ALGEBRAIC GROUPS AND INVARIANT THEORY 1.

1 Let $U \subset V$ are the vector spaces over $k = \overline{k}$, and $G \subset GL(n, k)$ is the closed algebraic subgroup. Prove that the set of elements $g \in G$, such that $gU \subset V$ form an algebraic subgroup of G.

2 Let U, V are the open nonempty subsets in irreducible algebraic group. Prove that $G = UV$ where UV is the pairwise product of elements of U and V. Let H be an abstract subgroup of G that contains a constructible set. Prove that H is equal to its Zariski closure. Prove that the Zariski closure $SL(n, \mathbb{Z})$ in $SL(n, \mathbb{C})$ is equal to $SL(n, \mathbb{C})$.

3 In the case when G is irreducible prove that $[G, G]$ is an algebraic group.

4 Prove that for any G there exists a Rad G - the unique maximal normal solvable irreducible algebraic subgroup. Prove that when char$k = 0$ there exists a unipotent radical i.e. maximal normal irreducible algebraic subgroup that consists of unipotent elements.

5 Prove that for Jordan decomposition of $g \in G \subset GL(n, k)$ i.e. $g = g_s g_u$ we have $g_s, g_u \in G$. Prove that in the case of characteristic zero g_u is contained in the unique subgroup of G isomorphic to $\mathbb{G}_a = k^+$. Is this statement true for g_s with $\mathbb{G}_m = k^\times$? Is it true when $k = \overline{k}$?

6 Prove that the G-orbit in algebraic variety X is open in its closure. Prove that an the closure of an orbit Gx contains a closed orbit. Is this closed orbit unique? Is it unique when G is reductive and X is affine?

7 Let G be a reductive group acting on an algebraic variety X over char $k = 0$. And $\pi : X \to Speck[X]^G$ is the categorical quotient. Prove that for Z_1, Z_2 invariant closed subsets such that $Z_1 \cap Z_2 = \emptyset$ we have $\overline{\pi(Z_1)} \cap \overline{\pi(Z_2)} = \emptyset$.

8 Let G be a reductive group acting on an algebraic variety X over char $k = 0$. Prove that the map $\pi : X \to Speck[X]^G$ is surjective. Is it true when G is not reductive but the algebra $k[X]^G$ is still finitely generated.

9 Let G be a reductive group acting on an algebraic variety X over char $k = 0$. Consider the categorical quotient $\pi : X \to Speck[X]^G$. Is it proper? equidimentional? smooth? flat?