ALGEBRAIC GEOMETRY 1. NOTIONS OF COMMUTATIVE ALGEBRA

1 Prove that A is a field if and only if there are no ideals in A, except (0) and (1).

2 Let the ideal I be simple (respectively, maximal). What can you say about the properties of quotient ring A/I? Prove that the maximal ideal is simple.

3 Prove the existence of maximal ideals, as well as the fact that any the ideal is contained in some maximal ideal (instruction: use the Zorn lemma).

4 Prove that in the domain of principal ideals the following conditions are equivalent: (1) I is simple; (2) I is maximal; (3) I is generated by an irreducible element.

5 Prove that the ring of polynomials in one variable $A[X]$ - is domain of principal ideals if and only if A is a field.

6 Describe simple ideals in $\mathbf{Z}[X]$ (it is convenient to first look at intersection of such an ideal with $\mathbf{Z}$).

7 The ideals I and J are said to be coprime if $I + J = A$. Prove that if $I_1, \dots, I_k$ are pairwise coprime, then $I_1 I_2 \dots I_k = \cap_{1 \leq l \leq k} I_l$.

8 The Chinese remainder theorem: if $I_1, \dots, I_k$ are pairwise coprime, then the mapping $A \to A/I_1 \times \dots \times A/I_k$ is surjective.

9 Prove that the prime ideal of a finite algebra over some field is maximal.

10 For a short exact sequence of modules

$$0 \to M_1 \xrightarrow{i} M \xrightarrow{\pi} M_2 \to 0,$$

The following conditions are equivalent:

(i) $M \cong M_1 \oplus M_2$

(ii) There exists a $\sigma : M_2 \to M$ homomorphism such that $\pi \circ \sigma = \mathrm{id}$.

(ii) There exists a $p : M \to M_1$ homomorphism such that $p \circ i = \mathrm{id}$.

Recall that $Spec(A)$ is the set of simple ideals in the A ring.

11 For any subset $E \subset A$ let $V(E)$ be a set of simple ideals containing E. Check that $V(E)$ satisfy axioms for closed sets in some topology (intersection and union). This topology is called the Zariski topology on $Spec(A)$. When is it Hausdorff? Show that for the ideal I we have:

$$\mathrm{rad(I)} = \cap_{\mathfrak{p} \in \mathrm{SpecA}, \mathfrak{p} \supset \mathrm{I}} \mathfrak{p},$$

Show that for two ideals I_1, I_2, $V(I_1 \cap I_2) = V(I_1 \cdot I_2)$. Is this equality true for arbitrary subsets?

12 Now let $f \in A$ and X_f be the complement of $V(f)$. For which f, is the set X_f empty? equal to $Spec(A)$?

Show that $Spec(A)$ is quasi-compact, that is, from any open cover $Spec(A)$ you can select the final subcover (the prefix " quasi " here because that $Spec(A)$ is usually not Hausdorff). Note: can be without loss of generality replace any open cover by the cover consisting of the subsets of the form X_f.

13 Show that in the ring A there are idempotents other than 0 and 1, if and only if $A \cong A_1 \times A_2$, $A_i \neq 0$ (indication: if e is an idempotent, then $1 - e$ is also idempotent). Decribe the prime ideals in $A_1 \times A_2$? What properties does the topological space $Spec(A_1 \times A_2)$ has?

14 The Jacobson radical $r(A)$ is the intersection of all maximal ideals in A. Show that $x \in r(A)$ if and only if when $1 - xy$ is invertible for any $y \in A$.

15 Let M be a square matrix $(n \times n)$ with coefficients in the ring A. Show, that there exists a matrix $\tilde{M}$ (with coefficients in A) such that $M\tilde{M} = \tilde{M}M = det(M)Id$, and that the given M mapping is A^n in a bijective way if and only if $det(M)$ is invertible in A.

16 (Nakayama's lemma) Let I be an ideal in A and M be a finetely generated A -module such that $IM = M$. Then there exists $x \in I$ such that $(1-x)M = 0$. In particular, if I is contained in the Jacobson radical, then $M = 0$. Note: Describe $(1-x)$ as the determinant of some matrix.

17 Show that in a finite algebra over a field there is only a finite the number of maximal ideals (use the Chinese theorem on residues).

18 Show that the Jacobson radical of finite algebra over a field is nilpotent as the ideal, i.e. $r(A)^n = 0$ for some n (use Nakayama's lemma).

19 Derive from the previous exercises that a finite algebra over a field is isomorphic the product of its quotines by some powers of maximal ideals. What can be said about the finite algebra over a field without nilpotents? What are the properties of the spectrum of a finite algebra over a field?