# Elliptic Functions

# Abel-Jacobi theorem

# §7.1 Abel-Jacobi theorem

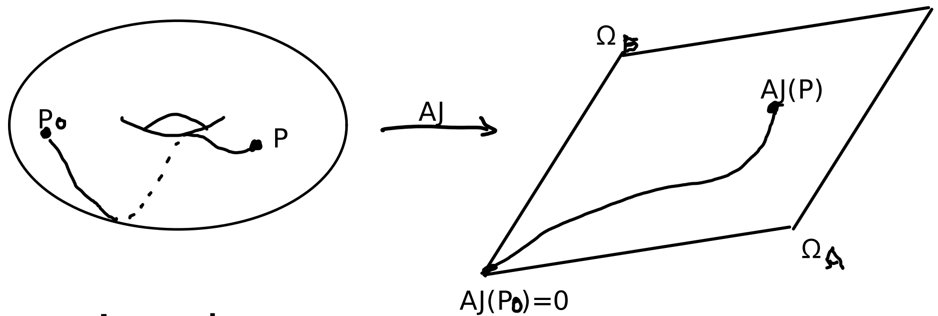
Recall: periods of $\omega_1 = \dfrac{dz}{\sqrt{\varphi(z)}} = \dfrac{dz}{w}$ belong to $\Gamma := \mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B$:

$$\Omega_A := \int_A \omega_1, \qquad \Omega_B := \int_B \omega_1.$$

$\Longrightarrow$ The *Abel-Jacobi map*:

$$AJ : \bar{\mathcal{R}} \ni P \mapsto \int_{P_0}^{P} \omega_1 \bmod \Gamma \in \mathbb{C}/\Gamma$$

is well-defined. ($P_0$: a fixed point in $\bar{\mathcal{R}}$.)




Remark: There is an "Abel-Jacobi map" associated to any compact Riemann surface. The above $AJ$ is a special case.

Theorem (Abel-Jacobi theorem)

(i) The Abel-Jacobi map $AJ$ is bijective.

(ii) It is an isomorphism of complex manifolds between $\bar{\mathcal{R}}$ and $\mathbb{C}/\Gamma$.

Proof of (ii) $\Longleftarrow$ (i):

- $AJ$ is holomorphic ($\Leftarrow$ definition).
- Complex analysis:

  The inverse map of a holomorphic bijection is holomorphic.

$\square$

The essential part of the theorem is *bijectivity* (i).

# §7.2 Surjectivity of $AJ$ (Jacobi's theorem)

Recall:

- The image of a compact set by a continuous map is compact.
- A compact subset of a Hausdorff space is closed.

$$\left.\begin{array}{l} AJ\text{: holomorphic} \Rightarrow \text{continuous.} \\ \bar{\mathcal{R}}\text{: compact.} \end{array}\right\} \Rightarrow AJ(\bar{\mathcal{R}})\text{: } \textit{compact.} \Rightarrow \textit{closed} \text{ in } \mathbb{C}/\Gamma.$$

On the other hand,

- A holomorphic map is open, i.e., the image of an open set is open.

$\Longrightarrow AJ(\bar{\mathcal{R}})$ is *open* in $\mathbb{C}/\Gamma$.

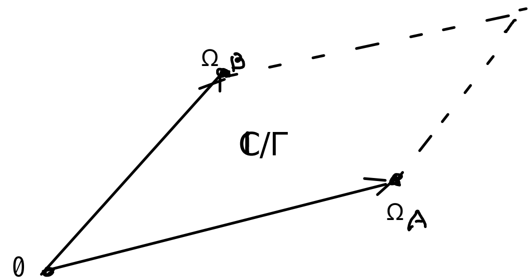
$AJ(\bar{\mathcal{R}})$ is closed & open in $\mathbb{C}/\Gamma$.

$$\Longrightarrow AJ(\bar{\mathcal{R}}) \text{ is a connected component of } \mathbb{C}/\Gamma.$$

*But $\mathbb{C}/\Gamma$ is connected!*

Hence,

$$AJ(\bar{\mathcal{R}}) = \mathbb{C}/\Gamma.$$




□

Corollary:

$\Omega_A$ and $\Omega_B$ are linearly independent over $\mathbb{R}$. In particular, $\Omega_A, \Omega_B \neq 0$.

Proof: $\bar{\mathcal{R}}$: compact $\Longrightarrow \mathbb{C}/\Gamma = AJ(\bar{\mathcal{R}})$: compact.

$\leftrightarrow$ If $\Omega_A$ & $\Omega_B$: linearly dependent/$\mathbb{R}$, $\Gamma = \mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B \subset \mathbb{R}\Omega_A$ or $\mathbb{R}\Omega_B$.

$\Longrightarrow \mathbb{C}/\Gamma$ is not compact. □

## §7.3 Injectivity of $AJ$ (Abel's theorem)

Assumption: $AJ(P_1) = AJ(P_2)$, but $P_1 \neq P_2$.

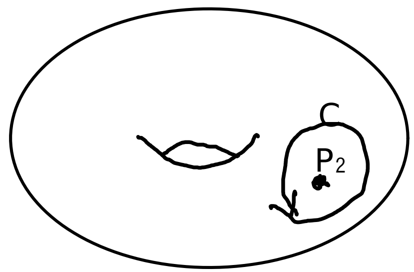
Goal: Construct a meromorphic function $f(z)$ on $\bar{\mathcal{R}}$ such that

- $f$ has a unique pole at $P_2$, which is simple.
- $f(z) = 0 \Leftrightarrow z = P_1$. (This property will not be used.)

*But such $f$ cannot exist!*

Because, as $\omega_1$ is a holomorphic nowhere vanishing differential,

- $f(z)\,\omega_1$ has a simple pole at $P_2$. $\Longrightarrow \int_C f(z)\,\omega_1 \neq 0$.
- $f(z)\,\omega_1$ is holomoprhic elsewhere. $\Longrightarrow \int_C f(z)\,\omega_1 = 0$.




($C$: a small circle around $P_2$; Figure)

Contradiction $\Longrightarrow P_1 = P_2$. □

Construction of $f(z)$:

We define $f(z)$ by

$$f(z) := \exp\left(\int_{Q_0}^{z} \omega_3(P_1, P_0) - \int_{Q_0}^{z} \omega_3(P_2, P_0) - \frac{2\pi i N}{\Omega_A} \int_{Q_0}^{z} \omega_1\right).$$

Notations:

- $Q_0$: a fixed point $\neq P_0, P_1, P_2$.
- $\omega_3(P, Q)$: an normalised Abelian differential of the third kind with simple poles at $P$ and $Q$ normalised by
  - $\mathrm{Res}_P\, \omega_3(P, Q) = 1$, $\mathrm{Res}_Q\, \omega_3(P, Q) = -1$.
  - $\displaystyle\int_A \omega_3(P, Q) = 0.$

  Existence of such $\omega_3$ shall be proved later.
- $N$: an integer determined later.

Need to show:

- $f(z)$ has a simple pole at $P_2$ (and a simple zero at $P_1$).
- $f(z)$ is a single-valued meromorphic function on $\bar{\mathcal{R}}$.

$\underline{f(z) \text{ has a simple pole at } P_2.}$ (The proof of $f(P_1) = 0$ is similar.)

$$\omega_3(P_2, P_0) = \left( \frac{1}{z - P_2} + \text{(holomorphic function)} \right) dz \text{ at } P_2.$$

$$\Longrightarrow \int_{Q_0}^{z} \omega_3(P_2, P_0) = \log(z - P_2) + \text{(holomorphic function)}.$$

When $z \to P_2$, only this term in the definition of $f(z)$ diverges.

$$\Longrightarrow \quad f(z) \sim \exp\left(-\log(z - P_2) + \text{(holomorphic function)}\right)$$
$$= \frac{1}{z - P_2} \times \text{(non-zero holomorphic function)}.$$

□

Single-valuedness of $f(z)$.

Possible multi-valuedness $\leftarrow$ ambiguity of integration contours.

Three types of contours should be checked.

(i) contours around singularities of $\omega_3(P_1, P_0)$ and $\omega_3(P_2, P_0)$.

(ii) contours around the $A$-cycle.

(iii) contours around the $B$-cycle.

• Case (i).

When $z$ goes around $P_1$: (The proofs for $P_2$ and $P_0$ are the same.)

$$\int_{Q_0}^{z \circlearrowleft_{P_1}} \omega_3(P_1, P_0) = \int_{Q_0}^{z} \omega_3(P_1, P_0) + 2\pi i.$$

($z \circlearrowleft_{P_1}$ means that the contour additionally goes around $P_1$.)

$\Longrightarrow f(z) \mapsto f(z) \times e^{2\pi i} = f(z)$. OK!

• Case (ii).

Recall $\int_A \omega_3(P,Q) = 0 \Longrightarrow \int_{Q_0}^{z \circlearrowleft_A} \omega_3(P_i, P_0) = \int_{Q_0}^{z} \omega_3(P_i, P_0)$.

($z \circlearrowleft_A$: the contour additionally goes around the $A$-cycle.)

On the other hand, $\int_{Q_0}^{z \circlearrowleft_A} \omega_1 = \int_{Q_0}^{z} \omega_1 + \Omega_A$.

$\Longrightarrow f(z) \mapsto f(z) \times \exp\left(-\frac{2\pi i N}{\Omega_A}\Omega_A\right) = f(z)$. OK!

• Case (iii).

Lemma: $\exists$ contour $C : Q \to P$ such that

$$\int_B \omega_3(P,Q) = \frac{2\pi i}{\Omega_A} \int_C \omega_1.$$

$\square$

We prove this lemma later.

$$\left(\int_{Q_0}^{z \circlearrowleft_B} \omega_3(P_1, P_0) - \int_{Q_0}^{z \circlearrowleft_B} \omega_3(P_2, P_0)\right)$$
$$- \left(\int_{Q_0}^{z} \omega_3(P_1, P_0) - \int_{Q_0}^{z} \omega_3(P_2, P_0)\right)$$
$$= \int_B \omega_3(P_1, P_0) - \int_B \omega_3(P_2, P_0) \overset{\text{Lemma}}{=} \frac{2\pi i}{\Omega_A} \left(\int_{P_0}^{P_1} \omega_1 - \int_{P_0}^{P_2} \omega_1\right).$$

Assumption $AJ(P_1) = AJ(P_2)$ means

$$\int_{P_0}^{P_1} \omega_1 - \int_{P_0}^{P_2} \omega_1 = M\Omega_A + N\Omega_B$$

for some $M, N \in \mathbb{Z}$. This is the "$N$" in the definition of $f(z)$.

$$
\begin{aligned}
\implies f(z) \mapsto f(z) \exp\left(\frac{2\pi i}{\Omega_A}(M\Omega_A + N\Omega_B) - \frac{2\pi i N}{\Omega_A}\int_B \omega_1\right) \\
= f(z) \exp\left(2\pi i M + \frac{2\pi i N \Omega_B}{\Omega_A} - \frac{2\pi i N}{\Omega_A}\Omega_B\right) \\
= f(z).
\end{aligned}
$$

Single-valuedness proved!! $=$ End of the proof of the Abel-Jacobi theorem.

$\square$

It remains to show:

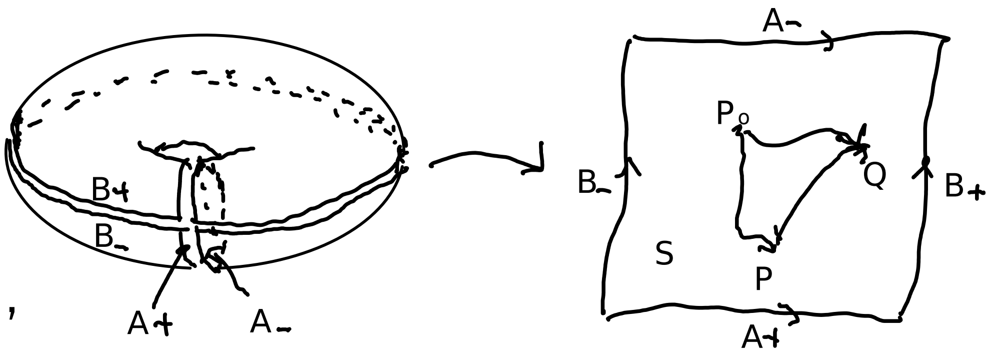
- Lemma.
- Existence of $\omega_3(P, Q)$.

• <u>Proof of the lemma.</u>

$F(z) := \int_{P_0}^{z} \omega_1$: multivalued holomorphic function on $\bar{\mathcal{R}}$

(incomplete elliptic integral of the first kind).

Cut $\bar{\mathcal{R}}$ along $A$- and $B$-cycles to a rectangle $S$: (Figure)




By the residue theorem,

$$\frac{1}{2\pi i}\int_{\partial S} F(z)\,\omega_3(P,Q) = \mathrm{Res}_P\, F(z)\,\omega_3(P,Q) + \mathrm{Res}_Q\, F(z)\,\omega_3(P,Q)$$

$$= F(P) - F(Q) = \int_{P_0}^{P}\omega_1 - \int_{P_0}^{Q}\omega_1 = \int_{Q}^{P}\omega_1.$$

(All the contours are in $S$.)

On the other hand,

$$\int_{\partial S} F(z)\,\omega_3(P,Q) = \left(\int_{A_-} - \int_{A_+} + \int_{B_+} - \int_{B_-}\right) F(z)\,\omega_3(P,Q).$$

From the multi-valuedness of $F(z)$,

$$\begin{aligned} &\int_{A_-} F(z)\,\omega_3(P,Q) - \int_{A_+} F(z)\,\omega_3(P,Q) \\ &= \int_A (F(z) - F(z \circlearrowleft_B))\,\omega_3(P,Q) \\ &= \int_A \left(-\int_B \omega_1\right)\,\omega_3(P,Q) = -\left(\int_B \omega_1\right)\,\left(\int_A \omega_3(P,Q)\right) \\ &= 0. \qquad \left(\text{Recall the normalisation : } \int_A \omega_3(P,Q) = 0.\right) \end{aligned}$$

Similarly,

$$\int_{B_+} F(z)\,\omega_3(P,Q) - \int_{B_-} F(z)\,\omega_3(P,Q)$$
$$= \left(\int_A \omega_1\right)\left(\int_B \omega_3(P,Q)\right) = \Omega_A \int_B \omega_3(P,Q).$$

As a result,

$$2\pi i \int_Q^P \omega_1 = \Omega_A \int_B \omega_3(P,Q).$$

□

- Proof of the existence of $\omega_3(P,Q)$.

We have only to show existence of $\tilde{\omega}_3(P,Q)$ with simple poles at $P$ and $Q$,

$$\mathrm{Res}_P\, \tilde{\omega}_3(P,Q) = 1, \qquad \mathrm{Res}_Q\, \tilde{\omega}_3(P,Q) = -1.$$

Because:

- $\tilde{\omega}_3(P,Q) + \lambda\omega_1$ has the same property for any $\lambda \in \mathbb{C}$.
- $\displaystyle\int_A \omega_1 = \Omega_A \neq 0$.

$\Longrightarrow$ If $\lambda = -\dfrac{1}{\Omega_A}\displaystyle\int_A \tilde{\omega}_3(P,Q)$,

$$\omega_3(P,Q) := \tilde{\omega}_3(P,Q) + \lambda\omega_1$$

satisfies all the conditions, including $\displaystyle\int_A \omega_3(P,Q) = 0$.

• <u>Construction of $\tilde{\omega}_3(P, Q)$.</u>

Recall: $\bar{\mathcal{R}}$ = compactification of $\mathcal{R} = \{(z, w) \mid w^2 = \varphi(z)\}$,

$$\varphi(z) = a(z - \alpha_0)(z - \alpha_1)(z - \alpha_2)(z - \alpha_3).$$

<u>Case I. $P, Q \neq \infty_\pm$.</u>

Denote $P = (z_1, w_1 = \sqrt{\varphi(z_1)})$, $Q = (z_2, w_2 = \sqrt{\varphi(z_2)})$.

(Branches of $\sqrt{\ }$ are defined appropriately.)

$$\tilde{\omega}_3(P, Q) := \frac{1}{2}\left(\frac{w + w_1}{z - z_1} - \frac{w + w_2}{z - z_2}\right)\frac{dz}{w}.$$

<u>Exercise</u>: Check that this $\tilde{\omega}_3(P, Q)$ satisfies the required properties: holomorphic on $\bar{\mathcal{R}} \setminus \{P, Q\}$, simple poles at $P$, $Q$, $\mathrm{Res}_P = 1$, $\mathrm{Res}_Q = -1$.

(Use an appropriate coordinate, especially at $\infty_\pm$ and $(z, w) = (\alpha_i, 0)$!)

Case II. $P = \infty_+$, $Q \neq \infty_\pm$.

Case III. $P = \infty_+$, $Q = \infty_-$.

Exercise: Find $\tilde{\omega}_3(P, Q)$ for the cases II and III.

(Hint: When $z_1 \to \infty$, $w_1 \sim \pm\sqrt{a}\, z_1^2$. $\Longrightarrow \tilde{\omega}_3(P, Q)$ of Case I diverges.

Find an appropriate $\lambda = \lambda(z_1)$ and take $\lim\limits_{z_1 \to \infty} (\tilde{\omega}_3(P, Q) - \lambda\omega_1)$.)

Exercise*: Find $\tilde{\omega}_3(P, Q)$ when $\deg \varphi = 3$.

Remark:

There is such $\omega_3(P, Q)$ on any compact Riemann surface.

The proof requires much analysis!