

Elliptic Functions

Applications of elliptic integrals

§2.1 Arithmetic-geometric mean

Well-known fact: geometric mean $\leq$ arithmetic mean:

$$(G < A) \qquad \sqrt{ab} \leq \frac{a+b}{2}$$

for $a, b > 0$.

Repeat the process: $a_0 := a$, $b_0 := b$ for $a \geq b > 0$,

$$a_{n+1} := \frac{a_n + b_n}{2}, \qquad b_{n+1} := \sqrt{a_n b_n}.$$

Because of the inequality (G < A),

$$b_n \leq b_{n+1} \leq a_{n+1} \leq a_n,$$

i.e.,

$$b = b_0 \leq b_1 \leq \cdots \leq b_n \leq \cdots \leq a_n \leq \cdots \leq a_1 \leq a_0 = a.$$

$\{a_n\}_{n=0,1,2,\dots}, \{b_n\}_{n=0,1,2,\dots}$: bounded monotone sequences.

$$\implies \exists \alpha := \lim_{n \rightarrow \infty} a_n, \beta := \lim_{n \rightarrow \infty} b_n.$$

In fact, $\alpha = \beta$, i.e., $c_n := \frac{1}{2}(a_n - b_n) \rightarrow 0$.

Proof:

$$0 \leq a_n - b_n \leq a_n - b_{n-1} = \frac{a_{n-1} + b_{n-1}}{2} - b_{n-1} = c_{n-1}.$$

$$\implies 0 \leq c_n \leq \frac{c_{n-1}}{2} \leq \dots \leq \frac{c_0}{2^n} \rightarrow 0 \quad (n \rightarrow \infty). \quad \square$$

$M(a, b) := \lim_{n \rightarrow \infty} a_n = \lim_{n \rightarrow \infty} b_n$: *arithmetic-geometric mean (AGM)*.

Easy to see:

- (i) (symmetry) $M(a, b) = M(b, a)$.
- (ii) (homogeneity) $M(\lambda a, \lambda b) = \lambda M(a, b)$.
- (iii) $M(a, a) = a$.
- (iv) $M(a, b) = M(a_n, b_n)$ ($n = 0, 1, 2, \dots$).
- (ii) $\implies$ enough to study $M(a, 1)$ or $M(1, b)$ for $a > 1 > b$.

Theorem: If $0 < k < 1$, then $M(1, k) = \frac{\pi}{2} \frac{1}{K(k')}$, where

- $k' := \sqrt{1 - k^2}$,
- $K(k')$: complete elliptic integral of the first kind,

$$K(k') = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}} = \int_0^1 \frac{dz}{\sqrt{(1 - z^2)(1 - k'^2 z^2)}}.$$

Proof (Newman (1985); Gauss used a different method.):

Lemma: for $a \geq b > 0$,

$$I(a, b) := \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} = \frac{\pi}{2} \frac{1}{M(a, b)}.$$

Once this lemma is proved, the theorem is its corollary:

$$M(1, k) = \frac{\pi}{2} \frac{1}{I(1, k)},$$

and

$$\begin{aligned} I(1, k) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{\cos^2 \theta + k^2 \sin^2 \theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - (1 - k^2) \sin^2 \theta}} = K(k'). \end{aligned}$$

□

Proof of the lemma:

Key:

$$I(a, b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

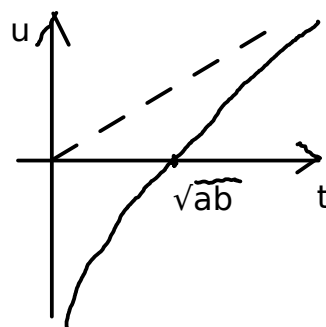
In fact,

$$\begin{aligned} I(a, b) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{a^2 \cos^2 \theta + b^2 \sin^2 \theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{\cos \theta \sqrt{a^2 + b^2 \tan^2 \theta}} \\ &= \int_0^\infty \frac{dt}{\sqrt{(a^2 + t^2)(b^2 + t^2)}}, \end{aligned} \quad \left(\begin{array}{l} t = b \tan \theta, \quad \frac{dt}{d\theta} = \frac{b}{\cos^2 \theta}, \\ b^2 + t^2 = \frac{b^2}{\cos^2 \theta} \end{array} \right).$$

Another change of the integration variable: $u = \frac{1}{2} \left(t - \frac{ab}{t} \right)$,

$$\frac{du}{dt} = \frac{1}{2} \left(1 + \frac{ab}{t^2} \right) = \frac{\sqrt{ab + u^2}}{t}, \quad (a^2 + t^2)(b^2 + t^2) = t^2((a + b)^2 + 4u^2).$$

(Figure: graph of $u = u(t)$.)



$$\begin{aligned} I(a, b) &= \int_{-\infty}^{\infty} \frac{du}{\sqrt{((a + b)^2 + 4u^2)(ab + u^2)}} \\ &= 2 \int_0^{\infty} \frac{du}{\sqrt{((a + b)^2 + 4u^2)(ab + u^2)}} \\ &= \int_0^{\infty} \frac{du}{\sqrt{\left(\left(\frac{a+b}{2}\right)^2 + u^2\right) ((\sqrt{ab})^2 + u^2)}} \end{aligned}$$

Comparing the expression of $I(a, b)$ in the last two pages, we have

$$I(a, b) = I\left(\frac{a+b}{2}, \sqrt{ab}\right).$$

Repeating the procedure,

$$I(a, b) = I(a_1, b_1) = \cdots = I(a_n, b_n) = \cdots \longrightarrow I(M(a, b), M(a, b)).$$

In general,

$$\begin{aligned} I(c, c) &= \int_0^{\pi/2} \frac{d\theta}{\sqrt{c^2 \cos^2 \theta + c^2 \sin^2 \theta}} \\ &= \int_0^{\pi/2} \frac{d\theta}{c\sqrt{\cos^2 \theta + \sin^2 \theta}} = \int_0^{\pi/2} \frac{d\theta}{c} = \frac{\pi}{2c}. \end{aligned}$$

$$\implies I(a, b) = \frac{\pi}{2M(a, b)}.$$

□

Similarly, when $k > 1$,

$$M(k, 1) = \frac{\pi}{2} \frac{1}{K(k')}, \quad k' := \sqrt{1 - k^2}.$$

Remark: $k' \notin \mathbb{R}$ but $K(k') = \int_0^{\pi/2} \frac{d\theta}{\sqrt{1 - k'^2 \sin^2 \theta}}$ makes sense.

Example: $k = \sqrt{2}$ ($k' = \sqrt{-1}$): $K(\sqrt{-1}) = \frac{\pi}{2M(\sqrt{2}, 1)}$, i.e.,

$$\int_0^1 \frac{dx}{\sqrt{1 - x^4}} = \frac{\pi}{2M(\sqrt{2}, 1)}.$$

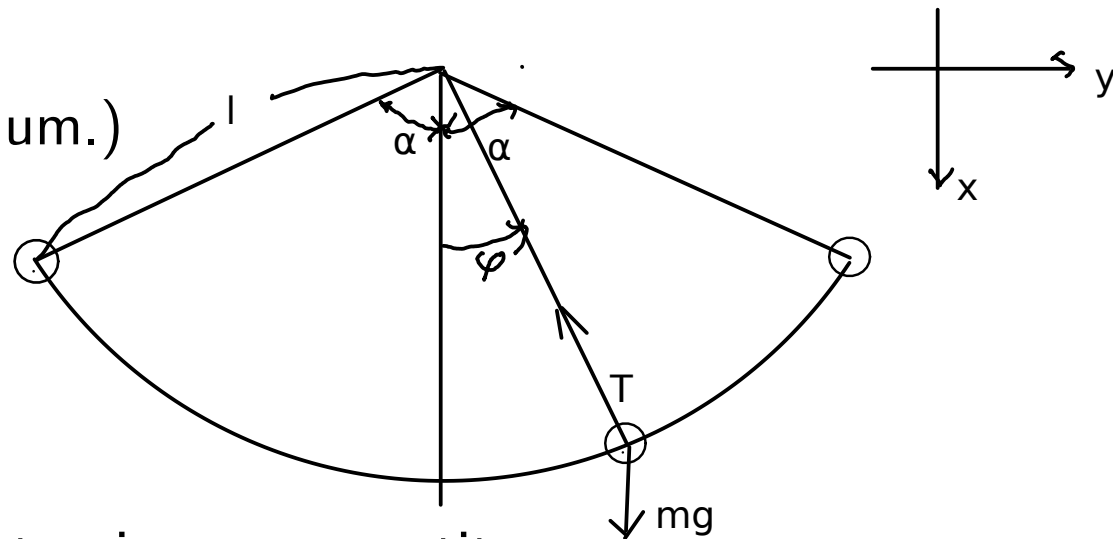
Gauss found this formula by computing the both sides *by hand*!

From a_4 and b_4 : $M(\sqrt{2}, 1) = 1.981402347\dots = \frac{\pi}{2 \int_0^1 \frac{dx}{\sqrt{1 - x^4}}}.$

(cf. Gauß, Gesammelte Werke, Tagebuch.)

§2.2 Motion of a simple pendulum

(Figure of a simple pendulum.)



l : length of the string, T : tension, mg : gravity.

$(x(t), y(t)) = (l \cos \varphi(t), l \sin \varphi(t))$: coordinates of the point mass.

$$\begin{aligned}\text{acceleration} &= \frac{d^2}{dt^2} (l \cos \varphi(t), l \sin \varphi(t)) \\ &= \frac{d}{dt} (-l\dot{\varphi} \sin \varphi, l\dot{\varphi} \cos \varphi) \\ &= (-l\ddot{\varphi} \sin \varphi - l\dot{\varphi}^2 \cos \varphi, l\ddot{\varphi} \cos \varphi - l\dot{\varphi}^2 \sin \varphi)\end{aligned}$$

The equation of motion:

$$ml \begin{pmatrix} -\sin \varphi \\ \cos \varphi \end{pmatrix} \ddot{\varphi} - ml \begin{pmatrix} \cos \varphi \\ -\sin \varphi \end{pmatrix} \dot{\varphi}^2 = \begin{pmatrix} -T \cos \varphi + mg \\ -T \sin \varphi \end{pmatrix}.$$

x -component $\times (-\sin \varphi) + y$ -component $\times \cos \varphi$: $ml\ddot{\varphi} = -mg \sin \varphi$, i.e.,

$$\frac{d^2\varphi}{dt^2} = -\omega^2 \sin \varphi, \quad \omega := \sqrt{\frac{g}{l}}.$$

Linear approximation: amplitude $\ll 1 \implies \sin \varphi \approx \varphi$.

$$\frac{d^2\varphi}{dt^2} = -\omega^2 \varphi.$$

A *LINEAR* differential equation of the second order.

General solution: $\varphi(t) = c_1 \cos \omega t + c_2 \sin \omega t$, $c_1, c_2 \in \mathbb{R}$.

Exact solutions without approximation “ $\sin \varphi \approx \varphi$ ”.

(eq. of motion) $\times \dot{\varphi}$:

$$\frac{d^2 \varphi}{dt^2} \frac{d\varphi}{dt} = -\omega^2 \sin \varphi \frac{d\varphi}{dt},$$

$$\frac{1}{2} \frac{d}{dt} \left(\frac{d\varphi}{dt} \right)^2 = \frac{d}{dt} (\omega^2 \cos \varphi)$$

$$\xrightarrow{\int(\dots)dt} \quad \frac{1}{2} \left(\frac{d\varphi}{dt} \right)^2 = \omega^2 \cos \varphi + (\text{const.}),$$

$$\tilde{E} := \frac{1}{2} \left(\frac{d\varphi}{dt} \right)^2 - \omega^2 \cos \varphi = (\text{const.}),$$

= conservation law of the energy.

Remark: the total energy of the pendulum is

$$(\text{kinetic energy}) + (\text{potential energy}) = \frac{ml^2}{2} \left(\frac{d\varphi}{dt} \right)^2 - mgl \cos \varphi = ml^2 \tilde{E}.$$

$\alpha :=$ maximum amplitude, i.e., if $\varphi(t_0) = \alpha$, $\dot{\varphi}(t_0) = 0$.

$\implies \tilde{E} = -\omega^2 \cos \alpha$, hence

$$\frac{1}{2} \left(\frac{d\varphi}{dt} \right)^2 = \omega^2 (\cos \varphi - \cos \alpha) = 2\omega^2 \left(\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\varphi}{2} \right),$$

$$\frac{d\varphi}{dt} = 2\omega \sqrt{\sin^2 \frac{\alpha}{2} - \sin^2 \frac{\varphi}{2}}.$$

Note: $\left| \sin \frac{\varphi}{2} \right| < \sin \frac{\alpha}{2}$, because $|\varphi| \leq \alpha < \pi$.

Notations: $k := \sin \frac{\alpha}{2}$, $\theta := \arcsin \left(k^{-1} \sin \frac{\varphi}{2} \right)$, i.e., $\sin \frac{\varphi}{2} = k \sin \theta$.

The equation of motion becomes

$$\frac{d\varphi}{dt} = 2\omega \sqrt{k^2 - k^2 \sin^2 \theta} = 2k\omega \cos \theta.$$

On the other hand, $\frac{d\varphi}{dt} = \frac{d\varphi}{d\theta} \frac{d\theta}{dt}$ and

$$\frac{d}{d\theta} k \sin \theta = \frac{d}{d\theta} \left(\sin \frac{\varphi}{2} \right), \text{ i.e., } k \cos \theta = \frac{\cos \frac{\varphi}{2}}{2} \frac{d\varphi}{d\theta} = \frac{\sqrt{1 - k^2 \sin^2 \theta}}{2} \frac{d\varphi}{d\theta}$$

Hence,

$$\begin{aligned} \frac{2k \cos \theta}{\sqrt{1 - k^2 \sin^2 \theta}} \frac{d\theta}{dt} &= 2k\omega \cos \theta, \\ \frac{1}{\sqrt{1 - k^2 \sin^2 \theta}} \frac{d\theta}{dt} &= \omega, \\ \int_{\theta(0)}^{\theta(t)} \frac{d\theta}{\sqrt{1 - k^2 \sin^2 \theta}} &= \int_0^t \omega dt = \omega t. \end{aligned}$$

(Normalisation: $\theta(0) = 0$, i.e., $\varphi(0) = 0$.)

The left hand side = $F(k, \theta(t))$ = elliptic integral of the first kind.

The motion of the pendulum:

$$t(\theta) = \frac{1}{\omega} F(k, \theta) = \sqrt{\frac{l}{g}} F\left(\sin \frac{\alpha}{2}, \theta\right).$$

Period = 4(time between $\varphi = 0$ and $\varphi = \alpha$)

$$= 4(\text{time between } \theta = 0 \text{ and } \theta = \frac{\pi}{2}) = 4\sqrt{\frac{l}{g}} K\left(\sin \frac{\alpha}{2}\right).$$

(The period depends on the amplitude!)

$$\text{Since } K\left(\sin \frac{\alpha}{2}\right) = \frac{\pi}{2} \frac{1}{M(1, \cos \frac{\alpha}{2})},$$

$$\text{Period}(\alpha) = \frac{\text{Period}(\alpha = 0)}{M(1, \cos \frac{\alpha}{2})}.$$

The pendulum knows the AGM!