Elliptic Functions

Complex Elliptic Integrals

§6.1 Complex elliptic integral of the first kind

Want: elliptic integrals $\int_C R(x, \sqrt{\varphi(x)})\, dx$ with complex variables.

C: curve on the Riemann surface $\mathcal{R}$ of $\sqrt{\varphi(z)}$,

or its compactification $\bar{\mathcal{R}} =$ the elliptic curve.

Let us begin with $\displaystyle\int \frac{dz}{\sqrt{\varphi(z)}}$, the elliptic integral of the first kind.

$$\omega_1 := \frac{dz}{\sqrt{\varphi(z)}} = \frac{dz}{w}.$$

We know that ω_1 is holomorphic on $\mathcal{R} = \bar{\mathcal{R}} \setminus \{\infty\}$ $(\deg \varphi = 3)$ or $\mathcal{R} = \bar{\mathcal{R}} \setminus \{\infty_\pm\}$ $(\deg \varphi = 4)$. (Problem **11** (ii).)

How about on neighbourhoods of infinities?

Assume $\deg \varphi = 4$: $\varphi(z) = a(z - \alpha_0)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3)$.

(The case $\deg \varphi = 3$ is similar.)

Recall:

- (a local coordinate at $\infty_\pm$) $= \xi = z^{-1}$.
- the equation of $\bar{\mathcal{R}}$ in the neighbourhood of $\infty_\pm$:

$$\eta^2 = a(1 - \alpha_0\xi)(1 - \alpha_1\xi)(1 - \alpha_2\xi)(1 - \alpha_3\xi),$$

  where $\eta = wz^{-2}$.
- $\infty_\pm = (\xi = 0, \eta = \pm\sqrt{a} \neq 0)$.

Consequently,

- $d\xi = -z^{-2}\, dz$.
- $\omega_1 = \dfrac{dz}{w} = -\dfrac{d\xi}{\eta}$.
- $\eta(\xi) = \sqrt{a(1-\alpha_0\xi)(1-\alpha_1\xi)(1-\alpha_2\xi)(1-\alpha_3\xi)}$ is holomorphic in ξ and $\eta(\xi) \neq 0$ in the neighbourhood of $\xi = 0$.

$\Longrightarrow$ ω_1 is holomorphic at $\infty_\pm$.

Conclusion: ω_1 is holomorphic everywhere on $\bar{\mathcal{R}}$.

Moreover, $\omega_1 \neq 0$ everywhere on $\bar{\mathcal{R}}$.

($\omega_1 = \dfrac{1}{w} dz$ on $\mathcal{R}$ and $\dfrac{1}{w} \neq 0$; $\omega_1 = -\dfrac{1}{\eta} d\xi$ at $\infty_\pm$ and $-\dfrac{1}{\eta} \neq 0$.)

Exercise: Show that ω_1 is holomorphic everywhere and nowhere-vanishing in the case $\deg \varphi = 3$.

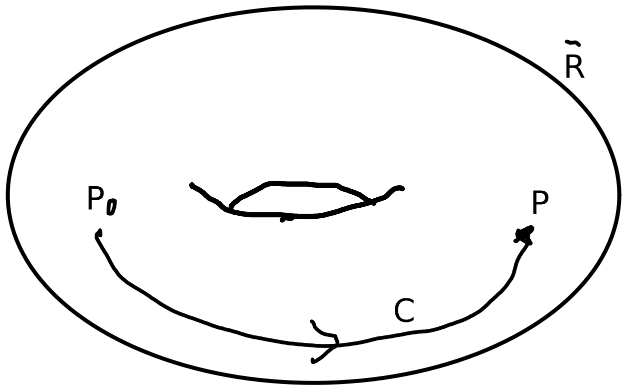
Fix $P_0 \in \bar{\mathcal{R}}$.

ω_1 is a holomorphic one-form on $\bar{\mathcal{R}}$.

$$\Longrightarrow F(P) := \int_{P_0 \to P} \omega_1 = \int_{C\text{: contour from } P_0 \text{ to } P} \omega_1$$

is "locally" well-defined.

Figure of $\bar{\mathcal{R}}$ and C:

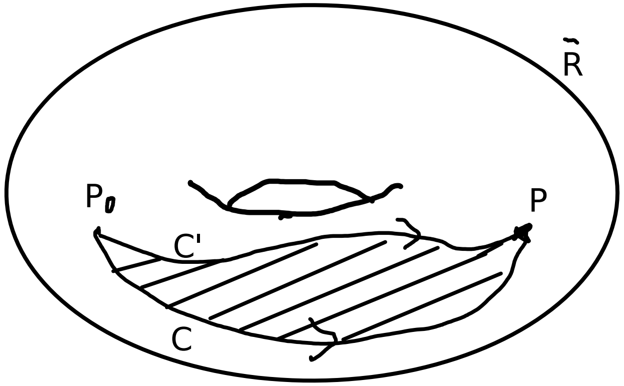


$\Leftrightarrow$ $F(P)$ does not change by "small perturbation of C."

Exactly speaking, by Cauchy's integral theorem,

$$[C - C'] = 0 \text{ in } H_1(\bar{\mathcal{R}}, \mathbb{Z}) \Longrightarrow \int_C \omega_1 = \int_{C'} \omega_1.$$

Figure: $[C - C'] = 0$.



Is $F(P)$ *"globally"* well-defined?

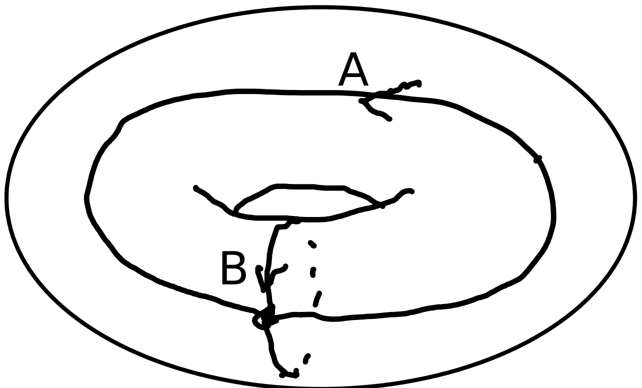
Need to know: How many "globally" different contours exist on $\bar{\mathcal{R}}$?

Answer from topology: $H_1(\bar{\mathcal{R}}, \mathbb{Z}) = \mathbb{Z}[A] \oplus \mathbb{Z}[B]$, which means:

for $\forall$ closed curve C on $\bar{\mathcal{R}}$, $\exists!\, m, n \in \mathbb{Z}$, such that

$$[C] = m[A] + n[B] \text{ in } H_1(\bar{\mathcal{R}}, \mathbb{Z}).$$

Figure: A-cycle and B-cycle.



C_0, C_1: curves from P_0 to P.

$\Longrightarrow [C_1 - C_0] = m[A] + n[B]$ for some $m, n \in \mathbb{Z}$.

$$\int_{C_1} \omega_1 = \int_{C_0} \omega_1 + m \int_A \omega_1 + n \int_B \omega_1.$$

We call

$$\int_A \omega_1\text{: } A\text{-}\textit{period} \text{ of 1-form } \omega_1, \quad \int_B \omega_1\text{: } B\text{-}\textit{period} \text{ of 1-form } \omega_1.$$

Let us compute A- and B-periods for the case $\varphi(z) = (1 - z^2)(1 - k^2 z^2)$, i.e.,

$$\omega_1 = \frac{dz}{\sqrt{(1 - z^2)(1 - k^2 z^2)}}.$$

For simplicity, assume $k \in \mathbb{R}$, $0 < k < 1$.

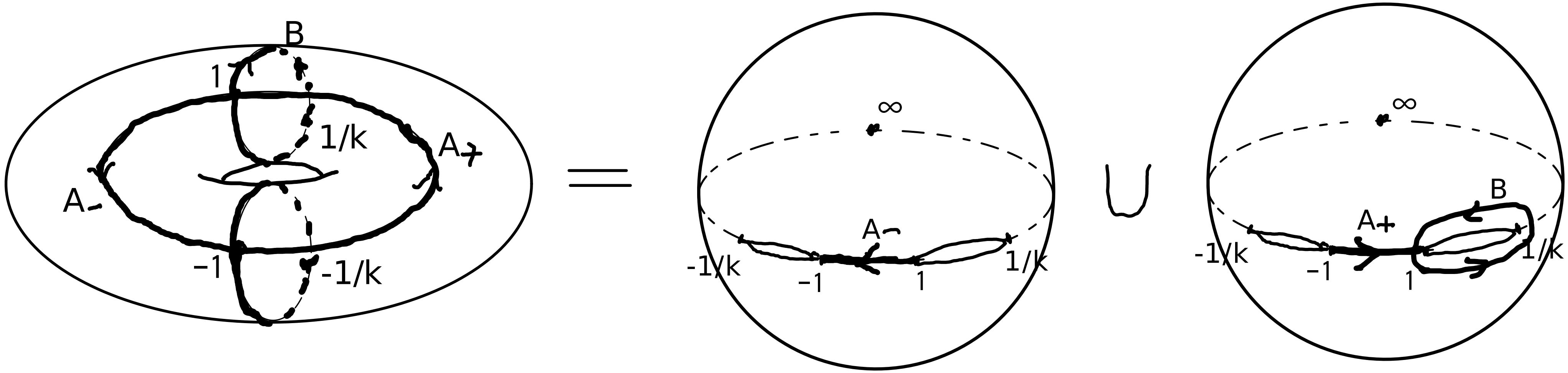
Recall <u>the construction of $\bar{\mathcal{R}}$</u>:

Two $\mathbb{P}^1$'s are glued together along cuts between two pairs of roots of $\varphi(z)$.

$$\text{roots of } \varphi(z) = \pm 1, \pm k^{-1}.$$

Cut $\mathbb{P}^1$'s along $[-k^{-1}, -1]$ and $[1, k^{-1}]$ and glue.

(Figure of A- and B-cycles on $\mathbb{P}^1$'s)



Periods of ω_1:

$$\int_A \omega_1 = 4\,K(k), \qquad \int_B \omega_1 = 2i\,K'(k),$$

where

- $K(k) = \displaystyle\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}}$: complete elliptic integral of the first kind.
- $K'(k) := K(k')$, $k' := \sqrt{1-k^2}$ (supplementary modulus).

Proof:

$$\int_A \omega_1 = \int_{-1}^1 \frac{dx}{+\sqrt{(1-x^2)(1-k^2x^2)}} + \int_1^{-1} \frac{dx}{-\sqrt{(1-x^2)(1-k^2x^2)}}$$

(Note: $\pm$ of the denominators are different because of branches.)

$$= 4\int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = 4\,K(k).$$

$$\int_B \omega_1 = \int_1^{1/k} \frac{dx}{+\sqrt{(1-x^2)(1-k^2x^2)}} + \int_{1/k}^{1} \frac{dx}{-\sqrt{(1-x^2)(1-k^2x^2)}}$$

$$= 2\int_1^{1/k} \frac{dx}{\sqrt{(1-x^2)(1-k^2x^2)}} = 2i\int_1^{1/k} \frac{dx}{\sqrt{(x^2-1)(1-k^2x^2)}}$$

(N.B.: $1 \le x \le 1/k \Rightarrow x^2 - 1 \ge 0$, $1 - k^2x^2 \ge 0$.)

Change of the variable: $x = \dfrac{1}{\sqrt{1-k'^2t^2}}$, i.e., $x^2 = \dfrac{1}{1-k'^2t^2}$,

$$dx = \frac{k'^2 t}{(1-k'^2t^2)^{3/2}}dt, \qquad (x^2-1)(1-k^2x^2) = \frac{k'^4t^2(1-t^2)}{(1-k'^2t^2)^2}.$$

Hence,

$$\int_B \omega_1 = 2i\int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}} = 2i\,K(k') = 2i\,K'(k).$$

□

Remark:

- Signs of $\sqrt{\ }$ should be chosen carefully.
- For general $k \in \mathbb{C}$, the results are the same (analytic continuation).

Recall:

$$\text{“}A\text{-period of } \frac{dz}{\sqrt{1-z^2}} = 2\pi = \text{period of } \sin(u)\text{.”}$$

Correspondingly,

$$A\text{-period of } \frac{dz}{\sqrt{(1-z^2)(1-k^2z^2)}} = 4\,K(k) = \text{period of } \mathrm{sn}(u)!$$

What is the role of the B-period $2i\,K'(k)$ for $\mathrm{sn}(u)$?

$\longrightarrow$ Another period of $\mathrm{sn}(u)$, i.e., $\mathrm{sn}(u)$ is doubly-periodic!

Details will be discussed later...

Recall ω_1 is holomorphic on $\bar{\mathcal{R}}$.

$$\Longrightarrow F(P) = \int_{P_0 \to P} \omega_1 \text{ defines a holomorphic function on } \bar{\mathcal{R}}.$$

Conclusion:

The integral of ω_1 is a multi-valued holomorphic function on $\bar{\mathcal{R}}$.

§6.2 Complex elliptic integral of the second kind

$$\int \sqrt{\frac{1-k^2z^2}{1-z^2}}\,dz = \int \frac{1-k^2z^2}{\sqrt{\varphi(z)}}\,dz, \qquad \varphi(z) = (1-z^2)(1-k^2z^2).$$

Corresponding Riemann surface $= \mathcal{R} = \{(z,w) \mid w^2 = \varphi(z)\}$ as before.

The compactification $= \bar{\mathcal{R}}$: elliptic curve.

$$\omega_2 := \sqrt{\frac{1-k^2z^2}{1-z^2}}\,dz = \frac{1-k^2z^2}{\sqrt{\varphi(z)}}\,dz = \frac{1-k^2z^2}{w}\,dz$$

is holomorphic on $\mathcal{R}$ as ω_1. (In particular, at $z = \pm 1, \pm k^{-1}$.)

How does this form behave at $\{\infty_\pm\} = \bar{\mathcal{R}} \setminus \mathcal{R}$?

Local coordinate at $\pm\infty$: $\xi = z^{-1}$.

$$\omega_2 = \sqrt{\frac{1-k^2\xi^{-2}}{1-\xi^{-2}}}\, d(\xi^{-1}) = \sqrt{\frac{\xi^2-k^2}{\xi^2-1}} \cdot (-\xi^{-2})\, d\xi$$

$$= -\xi^{-2}(\pm k + O(\xi^2))\, d\xi = \left(\frac{\mp k}{\xi^2} + (\text{holomorphic at } \xi = 0) \right) d\xi.$$

$\Longrightarrow$ ω_2 has double poles at $\infty_\pm$ without residues: $\operatorname{Res}_{\infty_\pm} \omega_2 = 0$.

$\Longrightarrow$ $G(P) := \displaystyle\int_{P_0 \to P} \omega_2 = \int_{C\text{: contour from } P_0 \text{ to } P} \omega_2$ is

- locally well-defined. (Cauchy's theorem & residues $= 0$.)
- holomorphic in P except at $\infty_\pm$.
- has a simple pole at $\infty_\pm$: $G(P) = \pm\dfrac{k}{\xi} +$ (holomorphic at $\xi = 0$).

Namely, $G(P)$ is a multi-valued meromorphic function on $\bar{\mathcal{R}}$.

Global multi-valuedness: similar to the case of ω_1.

C_0, C_1: curves from P_0 to P.

$\Longrightarrow [C_1 - C_0] = m[A] + n[B]$ for some $m, n \in \mathbb{Z}$.

$$\int_{C_1} \omega_2 = \int_{C_0} \omega_2 + m \int_A \omega_2 + n \int_B \omega_2.$$

$$\int_A \omega_2\text{: } A\textit{-period} \text{ of } \omega_2, \quad \int_B \omega_2\text{: } B\textit{-period} \text{ of } \omega_2.$$

<u>Exercise</u>: Express the A-period of ω_2 in terms of the complete elliptic integral of the second kind.

§6.3 Complex elliptic integral of the third kind

$$\int \frac{dz}{(z^2-a^2)\sqrt{\varphi(z)}}, \qquad \varphi(z) = (1-z^2)(1-k^2z^2).$$

$$\omega_3 := \frac{dz}{(z^2-a^2)\sqrt{\varphi(z)}} = \frac{dz}{(z^2-a^2)w}$$

is holomorphic on the elliptic curve (including $\infty_\pm$) except at four points:

$$(z,w) = (\pm a, \pm\sqrt{(1-a^2)(1-k^2a^2)}).$$

These are *simple poles*.

Exercise: (i) Check these facts. (ii) Compute the residues at poles.

$$H(P) := \int_{P_0 \to P} \omega_3$$

is multi-valued in the neighbourhood of simple poles because of the residue.

And, of course, globally multi-valued because of the A- and B-periods.

$\Longrightarrow H(P)$ is a very complicated multi-valued function.

Remark:

A meromorphic 1-form ω on a Riemann surface is called an *Abelian differential*. It is

- *of the first kind*, when ω is holomorphic everywhere.
- *of the second kind*, when the residue is zero at any pole.
- *of the third kind*, otherwise.

$\Longrightarrow$ ω_1: the first kind, ω_2: the second kind, ω_3: the third kind.

(There are several differnent definitions; e.g.,

- "'an Abelian differential of the third kind' has only simple poles",
- "'an Abelian differential of the second kind' has only one pole of order $\geqq 2$ without residue", etc.)