Elliptic Functions

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- If there are errors in the problems, please fix *reasonably* and solve them.
- The rule of evaluation is:

$$(\text{your final mark}) = \min\{\text{integer part of total points you get}), 10\}$$

- About twenty problems will be given till the end of the semester.
- This rule is subject to change and the latest rule applies.
- The deadline of **16** – **17**: 30 April 2020. (Send the scan or the photo to Takebe.)

16. Check that the 1-form $\tilde{\omega}_3(P,Q)$ on the elliptic curve $\bar{\mathcal{R}}_\varphi$ ($\deg\varphi = 4$) defined by

$$\tilde{\omega}_3(P,Q) := \frac{1}{2}\left(\frac{w+w_1}{z-z_1} - \frac{w+w_2}{z-z_2}\right)\frac{dz}{w}$$

satisfies the properties required in the seminar (16 April 2020) in the following cases (i) and (ii). Here (z_i, w_i) $(i = 1, 2)$ are coordinates of the points P and Q on $\bar{\mathcal{R}}_\varphi$.

(i) (1 pt.) $P = (z_1, w_1)$, $Q = (z_2, w_2)$ are not neither branch points nor $\infty_\pm$.
(Hint: You have to check the holomorphicity separately at (1) $(z, w) \in \mathcal{R}_\varphi$, $z \notin \{z_1, z_2, \alpha_0, \dots, \alpha_3\}$; (2) $(z_i, -w_i) \in \mathcal{R}_\varphi$ $(i = 1, 2)$; (3) $(\alpha_j, 0) \in \mathcal{R}_\varphi$; (4) $\infty_\pm$ (use the coordinate $\xi = 1/z$).)

(ii) (1 pt.) At least one of P and Q is a branch point.

(iii) (1 pt.) Find $\tilde{\omega}_3(P,Q)$ with the same properties when $P = \infty_\pm$ or $Q = \infty_\pm$ or both of them are at infinity. (Hint: When $z_1 \to \infty$, $w_1 \sim \pm\sqrt{a}z_1^2$. Hence naive limit $\lim\limits_{z_1\to\infty} \tilde{\omega}_3(P,Q)$ diverges. Find an appropriate $\lambda = \lambda(z_1)$ and take the limit $\lim\limits_{z_1\to\infty} (\tilde{\omega}_3(P,Q) - \lambda\omega_1)$.)

17*. (2 pt.) Find meromorphic 1-forms as in **16** on the elliptic curve $\bar{\mathcal{R}}_\varphi$ when $\deg\varphi = 3$.