Elliptic Functions

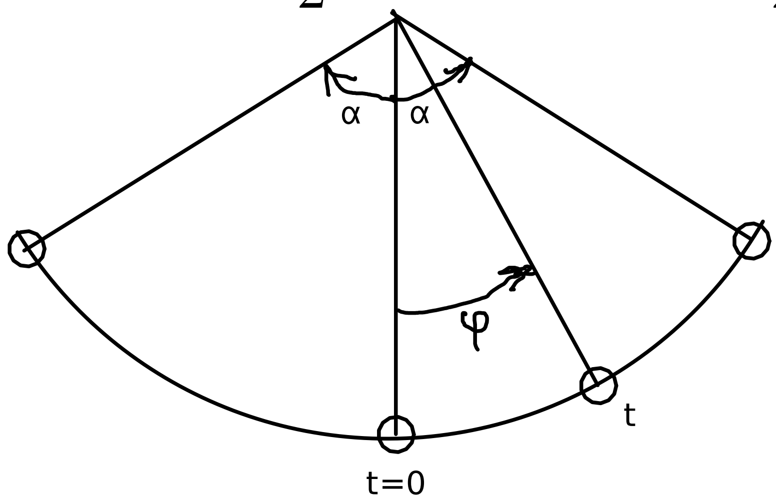
Jacobi's elliptic functions over $\mathbb{R}$

§3.1 Jacobi's elliptic functions

Recall: the motion of a simple pendulum is describe by

$$t(\theta) = \sqrt{\frac{l}{g}} F\left(\sin\frac{\alpha}{2}, \theta\right) = \sqrt{\frac{l}{g}} \int_0^\theta \frac{d\phi}{\sqrt{1 - k^2 \sin^2\phi}},$$

where $\alpha =$ max. amplitude, $k = \sin\frac{\alpha}{2}$, $\sin\theta = k^{-1}\sin\frac{\varphi}{2}$.



Better description of motion: "amplitude $\varphi =$ function of time t".

$\Longrightarrow$ Consider the *inverse function*!

Assume $0 \leqq k < 1$.

Definition:

Jacobi's elliptic function $\operatorname{sn}(u) = \operatorname{sn}(u, k) :=$ the inverse function of

$$u(x) = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

on $-1 \leqq x \leqq 1$ and, consequently,

$-K(k) \leqq u \leqq K(k)$ (= the complete elliptic integral of the first kind.)

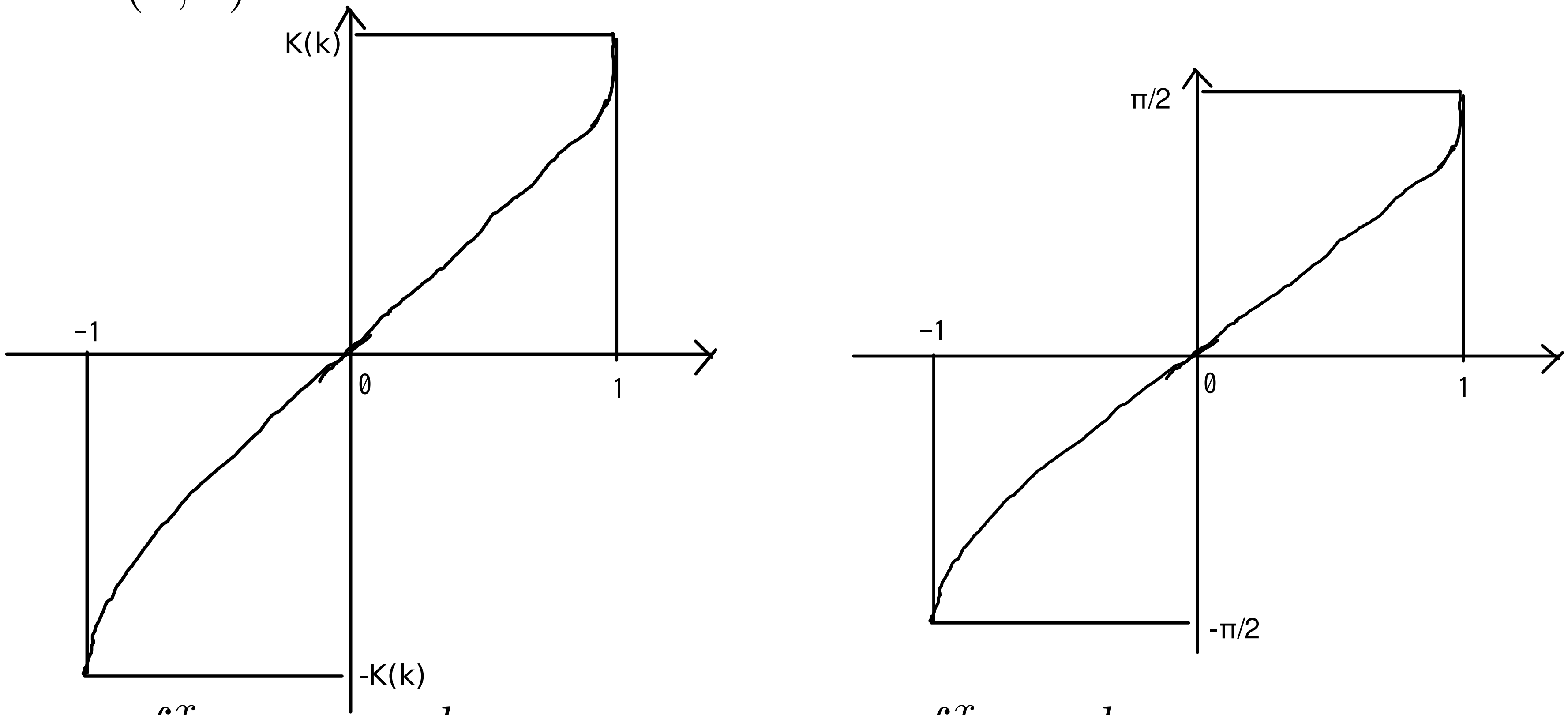
$\operatorname{sn}(u, k) =$ analogue of $\sin u$: in fact, when $k = 0$,

$$\operatorname{sn}(u, k) = \text{the inverse function of } \left(u(x) = \int_0^x \frac{dz}{\sqrt{(1-z^2)}} = \arcsin x \right).$$

Namely,

$$\operatorname{sn}(u, 0) = \sin u, \qquad K(0) = \frac{\pi}{2}.$$

Graphs of $F(x,k)$ and $\arcsin x$:



$$F(x,k) = \int_0^x \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \xrightarrow{k \longrightarrow 0} \int_0^x \frac{dz}{\sqrt{(1-z^2)}} = \arcsin x$$

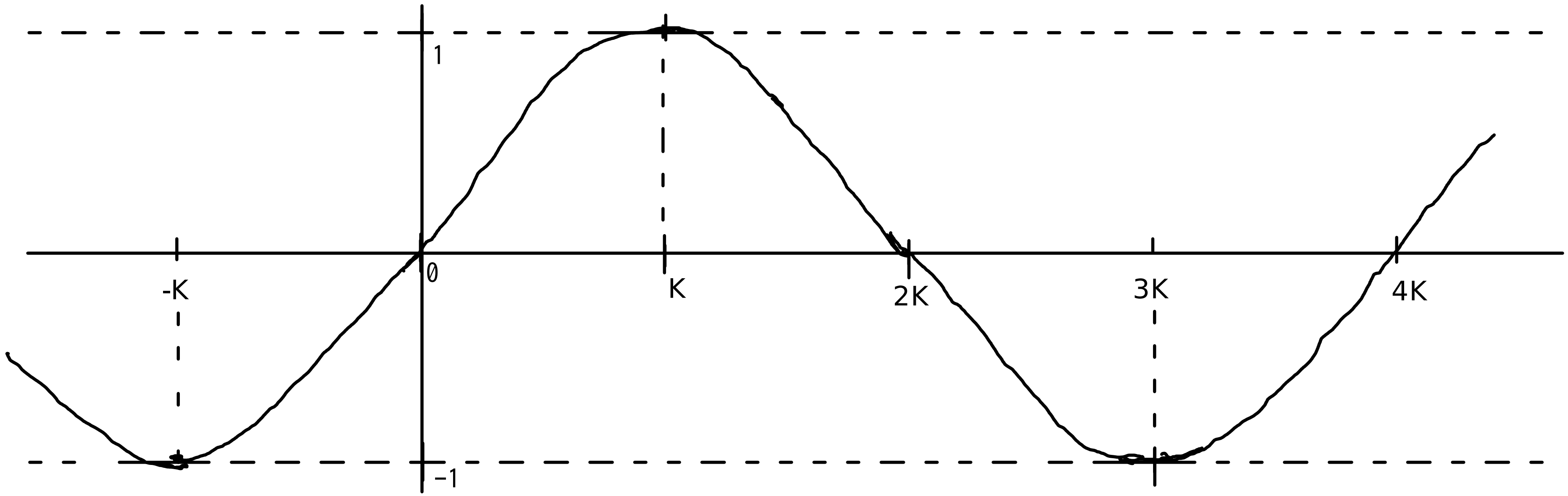
Important property of $\sin$ = periodicity: $\sin(u + 2\pi) = \sin u$.

$\Longrightarrow$ Extend sn to $\mathbb{R}$ by periodicity:

$$\mathrm{sn}(u + 2K(k), k) = -\,\mathrm{sn}(u, k), \qquad \mathrm{sn}(u + 4K(k), k) = \mathrm{sn}(u, k).$$

(Justification given in § "Complex elliptic integrals/functions".)

Graph of sn



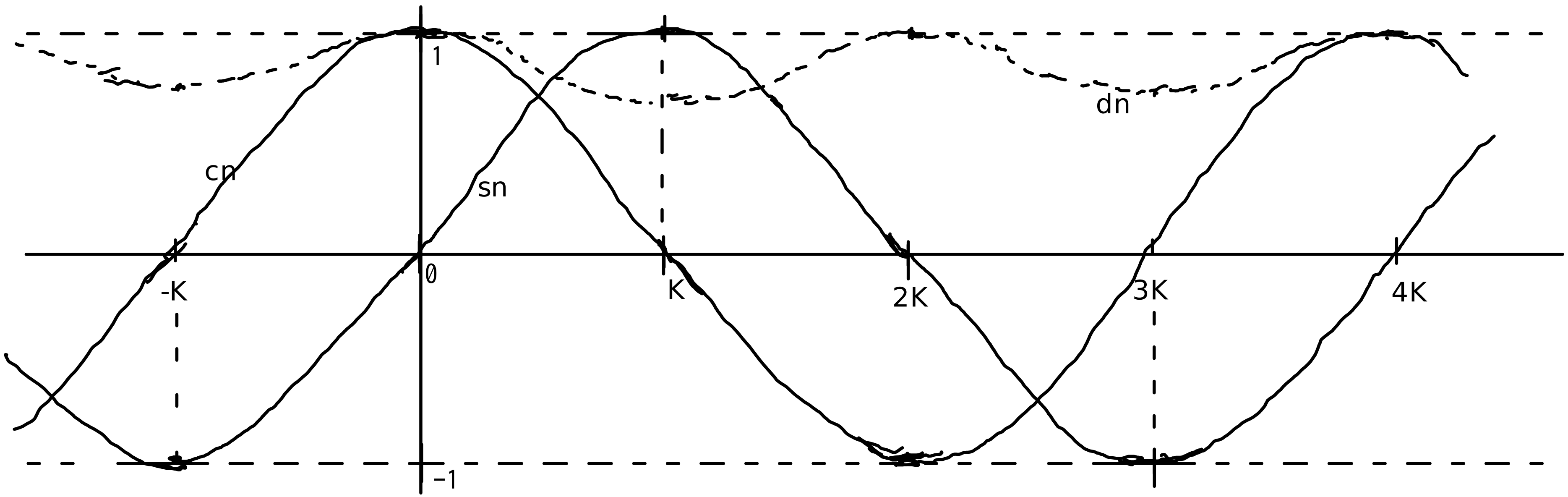
Introduce cn (analogue of $\cos$) and dn:

$$\mathrm{cn}(u) = \mathrm{cn}(u,k) := \sqrt{1-\mathrm{sn}^2(u,k)}, \qquad (\mathrm{cn}(0)=1),$$
$$\mathrm{dn}(u) = \mathrm{dn}(u,k) := \sqrt{1-k^2\,\mathrm{sn}^2(u,k)}, \qquad (\mathrm{dn}(0)=1),$$

and extend by periodicity.

$k \to 0$: $K(k) \to \pi$, $\mathrm{sn}\,u \to \sin u$, $\mathrm{cn}\,u \to \cos u$, $\mathrm{dn}\,u \to 1$.

Graphs of sn, cn and dn:



Exercise: Show that, when $k \to 1$,

$$
\begin{aligned}
K(k) &\to \infty, \\
\mathrm{sn}(u,k) &\to \tanh u = \frac{\sinh u}{\cosh u}, \\
\mathrm{cn}(u,k), \mathrm{dn}(u,k) &\to \mathrm{sech}\, u = \frac{1}{\cosh u}.
\end{aligned}
$$

$\operatorname{sn} u =$ the inverse function of elliptic integral $F(k,x)$, i.e.,

$$u = F(k, \operatorname{sn} u) = \int_0^{\operatorname{sn} u} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.$$

$$\frac{d}{du}\operatorname{sn} u = \frac{1}{\frac{\partial}{\partial x}F(k,x)\Big|_{x=\operatorname{sn} u}} = \frac{1}{\frac{1}{\sqrt{(1-x^2)(1-k^2x^2)}}\Big|_{x=\operatorname{sn} u}}$$

$$= \sqrt{1-\operatorname{sn}^2 u}\sqrt{1-k^2\operatorname{sn}^2 u} = \operatorname{cn} u \, \operatorname{dn} u.$$

As corollaries,

$$\frac{d}{du}\operatorname{cn} u = \frac{d}{du}\sqrt{1-\operatorname{sn}^2 u} = \frac{-\operatorname{sn} u \frac{d \operatorname{sn} u}{du}}{\sqrt{1-\operatorname{sn}^2 u}} = -\operatorname{sn} u \, \operatorname{dn} u.$$

$$\frac{d}{du}\operatorname{dn} u = \frac{d}{du}\sqrt{1-k^2\operatorname{sn}^2 u} = \frac{-k^2 \operatorname{sn} u \frac{d \operatorname{sn} u}{du}}{\sqrt{1-k^2\operatorname{sn}^2 u}} = -k^2 \operatorname{sn} u \, \operatorname{cn} u.$$

Summarising,

$$\frac{d\,\mathrm{sn}\,u}{du} = \mathrm{cn}\,u\ \mathrm{dn}\,u, \qquad \xrightarrow{k\to 0} \qquad \frac{d\sin u}{du} = \cos u,$$

$$\frac{d\,\mathrm{cn}\,u}{du} = -\,\mathrm{sn}\,u\ \mathrm{dn}\,u, \qquad \xrightarrow{k\to 0} \qquad \frac{d\cos u}{du} = -\sin u,$$

$$\frac{d\,\mathrm{dn}\,u}{du} = -k^2\,\mathrm{sn}\,u\ \mathrm{cn}\,u.$$

Addition formulae:

Addition formula of $\sin$: $\sin(u+v) = \sin u \, \cos v + \cos u \, \sin v$.

Addition formula of $\tanh$: $\tanh(u+v) = \dfrac{\tanh u + \tanh v}{1 + \tanh u \, \tanh v}$.

$\mathrm{sn}(u,k)$ interpolates $\sin$ ($k=0$) and $\tanh$ ($k=1$).

$\Longrightarrow$ A natural guess is "sn has an addition formula."

Let us "interpolate" the above formulae!

Addition formula of $\sin$ without $\cos$:

$$\sin(u+v) = \sin u \frac{d \sin v}{dv} + \frac{d \sin u}{du} \sin v.$$

Note $\dfrac{d\tanh u}{du} = 1 - \tanh^2 u$. Hence

$$\tanh u \frac{d\tanh v}{dv} + \frac{d\tanh u}{du} \tanh v = (\tanh u + \tanh v)(1 - \tanh u \, \tanh v).$$

Addition formula of $\tanh$ can be rewritten as

$$\tanh(u+v) = \frac{\tanh u \frac{d\tanh v}{dv} + \frac{d\tanh u}{du} \tanh v}{1 - \tanh^2 u \, \tanh^2 v}.$$

A possible interpolation of the addition formulae of $\sin$ and $\tanh$:

$$\begin{aligned}\mathrm{sn}(u+v) &= \frac{\mathrm{sn}\, u \frac{d\,\mathrm{sn}\, v}{dv} + \frac{d\,\mathrm{sn}\, u}{du} \mathrm{sn}\, v}{1 - k^2 \mathrm{sn}^2 u \, \mathrm{sn}^2 v} \\ &= \frac{\mathrm{sn}\, u \, \mathrm{cn}\, v \, \mathrm{dn}\, v + \mathrm{sn}\, v \, \mathrm{cn}\, u \, \mathrm{dn}\, u}{1 - k^2 \mathrm{sn}^2 u \, \mathrm{sn}^2 v}.\end{aligned}$$

In fact this is true!

Proof: $u+v\to c$, $v\to c-u$,

$$F(u) := \frac{\operatorname{sn} u\ \operatorname{cn}(c-u)\ \operatorname{dn}(c-u) + \operatorname{sn}(c-u)\ \operatorname{cn} u\ \operatorname{dn} u}{1-k^2\operatorname{sn}^2 u\ \operatorname{sn}^2(c-u)}.$$

Claim: $\dfrac{dF}{du}=0$, when c is fixed.

Claim $\Longrightarrow F(0)=F(u)$ and, since $\operatorname{sn} 0=0$, $\operatorname{cn} 0=\operatorname{dn} 0=1$,

$$\operatorname{sn} c = \frac{\operatorname{sn} u\ \operatorname{cn}(c-u)\ \operatorname{dn}(c-u) + \operatorname{sn}(c-u)\ \operatorname{cn} u\ \operatorname{dn} u}{1-k^2\operatorname{sn}^2 u\ \operatorname{sn}^2(c-u)}.$$

Substitutig $c=u+v$, we obtain the addition formula. □

Proof of the claim:

$N :=$ numerator of $F(u) = \operatorname{sn} u\ \operatorname{cn}(c-u)\ \operatorname{dn}(c-u) + \operatorname{sn}(c-u)\ \operatorname{cn} u\ \operatorname{dn} u$

$D :=$ denominator of $F(u) = 1-k^2\operatorname{sn}^2 u\ \operatorname{sn}^2(c-u)$.

Long computation (Exercise!) shows

$$\frac{dN}{du} D = N \frac{dD}{du}.$$

Therefore

$$\frac{dF}{du} = \frac{\frac{dN}{du} D - N \frac{dD}{du}}{D^2} = 0.$$

$\square$

Addition formulae of cn and dn:

$$\mathrm{cn}(u+v) = \frac{\mathrm{cn}\, u\ \mathrm{cn}\, v - \mathrm{sn}\, u\ \mathrm{sn}\, v\ \mathrm{dn}\, u\ \mathrm{dn}\, v}{1 - k^2 \,\mathrm{sn}^2 u\ \mathrm{sn}^2 v},$$

$$\mathrm{dn}(u+v) = \frac{\mathrm{dn}\, u\ \mathrm{dn}\, v - k^2 \,\mathrm{sn}\, u\ \mathrm{sn}\, v\ \mathrm{cn}\, u\ \mathrm{cn}\, v}{1 - k^2 \,\mathrm{sn}^2 u\ \mathrm{sn}^2 v}.$$

§3.2 Jacobi's function in physics

The motion of the simple pendulum revisited:

$$
\begin{aligned}
t(\theta) &= \sqrt{\frac{l}{g}} \int_0^{\theta} \frac{d\phi}{\sqrt{1-k^2\sin^2\phi}} \qquad \left(k = \sin\frac{\alpha}{2}\right) \\
&= \sqrt{\frac{l}{g}} \int_0^{\sin\theta} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} \qquad (z = \sin\phi) \\
&= \sqrt{\frac{l}{g}} \int_0^{k^{-1}\sin\frac{\varphi}{2}} \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}}.
\end{aligned}
$$

Using Jacobi's sn function,

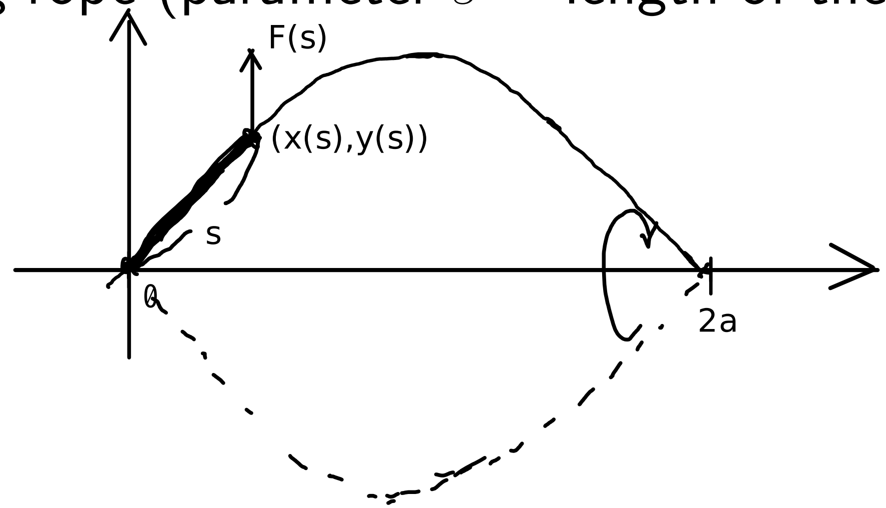
$$
\sin\frac{\varphi(t)}{2} = k\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}t, k\right), \qquad \varphi(t) = 2\arcsin\left(k\,\mathrm{sn}\left(\sqrt{\frac{g}{l}}t, k\right)\right).
$$

Another application: form of a skipping rope

Assumptions:

- The rope rotates fast enough. $\Longrightarrow$ centrifugal force $\gg$ gravity.
- The density ρ ($=$ mass/length) of the rope is constant.
- The ends are fixed at $(0,0)$ and $(2a,0)$ in the xy-plane.
- The rope rotates around the x-axis with the angular velocity ω.

Figure of the skipping rope (parameter $s =$ length of the rope from $(0,0)$):



The centrifugal force at (x,y) =(mass) $\times\omega^2 y$.

Potential at (x,y):

$$V(y) := -\frac{\text{mass}}{2}\omega^2 y^2, \text{ i.e., } -\frac{\partial V(x,y)}{\partial y} = \text{centrifugal force.}$$

mass of the segment of length $ds = \rho\, ds$.

Total potential: $U = -\frac{1}{2}\int_0^l \omega^2\, y(s)^2\, \rho\, ds = -\frac{\rho\omega^2}{2}\int_0^l y(s)^2\, ds.$

When the form of the skipping rope is stable, U is minimum.

Constraint: the length of the rope $= l$.

The problem to be solved: a variational problem for $y = y(x)$:

- Maximise $\displaystyle\int_0^l y(s)^2\,ds = \int_0^{2a} y(x)^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$

- Under the constraint $\displaystyle l = \int_0^l ds = \int_0^{2a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx.$

Calculus of variations with the Lagrange multiplier:

$$
\begin{aligned}
\mathcal{L}[y(x)] &:= \int_0^{2a} y(x)^2\sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx - \lambda\int_0^{2a} \sqrt{1+\left(\frac{dy}{dx}\right)^2}\,dx \\
&= \int_0^{2a} L[y,y']\,dx, \qquad L[y,y'] := (y^2-\lambda)\sqrt{1+y'^2}.
\end{aligned}
$$

Want: $y_0(x)$, such that

$$\left.\frac{\delta \mathcal{L}}{\delta y}\right|_{y_0} = 0, \qquad y_0(0) = y_0(2a) = 0, \qquad l = \int_0^{2a} \sqrt{1 + {y_0'}^2}\, dx.$$

Variation $\delta\mathcal{L} = \mathcal{L}[y + \delta y] - \mathcal{L}[y]$

$$\begin{aligned}
&= \int_0^{2a} \left(L[y + \delta y, y' + \delta y'] - L[y, y']\right) dx \\
&= \int_0^{2a} \left(\frac{\partial L}{\partial y}\delta y + \frac{\partial L}{\partial y'}\delta y' + o(\delta y, \delta y')\right) dx \\
&= \int_0^{2a} \left(\frac{\partial L}{\partial y}\delta y - \frac{d}{dx}\frac{\partial L}{\partial y'}\delta y + o(\delta y, \delta y')\right) dx \quad \text{(integration by parts)} \\
&= \int_0^{2a} \left(\frac{\partial L}{\partial y} - \frac{d}{dx}\frac{\partial L}{\partial y'}\right) \delta y\, dx + o(\delta y, \delta y')
\end{aligned}$$

$\delta\mathcal{L}$ should be $o(\delta y, \delta y')$ for any variation $\delta y(x)$

$\Longrightarrow$ *Euler-Lagrange equation*: $\dfrac{\partial L}{\partial y} - \dfrac{d}{dx}\dfrac{\partial L}{\partial y'} = 0.$

In our case ($L = (y^2 - \lambda)\sqrt{1 + y'^2}$),

$$\frac{\partial L}{\partial y} = 2y\sqrt{1 + y'^2},$$

$$\frac{d}{dx}\frac{\partial L}{\partial y'} = \frac{d}{dx}\left((y^2 - \lambda)\frac{y'}{\sqrt{1 + y'^2}}\right)$$

$$= 2y\frac{{y'}^2}{\sqrt{1 + y'^2}} + (y^2 - \lambda)\left(\frac{y''}{\sqrt{1 + y'^2}} - \frac{y'^2 y''}{(1 + y'^2)^{3/2}}\right).$$

Euler-Lagrange equation $\times\sqrt{1 + y'^2}$:

$$2y - (y^2 - \lambda)\frac{y''}{1 + y'^2} = 0, \qquad \text{i.e., } \frac{y''}{1 + y'^2} = \frac{2y}{y^2 - \lambda}.$$

Note: from "physical" point of view, $y'' < 0$ in the region $\{x > 0, y > 0\}$.
Hence $\lambda > y^2$.

Integral of the Euler-Lagrange equation:

$$\frac{1}{2}\log\bigl(1 + {y'}^2\bigr) = \log(\lambda - y^2) + (\text{const.}),$$

$$\text{therefore,} \qquad 1 + {y'}^2 = C(\lambda - y^2)^2.$$

Let $x = b$ be the maximum value of $y(x)$:

$$b = y(x_0), \text{ where } y'(x_0) = 0.$$

$\Longrightarrow C = (\lambda - b^2)^{-2}$.

$$\left(\frac{dy}{dx}\right)^2 = \left(\frac{\lambda - y^2}{\lambda - b^2}\right)^2 - 1$$
$$= \frac{(\lambda - y^2)^2 - (\lambda - b^2)^2}{(\lambda - b^2)^2} = \frac{(b^2 - y^2)(2\lambda - b^2 - y^2)}{(\lambda - b^2)^2}.$$

$\eta := y/b$ satisfies

$$\frac{d\eta}{dx} = c\sqrt{(1-\eta^2)(1-k^2\eta^2)}, \text{ where } c^2 = \frac{b^2(2\lambda - b^2)}{(\lambda - b^2)^2},\ k^2 = \frac{b^2}{2\lambda - b^2}.$$

Integrate by x:

$$\int \frac{d\eta}{\sqrt{(1-\eta^2)(1-k^2\eta^2)}} = \int c\,dx.$$

The integral of the LHS = elliptic integral of the first kind!

$$\eta = \operatorname{sn}(cx), \text{ i.e., } y = b\operatorname{sn}(cx).$$

b, c: to be determined by the condition "length of the rope $= l$".