Problem sheet 2. Algebraic geometry 2019

1 Prove that the hyperbola $xy = 1$ in the affine plane is not isomorphic affine line. Are they birationally isomorphic?

2 Let $k = \bar{k}$, $f : \mathbb{A}^1_k \to \mathbb{A}^2_k$ be a mapping defined by the formula $f(t) = (t^2, t^3)$. Show that f is a bijection to its image, but the image of f is not isomorphic to $\mathbb{A}^1$ (indication: you can recall the definition of integer closure). Show that f defines a birational and finite morphism onto your form.

3 Let $X \subset \mathbb{A}^3$ be a set of points of the form (t, t^2, t^3). Show that X is an algebraic set and find the system of generators of ideal $I(X)$. Show that X is isomorphic to $\mathbb{A}^1$.

4 Decompose the algebraic set in $\mathbb{A}^3$ defined by the equations $y^2 = xz,\ z^2 = y^3$ into irreducible components and describe these components.

5 The dimension of an affine algebraic set is called the dimension of its ring of regular functions. Show that irreducible algebraic subset in $\mathbb{A}^n$ (over an algebraically closed field) then and only then it has the dimension at most $n-1$, and the equality holds when it is a set of zeros irreducible polynomial f (use the Krull theorem on principal ideals).

6 Show that an irreducible nondegenerate quadric in $\mathbb{A}^{n+1}_k$ is birationally isomorphic to $\mathbb{A}^n_k$ if and only if it has a k -point.

The following exercises assume that the base field algebraically closed; it is recommended to check what is happening over arbitrary field.

7 Prove that an irreducible affine variety cannot be isomorphic to projective if this is not the point. Deduce from here that any subvariety of positive dimension in $\mathbb{P}^n$ intersects any hyperplane.

8 Let $F_0, \dots, F_N$ be the set of all monomials of degree d of variables $x_0, \dots, x_n$. Define Veronese mapping by $v_d : \mathbb{P}^n \to \mathbb{P}^N$ by the formula $v_d(a_0 : \dots : a_n) = (F_0(a) : \dots : F_N(a))$, where $a = (a_0, \dots, a_n)$. Prove that the image of v_d is projective variety and v_d defines an isomorphism on its image.

9 Show that $\mathbb{P}^n \times \mathbb{P}^m$ is a projective variety (construct a closed embedding in $\mathbb{P}^{nm+n+m}$, it is called a Segre embedding). Is it true that the image of this map is not contained in any hyperplane?

10 Using the Veronese mapping, prove that any two curves in $\mathbb{P}^2$ intersect.

11 Show that a straight line with a double point (gluing two copies of affine straight lines over the subsets of $\mathbb{A}^1 \setminus 0$ with the help of the identity morphism) is not a separable scheme. (Describe the diagonal using the universal property).

12 Let $f : X \to Y$ be a morphism of algebraic schemes, with Y separable. Define a graph of morphism using the universal property. Show that it is obtained by the base change from the diagonal morphism. Show that it is closed and describe it with equations in the case of affine manifolds $f : SpecA \to SpecB$ (start by describing his ideal in $A \otimes B$ as the kernel of the morphism $a \otimes b \to af^*(b)$, then describe the equations as in the case of a diagonal).

13 Show that in a separable scheme, the intersection of two open affine subsets is also affine (realize their intersection as the intersection with a diagonal).

14 Show that the image of a complete variety is also a complete variety.