqslant m$. Then $\det A[1,s;1,2] > 0$ and $\det A[1,t;1,2] > 0$ by Lemma 10.1. By the inductive assumption, $(-1)^{s-1} \det A[1;2] > 0$, and Eq. (1) shows that $(-1)^s \det A > 0$. $\square$

Lemma 10.3. *Suppose A is a $k+1 \times k$ matrix which is TNN. Suppose that for some pair of integers s and t with $1 \leqslant s < t \leqslant k+1$,*
(i) $\det A[s;\phi] = 0$,
(ii) $\det A[t;\phi] \neq 0$.
Then the rank of $A(s+1,\dots,k+1;1,\dots,k)$ is $\leqslant k-s$.

Proof. For each $i = 1,\dots,k+1$, let R_i be the ith row of A, considered as a vector in $\mathbf{R}^k$. Assumption (ii) implies that $\{R_1,\dots,\hat{R}_t,\dots,R_{k+1}\}$ form a basis for $\mathbf{R}^k$. Hence,

$$R_t = \sum_{i \neq t} x_i R_i.$$

In this sum, $x_s = 0$, else $\{R_1,\dots,\hat{R}_s,\dots,R_{k+1}\}$ would also be a basis for $\mathbf{R}^k$, contradicting assumption (i). Then

$$\det A[1;\phi] = (-1)^t \cdot x_1 \cdot \det A[t;\phi] \geqslant 0.$$

Hence $(-1)^t x_1 \geqslant 0$, because $\det A[t;\phi] > 0$ $A[s,t;s]$ is a principal minor of $A[t;\phi]$, so $\det A[s,t;s] > 0$ by Lemma 10.1. Then

$$\det A[1,s;s] = (-1)^{t-1} \cdot x_1 \cdot \det A[s,t;s] \geqslant 0.$$

Hence $(-1)^{t-1} x_1 \geqslant 0$, Thus $x_1 = 0$. Similarly, $x_2 = 0,\dots,x_{s-1} = 0$. Thus

$$R_t = \sum_{\substack{i>s \\ i \neq t}} x_i R_i.$$

This implies rank $A(s+1,\dots,k+1;1,\dots,k) \leqslant k-s$. □

Notation. Let $P = (p_1, p_2,\dots,p_k)$ be a sequence of distinct indices. If $p \in P$, then $P-p$ denotes the sequence obtained by deleting the index p from P. If $p \notin P$, then $p+P$ denotes the sequence $(p_1, p_2,\dots,p_k)$. Also $\mu(P;Q)$ stands for $\det M(P;Q)$, and $\mu'(P;Q)$ stands for $\det M'(P;Q)$.

Recall the definition of the set Ω_n from Section 1. With our conventions, this means that if $M \in \Omega_n$ and $(P;Q)$ is a circular pair of indices, then the matrix $-M(P;Q)$ is TNN.

Lemma 10.4. *Let $M \in \Omega_n$ and suppose that $m_{h,h}$ is a non-zero diagonal entry. Then the Schur complement $M' = M/[m_{h,h}]$ is in Ω_{n-1}.*

Proof. If $(1,\dots,n)$ is the indexing set for M, it is convenient to regard the deleted set $(1,\dots,\hat{h},\dots,n)$ as the indexing set for M'. Let $(P;Q) = (p_1,\dots,p_k;q_1,\dots,q_k)$ be a circular pair of indices for M'. Then $h \notin P \cup Q$. By interchanging P and Q if necessary, and by a cyclic re-labelling of the indices, we may assume that $1 \leqslant h < q_k$ in the circular order. Let $B = (b_1, \ldots, b_{k+1})$, be the set $P \cup h$ with the circular ordering, where $b_s = h$ with $1 \leqslant s \leqslant k+1$. Thus $1 \leqslant b_1 < \cdots < b_{k+1} < q_k < \cdots < q_1 \leqslant n$. The matrix

$$A = -M(B; b_s + Q)$$

satisfies the conditions of Lemma 10.2. Hence $(-1)^s \det A \geqslant 0$, so

$$(-1)^{s+1+k} \mu(B; b_s + Q) \geqslant 0.$$

Taking the Schur complement with respect to the entry $m_{h,h}$, which is in the $(s,1)$ position of $M(B; b_s + Q)$, we find that $(-1)^K \mu'(P; Q) \geqslant 0$. □

Remark 10.5. If $(-1)^k \mu(P; Q) > 0$, then part (2) of Lemma 10.2 shows that $(-1)^{s+1+k} \mu(B; b_s + Q) > 0$. Therefore $(-1)^k \mu'(P; Q) > 0$.

Lemma 10.6. *Suppose $M \in \Omega_n$. Let $B = (b_1, \ldots, b_{k+1})$, and $Q = (q_1, \ldots, q_k)$ be two sequences of indices, with $1 \leqslant b_1 < \cdots < b_k < b_{k+1} < q_k < \cdots q_1 \leqslant n$. Suppose for some pair of indices (s,t) with $1 \leqslant s < t \leqslant k+1$, that $\mu(B - b_s; Q) = 0$ and $\mu(B - b_t; Q) \neq 0$. Let $B_0 = (b_{s+1}, \ldots, b_{k+1})$, and let $Q_0 = (q_{s+1}, \ldots, q_k)$. Then $\mu(B_0 - b_t; Q_0) \neq 0$, and*

$$\mu(B_i b_s + Q) = (-1)^s \frac{\mu(B - b_t; Q) \cdot \mu(B_0; b_s + Q_0)}{\mu(B_0 - b_t; Q_0)}$$

Proof. For $0 \leqslant r \leqslant s$, let

$$\begin{aligned} B_r &= (b_1, \ldots, b_r, b_{s+1}, \ldots, b_{k+1}) \\ Q_r &= (q_1, \ldots, q_r, q_{s+1}, \ldots, q_k) \end{aligned}.$$

Then $\mu(B_r - b_t; Q_r) \neq 0$ because $M(B_r - b_t; Q_r)$ is a principal minor of $M(B - b_t; Q)$. Dodgson's identity (Lemma 2.1) gives

$$\begin{aligned} \mu(B_{r+1}; b_s + Q_{r+1}) \cdot \mu(B - b_t; Q_r) &= \mu(B_r; Q_{r+1}) \cdot \mu(B - b_t; b_s + Q_r) \\ &\quad - \mu(B_{r+1} - b_t; Q_r) \cdot \mu(B_r; b_s + Q_r). \end{aligned}$$

$\mu(B_r; Q_{r+1}) = 0$ by Lemma 10.3, so the first term on the RHS is 0, and

$$\frac{\mu(B_{r+1}; b_s + Q_{r+1})}{\mu(B_{r+1} - b_t; Q_{r+1})} = -\frac{\mu(B_r; b_s + Q_r)}{\mu(B_r - b_t; Q_r)}.$$

Repeated use of this identity gives the result. □

Lemma 10.7. *Suppose $M \in \Omega_n$, p and q are adjacent indices, and $\xi > 0$. Let $T_\xi(M)$ be the matrix constructed in Section 8 (see also Remark 8.1). Then $T_\xi(M) \in \Omega_n$.*

Proof. The circular determinants in $M' = T_\xi(M)$ are equal to the circular determinants in M except for the ones which correspond to circular pairs $(P;Q) = (p_1,\dots,p_k;q_1,\dots,q_k)$ where $p = p_k$ and $q = q_k$, or $p = p_1$ and $q = q_1$. Each of these determinants has the form

$$\begin{aligned}\mu'(P;Q) &= \det\begin{bmatrix} C & a \\ b & d-\xi\end{bmatrix} = \det\begin{bmatrix} C & a \\ b & d\end{bmatrix} - \xi\det(C) \\ &= \mu'(P;Q) - \xi\mu(P-p;Q-q).\end{aligned}$$

Hence

$$(-1)^k\mu'(P;Q) = (-1)^k\mu(P;Q) - \xi(-1)^{k-1}\mu(P-p;Q-q) \geqslant 0. \tag{2}$$

Remark 10.8. If either $(-1)^k\mu(P;Q) > 0$ or $(-1)^{k-1}\mu(P-p;Q-q) > 0$, then $(-1)^k\mu'(P;Q) > 0$; otherwise $\mu'(P;Q) = 0$. Thus the signs of the circular determinants in M' are determined by the signs of the circular determinants in M.

Lemma 10.9. *Suppose $M \in \Omega_n$, and $\xi > 0$. Let $P_\xi(M)$ be the matrix constructed in Section 8 . Then $P_\xi(M) \in \Omega_{n+1}$.*

Proof. Let $M' = P_\xi(M)$, and let $(P;Q) = (p_1,\dots,p_k;q_1,\dots,q_k)$ be a circular pair of indices from the set $(0,1,\dots,n)$.
1. If $0 \notin P\cup Q$, then $\mu'(P;Q) = \mu(P;Q)$.
2. If $0 \in P$ and $1 \notin Q$, then $\mu'(P;Q) = 0$.
3. If $0 \in P$ and $1 \in Q$, then $0 = p_k$, $1 = q_k$, and $\mu'(P;\ Q) = -\xi\mu(P-p_k;\ Q-q_k)$.
4. The situation is similar if $0 \in Q$. □

Lemma 10.10. *Suppose $M \in \Omega_n$, and $\xi > 0$. Let $S_\xi(M)$ be the matrix constructed in Section 8. Then $S_\xi(M) \in \Omega_n$.*

Proof. Let $(P;\ Q) = (p_1,\dots,p_k;\ q_1,\dots,q_k)$ be a circular pair of indices. Let p be the index where the adjunction is made (see Remark 8.1). By interchanging P and Q if necessary, and by a circular re-labelling of the indices, we may assume that $1 \leqslant p < q_k$ in the circular order. Let $M' = S_\xi(M)$.
1. If $p \in P$, then the formula for $S_\xi(M)$, shows that

$$\mu'(P;\ Q) = \left(\frac{\xi}{\xi + m_{p,p}}\right)\mu(P;\ Q).$$

2. Suppose that $p \notin P$ and $(-1)^k\mu(P;\ Q) > 0$. Then $(-1)^k S_\xi(M)(P;\ Q) > 0$, by Remark 10.5.
3. Suppose that $p \notin P$, $\mu(P;\ Q) = 0$, and $\mu(P - p_j + p;\ Q) = 0$, for all $1 \leqslant j \leqslant k$. Then the proof of Lemma 10.2 shows that $\mu'(P;\ Q) = 0$.

4. Finally, suppose that $p \notin P$, $\mu(P;\ Q) = 0$, and that $\mu(P - p_j + p;\ Q) \neq 0$ for some j with $1 \leqslant j \leqslant k$. Let $B = (b_1, \dots, b_{k+1})$ be the set $P \cup p$ with the circular ordering. That is, $p = b_s$ for some s, and $p_j = b_t$ for some t, and w.l.o.g., may assume $s < t$. $P_\xi(M)(B, b_s + Q)$ and $M(B, b_s + Q)$ differ only at the $(s,1)$ position, and the cofactor of that entry is $\mu(P;\ Q)$, assumed to be 0. Therefore,

$$\det P_\xi(M)(B;\ b_s + Q) = \mu(B;\ b_s + Q).$$

Recall that $S_\xi(M)$ is the Schur complement of $P_\xi(M)$ with respect to the entry $m_{p,p} + \xi$, which is in the $(s,1)$ position of $P_\xi(M)(B;\ b_s + Q)$. Then

$$\begin{aligned}(-1)^{s+1}(m_{p,p} + \xi) \cdot \mu'(P{:}\,Q) &= \det P_\xi(M)(B; b_s + Q) \\ &= \mu(B;\ b_s + Q) \\ &= (-1)^s \frac{\mu(B - b_t;\ Q) \cdot \mu(B_0;\ b_s + Q_0)}{\mu(B_0 - b_t;\ Q_0)}.\end{aligned}$$

The last equality uses Lemma 10.6. Thus $(-1)^k M'(P;\ Q) \geqslant 0$ and if $\mu(B_0;\ b_s + Q_0) \neq 0$, then $(-1)^k M'(P;\ Q) > 0$. □

Remark 10.11. Parts (1) and (2) show that if $(-1)^k \mu(P;\ Q) > 0$, then $(-1)^k \mu'(P;\ Q) > 0$. Together with parts (3) and (4), this shows that the signs of the circular determinants in M' are determined by the signs of the circular determinants in M.

Lemma 10.12. *Let Γ be a circular planar graph with n boundary nodes.*
1. *Suppose a boundary edge pq is adjoined to Γ, as in Section 8. Let $\Gamma' = \mathscr{T}_\xi(\Gamma)$ and $\pi' = \pi(\Gamma')$. If $M \in \Omega(\pi)$, then $T_\xi(M) \in \Omega(\pi')$.*
2. *Suppose a boundary spike rp is adjoined to Γ at node p, without interiorizing as in Section 8. Let $\Gamma' = \mathscr{P}_\xi(\Gamma)$ and $\pi' = \pi(\Gamma')$. If $M \in \Omega(\pi)$, then $P_\xi(M) \in \Omega(\pi')$.*
3. *Suppose p is a boundary node of Γ, and a boundary spike rp is adjoined with p then declared interior, as in Section 8. Let $\Gamma' = \mathscr{P}_\xi(\Gamma)$ and $\pi' = \pi(\Gamma')$. If $M \in \Omega(\pi)$, then $S_\xi(M) \in \Gamma(\pi')$.*

Proof. The three processes are similar, so for definiteness, suppose that the operation is $\mathscr{S}_\xi$. Let γ be an arbitrary conductivity on Γ. By Section 8, statement (1) is true if $M = \Lambda(\Gamma, \gamma)$. Next, suppose M is any matrix in $\Omega(\pi)$, and let $M' = S_\xi(M)$. By Remark 10.11, the signs of the circular determinants in M' are determined by the signs of the circular determinants in M. Hence they have the same signs as the circular determinants in $S_\xi(\Lambda(\Gamma,\gamma))$. Since $S_\xi(\Lambda(\Gamma,\gamma)) \in \Omega(\pi')$ we have $M' \in \Omega(\pi')$ also. □

11. Removing edges

Suppose that Γ is a circular planar graph with n boundary nodes. Recall from Section 1, that there are two ways to remove an edge from Γ called deletion and contraction. In either case the new graph will be a circular planar graph with n boundary nodes.

Lemma 11.1. *Suppose Γ is a critical circular planar graph and pq is a boundary edge. Let Γ_1 be the graph obtained after deletion of pq. Then Γ_1 is also critical.*

Proof. Let $e \neq pq$ be an edge in Γ. Since Γ is critical, removal of e will break some connection in Γ. If this connection also exists in Γ_1, then removal of e from Γ_1 breaks this connection in Γ_1. Suppose that removal of e from Γ breaks a connection $(P; Q)$ that is not present in Γ_1. This connection must use the edge pq, so $(P; Q)$ has the form $(P; Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$, where $p_k = p$ and $q_k = q$. Thus removal of e breaks the connection of $(P'; Q') = (p_1, \dots, p_{k-1}; q_1, \dots, q_{k-1})$ in Γ_1. □

Lemma 11.2. *Suppose Γ is a critical circular planar graph with a boundary spike rp where r is a boundary node of Γ. Let Γ_1 be the graph obtained after contracting rp to p. Then Γ_1 is also critical.*

Proof. Let e be an edge in Γ with $e \neq pr$. Let Γ' be the graph with e removed, either by deletion or contraction. Similarly, let Γ'_1 be the graph Γ_1 with e removed. Let γ be a conductivity on Γ, and by restriction γ gives a conductivity on Γ_1, Γ' and Γ'_1. Let $(P; Q)$ be a pair of sequences of boundary nodes. Then $\lambda(P; Q)$, $\lambda'(P; Q)$, $\lambda_1(P; Q)$ and $\lambda'_1(P; Q)$ will denote the subdeterminants of $\Lambda(\Gamma)$, $\Lambda(\Gamma')$, $\Lambda(\Gamma 1)$ and $\Lambda(\Gamma'_1)$, respectively.

Suppose that removal of e breaks a connection in Γ that persists in Γ_1. Then removal of e from Γ_1 breaks the same connection in Γ_1.

Suppose removal of e from Γ breaks a connection $(P; Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$ in Γ which does not persist in Γ_1. Then $r \notin P \cup Q$. w.l.o.g., assume that $q_1 < p < q_k$ in the circular order around Γ_1. Let $B = (b_1, \dots, b_{k+1})$ be the set $P \cup p$ with the circular ordering around the boundary of Γ_1, and suppose $p = b_s$. The assumptions that $\lambda(P; Q) \neq 0$ and $\lambda_1(P; Q) = 0$ imply that each connection from Q to P through Γ must use $p = b_s$. Such a connection either connects q_{s-1} to b_{s-1} through b_s or connects q_s to b_{s+1} through b_s. w.l.o.g., assume the latter. Let $B_0 = (b_{s+1}, \dots, b_{k+1})$, and $Q_0 = (q_{s+1}, \dots, q_k)$. Hence $\lambda_1(B - b_{s+1}; Q) \neq 0$ and $\lambda_1(B_0; b_s + Q_0) \neq 0$. Both $(B - b_{s+1}; Q)$ and $(B_0; b_s + Q_0)$ are circular pairs. Suppose removal of e from Γ_1 does not break either connection. Then $\lambda'_1(B - b_{s+1}; Q) \neq 0$ and $\lambda'_1(B_0; b_s + Q_0) \neq 0$. We have assumed $\lambda_1(P; Q) = 0$; that is $\lambda_1(B - b_s; Q) = 0$. Hence $\lambda'_1(B - b_s; Q) = 0$. By Lemma 10.6, with $t = s + 1$,

$$\lambda_1'(p+P;\ p+Q) = (-1)^{s-1}\lambda_1'(B; b_s + Q)$$
$$= -\frac{\lambda_1'(B - b_{s+1}; Q)\lambda_1'(B_0; b_s + Q_0)}{\lambda_1'(B_0 - b_{s+1}; Q_0)} \neq 0.$$

Let $\xi = \gamma(pr)$. Then A' is the Schur complement of $P_\xi(A_1')$ with respect to the entry $A_1'(p,p) + \xi$. Part (4) of the proof of Lemma 10.10 shows that $\lambda'(P; Q) \neq 0$. This would contradict the assumption that removal of e from Γ breaks the connection $(P; Q)$. □

Lemma 11.3. *Suppose Γ is a non-trivial circular planar graph for which $\mathscr{M}(\Gamma)$ is lensless. Then Γ has either a boundary edge or a boundary spike.*

Proof. Refer to Section 7 for the notation. Let t be a number in the z-sequence for $\mathscr{M}(\Gamma)$ such that there are no repetitions of any other number between two occurrences of t. w.l.o.g., assume that $t = 1$, so that a portion of the z-sequence is

$$1, 2, \ldots, m, 1, z_{m+2}, \ldots$$

Let k be the portion of the outer circle C and Γ which lies between x_1 and y_1. Then h contains the points $x_2, \ldots, x_m$. Consider h, g_1 and the family $\{g_2, \ldots, g_m\}$. The proof of Lemma 6.2 shows that there is a triangle T formed by h and two of the geodesics from the set $\{g_1, \ldots, g_m\}$. The triangle T in $\mathscr{M}(\Gamma)$ corresponds in Γ either to a boundary spike (if there is a vertex of Γ inside T), or to a boundary to boundary edge (if there is no vertex of Γ inside T). □

Lemmas 11.3, 11.1 and 11.2, together with Corollaries 4.3 and 4.4 show that there is an algorithm for calculating the conductivity of any critical circular planar graph.

12. Surjectivity

Theorem 12.1. *Suppose Γ is a critical circular planar graph with n boundary nodes and $\pi = \pi(\Gamma)$. Let M be any matrix in $\Omega(\pi)$. Then there is a conductivity γ on Γ with $\Lambda(\Gamma, \gamma) = M$.*

Proof of Theorem 12.1. We first consider the case where $n = 4m + 3$ and the z-sequence for the medial graph $\mathscr{M}(\Gamma)$ is $1, \ldots, n, 1, \ldots, n$. Corollary 7.4 shows that Γ in Y–Δ equivalent to the graph $C(m, n)$ of Ref. [7]. By Theorem 6.2 of Ref. [7] there is a conductivity γ' on $C(m, n)$ with $\Lambda(C(m, n), \gamma') = M$. By Lemma 5.3, there is a conductivity γ on Γ with $\Lambda(\Gamma, \gamma) = M$.

Next suppose (Γ, γ) is any connected critical circular planar resistor network with n boundary nodes. If n is not of the form $4m + 3$, first adjoin 1, 2, or 3 boundary spikes without interiorising as in Section 8, to obtain a resistor network which does have $4m + 3$ boundary nodes. Combining this with Lemma 9.2, we obtain a sequence of circular planar resistor networks $\Gamma = \Gamma_0$, $\Gamma_1, \Gamma_2, \dots, \Gamma_k$, where Γ_k is a graph with $4m + 3$ boundary nodes, and which is Y–Δ equivalent to Σ_{4m+3}. Each Γ_{i+1} is obtained from Γ_i by one of the operation $\mathcal{T}$, $\mathcal{P}$ or $\mathcal{S}$. For each $i = 0, 1, \dots, k$, let $\pi_i = \pi(\Gamma_i)$. Given a matrix M in $\Omega(\pi)$, there is an analogous sequence of matrices $M = M_0, M_1, \dots, M_k$, where each matrix M_{i+1} is obtained from M_i by one of the operation $M_{i+1} = T_\xi(M_i)$, $M_{i+1} = P_\xi(M_i)$ or $M_{i+1} = S_\xi(M_i)$.

Let σ_n denote the set of connections in a well-connected circular planar graph with n boundary nodes. By Lemma 5.1 and Proposition 7.3, $\pi(\Sigma_n) = \sigma_n$. By Lemma 10.12, $M_k \in \Omega(\sigma_n)$. Using the first part of the proof, there is a conductivity γ_k on Γ_k so that $\Lambda(\Gamma_k, \gamma_k) = M_k$. The graph Γ_k is obtained from Γ_{k-1} by one of the operations $\mathcal{T}$, $\mathcal{P}$ or $\mathcal{S}$. The processes are similar, so for definiteness, suppose that the operation is S_ξ and $M_k = S_\xi(M_{k-1})$.

In going from Γ_k to Γ_{k-1}, removal of the spike breaks a connection in Γ_k. By Lemma 4.4, the value of this spike can be calculated as the ratio of two non-zero subdeterminants of $\Lambda(\Gamma_k) = M_k$. Moreover, the computed value is the same as the value ξ that was used to construct M_k from M_{k-1}. By Section 11, removal of the spike with conductivity ξ from Γ_k results in a critical graph Γ_{k-1}, with $\Lambda(\Gamma_{k-1})$. Continuing the argument on $\Gamma_{k-1}, \dots, \Gamma_0 = \Gamma$, we find that $\Lambda(\Gamma) = M$. □

Proof of Theorem 4. As in the proof of Theorem 12.1, there is a sequence of the operations $\mathcal{T}$, $\mathcal{P}$, and $\mathcal{S}$ which, when applied to the graph Γ, give a graph Γ_k which is Y–Δ equivalent to the graph $C(m, 4m + 3)$ of Ref. [7]. Let $\mathcal{U}$ be the composite of these operations, and let U be the composite of the corresponding operations T, P and S applied to the matrix $\Lambda(\Gamma, \gamma)$. With an ordering of the N edges in Γ, the conductivity γ is represented by a point in $(R^+)^N$. Similarly, with an ordering of the N_k edges in Γ_k, the conductivity γ_k is represented by a point in $(R^+)^{N_k}$. Let $\pi = \pi(\Gamma)$ and $\pi_k = \pi(\Gamma_k)$. With these conventions, there is a communicative diagram shown in Fig. 4. By Theorem 12.1, the map Λ is surjective. By Theorems 4.2 and 5.2 of Ref. [7], the map Λ_k is a diffeomorphism. For the differentials, we have

$$\begin{tikzcd}
(R^+)^N \arrow[r, "\mathcal{U}"] \arrow[d, "\Lambda"] & (R^+)^{N_k} \arrow[d, "\Lambda_k"] \\
\Omega(\pi) \arrow[r, "U"] & \Omega(\pi_k)
\end{tikzcd}$$

Fig. 4. Commutative diagram.

$$\mathrm{d}\Lambda_k \circ \mathrm{d}\mathscr{U} = \mathrm{d}U \circ \mathrm{d}\Lambda.$$

Since $\mathrm{d}\lambda_k$ and $\mathrm{d}\mathscr{U}$ are 1-1, $\mathrm{d}\Lambda$ is 1-1. By Theorem 2, Λ is 1-1. $\mathscr{U}^{-1}$ is the inverse of $\mathscr{U}$ which is well-defined and continuous on its image in $(R^+)^{N_k}$. Then

$$\Lambda^{-1} = \mathscr{U}^{-1} \circ \Lambda_k^{-1} \circ U.$$

Thus Λ^{-1} is continuous. It follows that Λ is a diffeomorphism of $(R^+)^N$ onto $\Omega(\pi)$. □

Lemma 12.2. *Suppose $M \in \Omega_n$, with at least one circular determinant equal to* 0. *Let $\epsilon > 0$ be given. Then there is a matrix $M' \in \Omega_n$, with $\|M' - M\|_\infty < \epsilon$, and*

(1) *$\mu'(P; Q) \neq 0$ whenever $\mu(P; Q) \neq 0$*

(2) *For at least one circular pair $(P; Q), \mu(P; Q) = 0$ and $\mu'(P; Q) \neq 0$.*

Proof. As in Section 10, $\mu(P; Q)$ stands for det $M(P; Q)$ and $\mu'(P; Q)$ stands for det $M'(P; Q)$. Let $(P; Q) = (p_1, \dots, p_k; q_1, \dots, q_k)$ be a circular pair of indices for which the minor $M(P; Q)$ has determinant 0, has minimum order k, and for which $q_k - p_k$ is a minimum.

(1) If $q_k - p_k = 1$, let $M' = T_\xi(M)$, where the chosen indices are p_k and q_k. By Remark 10.8, $\mu'(P; Q) \neq 0$. Also by Remark 10.8, $\mu'(R; S) \neq 0$ whenever $(R; S)$ is a circular pair for which $\mu(R; S) \neq 0$. If ξ is sufficiently small, then $\|M' - M\|_\infty < \epsilon$.

(2) If $q_k - p_k > 1$, let $p = p_k + 1$ and $M' = S_\xi(M)$ where the chosen index is p. By Remark 10.11, $\mu'(R; S) \neq 0$ whenever $(R;S)$ is a circular pair for which $\mu(R;S) \neq 0$. Dodgson's identity (Lemma 2.1) gives

$$\mu(P + p; Q + p) \cdot \mu(P - p_k; Q - q_k) = \mu(P - p_k + p; Q - q_k + p) \cdot \mu(P; Q) - \mu(P - p_k + p; Q) \cdot \mu(P; Q - q_k + p).$$

Using the assumption $\mu(P; Q) = 0$, we have

$$\mu(P + p; Q + p) = -\frac{\mu(P - p_k + p; Q) \cdot \mu(P; Q - q_k + p)}{\mu(P - p_k; Q - q_k)}. \tag{3}$$

Each of the factors on the RHS of Eq. (3) is non-zero because of the assumption of the minimality of $(P; Q)$. Therefore $\mu'(P; Q) \neq 0$. If ξ is taken sufficiently large, then $\|M' - M\|_\infty < \epsilon$. □

Proof of Theorem 3. Recall from Section 7 the graph $\Sigma_n = (V, V_B, E)$, with n boundary nodes, and let $\sigma = \pi(\Sigma_n)$. Since Σ_n is well-connected, $\Omega(\sigma)$ is the subset of Ω_n, consisting of those M which satisfy $(-1)^k \det M(P; Q) > 0$ for each $k \times k$ circular subdeterminant of M.

Lemma 12.2 implies that Ω_n is the closure of $\Omega(\sigma)$ in the space of $n \times n$ matrices. Thus for any $M \in \Omega_n$, there is a sequence of matrices $M_i \in \Omega(\sigma)$ which converge to M. Theorem 4 shows that for each integer i, there is a conductivity

γ_i on Σ_n with $M_i = \Lambda(\Sigma_n, \gamma_i)$. By taking a subsequence if necessary, we may assume for each edge $e \in E$ that $\lim_{i\to\infty} \gamma_i(e)$ is either 0, a finite non-zero value or ∞.

Let E_0 be the subset of E for which $\lim_{i\to\infty} \gamma_i(e) = 0$.

Let E_1 be the subset of E for which $\lim_{i\to\infty} \gamma_i(e) = \gamma(e)$ is a finite non-zero value.

Let E_∞ be the subset of E for which $\lim_{i\to\infty} \gamma_i(e) = \infty$.

Let $\Gamma = (W, V_B, E_1)$ be the graph obtained from $\Sigma_n = (V, V_B, E)$ by deleting the edges of E_0 and contracting each edge of E_∞ to a point. The vertex set W for Γ is the set of equivalence classes of vertices in V, where $p \sim q$ if $pq \in E_\infty$. Note that distinct boundary nodes of V_B cannot belong to the same equivalence class, because the M_i are bounded. Thus we may consider V_B as a subject of W. Each edge $e \in E_1$ joins a pair of points of W, so the edge-set of Γ is E_1. The restrictions of γ_i and γ to E_1 give conductivities on Γ. We shall show that $M = \Lambda(\Gamma, \gamma)$.

Suppose f is a function defined on the set of boundary nodes V_B of Γ. Let

$$Q(f) = \inf \sum_{e\in E_1} \gamma(e)(\Delta w(e))^2,$$

where $\Delta w(pq) = w(p) - w(q)$, and the infimum is taken over all functions w defined on the nodes of Γ which agree with f on V_B. Thus infimum is attained when $w = u$ is the potential function on the resistor network (Γ, γ), with boundary values f. Similarly, for each integer i, let

$$Q_i(f) = \inf \sum_{e\in E_1} \gamma_i(e)(\Delta w(e))^2.$$

This infimum is attained when $w = u_i$ is the potential function on (Γ, γ_i) with boundary values f. Then $\lim_{i\to\infty} u_i = u$, because the γ_i and γ are conductivities (non-zero, and finite) on Γ, with $\lim_{i\to\infty} \gamma_i = \gamma$. Therefore $Q(f) = \lim_{i\to\infty} Q_i(f)$

For each integer i, let

$$S_i(f) = \inf \sum_{e\in E} \gamma_i(e)(\Delta w(e))^2,$$

where the infimum is taken over all functions w defined on the nodes of Σ_n which agree with f on V_B. This infimum is attained when $w = w_i$ is the potential function on the resistor network (Σ_n, γ_i), with boundary values f. The maximum principle implies that $|w_i(p)| \leqslant \max |f(p)|$. By taking a subsequence if necessary, we may assume that for each node p, $w_i(p)$ converges to a finite value $w(p)$. The assumption that the M_i converge to M guarantees that for each function f, the $S_i(f)$ are bounded. Thus for each edge $e = pq \in E_\infty$, we have $w(p) = w(q)$. Let

$$R_i(f) = \sum_{e\in E_1} \gamma_i(e)(\Delta w_i(e))^2,$$

$$R(f) = \lim_{i\to\infty} R_i(f) = \sum_{e\in E_1} \gamma(e)(\Delta w(e))^2.$$

Let $\mathscr{F}$ be the set of functions $v = \{v(p)\}$ defined for all nodes of Σ_n, which agree with f on V_B, and for which $v(p) = v(q)$ whenever $pq \in E_\infty$. Let

$$P_i(f) = \inf_{v\in\mathscr{F}} \sum_{e\in E} \gamma_i(e)(\Delta v(e))^2.$$

We have

$$P_i(f) \geqslant S_i(f) \geqslant R_i(f),$$

$$Q_i(f) + \sum_{e\in E_0} \gamma_i(e)(\Delta u_i(e))^2 \geqslant P_i(f) \geqslant Q_i(f).$$

The maximum principle implies that the $|u_i(p)|$ are bounded by max $|f(p)|$. For each edge $e \in E_0$, we have $\lim_{i\to\infty} \gamma_i(e) = 0$, so

$$Q(f) = \lim_{i\to\infty} Q_i(f) = \lim_{i\to\infty} P_i(f) \geqslant \lim_{i\to\infty} R_i(f) = R(f).$$

But $R(f) \geqslant Q(f)$, so $R(f) = Q(f)$. Thus

$$\lim_{i\to\infty} S_i(f) = Q(f) = \lim_{i\to\infty} \langle f, M_i(f)\rangle = \langle f, M(f)\rangle.$$

13. Equivalence

Lemma 13.1. *Suppose that Γ is a circular planar graph. Then Γ is critical if and only if the medial graph $\mathscr{M}(\Gamma)$ is lensless.*

Proof. Lemma 6.4, shows that if Γ is critical, then $\mathscr{M}(\Gamma)$ is lensless. Conversely, suppose $\mathscr{M}(\Gamma)$ is lensless. Let $z = z_1z_2\ldots z_{2n}$ be the z-sequence for $\mathscr{M}(\Gamma)$ as in Section 8. If $z = 1,\ldots,n,1,\ldots,n$, then Γ is Y–Δ equivalent to the graph Σ_n of Section 8, which is critical and well-connected. Suppose that z is not the sequence $1,\ldots,n,1,\ldots,n$. By Lemma 9.2, there is a sequence of graphs $\Gamma_0, \Gamma_1, \ldots \Gamma_k$, where $\Gamma_0 = \Gamma$, each Γ_{i+1} is obtained from Γ_i by adjoining a boundary edge or a boundary spike, and Γ_k is Y–Δ equivalent to the standard graph Σ_n. By Lemmas 5.2 and 7.3, Γ_k is critical. By Lemmas 11.1 and 11.2, each of the graphs $\Gamma_{k-1}, \Gamma_{k-2}, \ldots, \Gamma_0$ is critical; in particular, $\Gamma = \Gamma_0$ is critical. □

Lemma 13.2. *A circular planar graph Γ is recoverable if and only if it is critical.*

Proof. By Theorem 2, if Γ is critical, then Γ is recoverable. Suppose that Γ is not critical. By Lemma 13.1, $\mathcal{M}(\Gamma)$ has a lens. By Lemma 6.3, Γ is Y–Δ equivalent to a graph Γ' with two edges in parallel or two edges in series. Γ' cannot be recoverable, so by Lemma 5.4, Γ is not recoverable either. □

Proof of Theorem 1. Suppose that Γ_1 and Γ_2 are two critical circular planar graphs with $\pi(\Gamma_1) = \pi(\Gamma_2)$. Let conductivities be put on both Γ_1 and Γ_2. By Lemma 9.2, and Lemma 13.1, there is a sequence of critical graphs $\Gamma_1 = F_0, F_1, \ldots, F_k$, each F_{i+1} is obtained from F_i by adjoining a boundary edge or a boundary spike, and F_k is Y–Δ equivalent to Σ_n. We perform the same operations on Γ_2 to produce a sequence $\Gamma_2 = H_0, H_1, \ldots, H_k$. For each i, let $\pi_i = \pi(F_i)$. We apply the results of Sections 8 and 12 to conclude that $\Lambda(H_1) \in \Omega(\pi_1)$. Hence $\pi(H_1) = \pi(F_1)$. Continuing, we see that $\pi(H_i) = \pi(F_i)$ for $i = 1, 2, \ldots, k$. Each F_{i+1} has more connections than F_i, so each H_{i+1} has more connections than H_i. By Corollaries 4.3 and 4.4, the edge adjoined to H_i is recoverable. Working back from H_k to H_0 which is critical and hence recoverable, we find that each H_k is recoverable, and hence critical.

Suppose the z-sequence for H_k were not $1, \ldots, n, 1, \ldots, n$. Then a boundary edge or boundary spike could be adjoined to H_k to give another graph H_{k+1} with more connections than H_k. But $\pi(H_k) = \pi(F_k)$ which is the maximal set of connections for circular planar graphs with n boundary nodes, so the z-sequence for $M(H_k)$ is $1, \ldots, n, 1, \ldots, n$.

The process of going from F_k to $F_0 = \Gamma_1$ by removing edges is the same as going from H_k to $H_0 = \Gamma_2$. Each step of this process preserves equality of the z-sequences of the medial graphs $\mathcal{M}(F_i)$ and $\mathcal{M}(H_i)$. Thus $\mathcal{M}(\Gamma_1)$ and $\mathcal{M}(\Gamma_2)$ have the same z-sequence, and by Lemma 7.2 are Y–Δ equivalent. □

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