Elliptic Integrals and Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

$$\text{(your final mark)} = \min \{\text{integer part of total points you get}), 10\}$$

- About twenty problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **7** – **8**: 20 February 2020.

7. (1 pt.) Fix $0 < k < 1$. Prove the following formulae for the complete elliptic integrals of the first kind, *using the arithmetic-geometric mean $M(a, b)$ and its properties* explained in the lecture on 6 February 2020:

$$K(k) = \frac{1}{1+k} K\left(\frac{2\sqrt{k}}{1+k}\right) = \frac{2}{1+k'} K\left(\frac{1-k'}{1+k'}\right),$$

where k' is defined by $k^2 + k'^2 = 1$, $0 < k' < 1$.

8. (1 pt.) If a simple pendulum is made of a stick of length l with negligibly small mass, then it can rotate around the centre. In this case the angle φ is a monotonically increasing function of the time t. (*Not a periodic function!*)

Express its period, namely, the time from $\varphi = 0$ till $\varphi = 2\pi$, in terms of elliptic integrals and the total energy E. (Hint: In the lecture we used a constant $\tilde{E}$ which is equal to E/ml^2. Although there is no "maximum amplitude" α for a rotating "pendulum", we can still use $\tilde{E}$, which plays the role of $-\omega^2 \cos\alpha$ in the lecture. Use $k_0 := \sqrt{\dfrac{2\omega^2}{\omega^2 + \tilde{E}}}$ as the modulus of the elliptic integral. The modulus k used in the lecture is equal to k_0^{-1}.)