Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

$$\text{(your final mark)} = \min\{\text{integer part of total points you get}), 10\}$$

- About twenty problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **1** – **3**: 6 February 2020.

1. (1 pt.) Express the arc length from $(0,0)$ to $(x_0, b\sin\frac{x_0}{a})$ $(x_0 > 0)$ of the graph of the sine curve $y = b\sin\frac{x}{a}$ $(a, b > 0)$ in terms of the elliptic integral of the second kind. Which arc corresponds to the complete elliptic integral $E(k)$?

2. We already know that the arc length of an ellipse is expressed in terms of elliptic integrals. How about the other conics? The answer is as follows.

(i) (1 pt.) Show that the arc length of the hyperbola $(x, y) = (a\cosh t, b\sinh t)$ from $t = 0$ to $t = t_0 > 0$ is formally expressed by an elliptic integral of the second kind as $-ib\,E\left(\frac{\sqrt{a^2+b^2}}{b}, it_0\right)$. ($i = \sqrt{-1}$; Of course, there is a formula without using complex numbers, but it is messy.)

(ii) (1 pt.) Find the formula for arc length of the parabola $y = ax^2$ from $x = 0$ to $x = x_0 > 0$. The result is an elementary function.

3. (1 pt.) Use the change of variables $\eta^2 := 1 - y^2$ and express the integrals

$$f(x) := \int_0^x \frac{dy}{\sqrt{1-y^4}}, \qquad L := \int_0^1 \frac{dy}{\sqrt{1-y^4}},$$

in terms of the elliptic integrals $F(k, \varphi)$ $(x = \cos\varphi)$ and $K(k)$ of the first kind with *real* $k \in \mathbb{R}$. (This is called the *lemniscate integral*.)