

Elliptic Functions

Elliptic functions (general theory)

§8.1 Definition of Elliptic Functions

At last we have come to the definition of elliptic functions!

Definition.

A meromorphic function on an elliptic curve is called an *elliptic function*.

A direct consequence:

f, g : elliptic functions $\implies f \pm g, fg, f/g$: elliptic functions.

Namely, {elliptic functions} is a field.

By the Abel-Jacobi theorem

$$\text{Elliptic curve} \cong \mathbb{C}/\Gamma, \quad \Gamma = \mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B.$$

Ω_A, Ω_B : linearly independent over \mathbb{R} .

\implies An alternative (*standard*) definition:

A meromorphic function on \mathbb{C} satisfying

$$f(u + \Omega_A) = f(u), \quad f(u + \Omega_B) = f(u)$$

is called an *elliptic function* with periods Ω_A and Ω_B .

Remark: For any Ω_A, Ω_B , \exists an elliptic curve, i.e.,

$\exists \varphi(z)$: polynomial of degree 3 or 4 such that

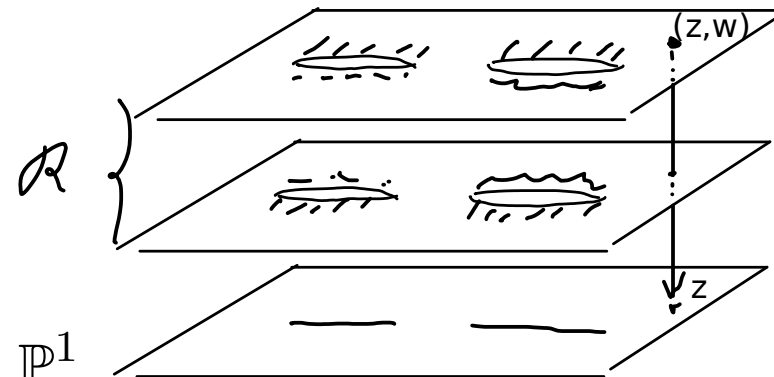
$$\mathbb{C}/\mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B \cong \overline{\{w^2 = \varphi(z)\}}.$$

Proved later. (\longleftarrow differential equation of $\wp(z)$)

Example:

$\varphi(z)$: as before.

$$\begin{aligned} \bar{\mathcal{R}} = \overline{\{(z, w) \mid w^2 = \varphi(z)\}} &\xrightarrow{\text{pr}} \mathbb{P}^1 \\ (z, w) &\mapsto z, \text{ (projection)} \\ \infty \text{ or } \infty_{\pm} &\mapsto \infty. \end{aligned}$$



pr: a holomorphic map (a meromorphic function with a pole at ∞).

\implies Composition

$$\mathbb{C} \xrightarrow{\pi} \mathbb{C}/\Gamma \xrightarrow{AJ^{-1}} \bar{\mathcal{R}} \xrightarrow{\text{pr}} \mathbb{P}^1$$

gives an elliptic function on \mathbb{C} : $f(u) = \text{pr} \circ AJ^{-1} \circ \pi(u)$.

Since AJ is defined by an elliptic integral, this means that

the inverse function of an elliptic integral is an elliptic function!

- $\varphi(z) = 4z^3 - g_2z - g_3$. ($g_2, g_3 \in \mathbb{C}$)

Fix the base point of the Abel-Jacobi map to ∞ :

$$AJ(z) = \int_{\infty}^z \frac{dz}{w} = \int_{\infty}^z \frac{dz}{\sqrt{4z^3 - g_2z - g_3}}.$$

$\wp(u) := \text{pr} \circ AJ^{-1} \circ \pi(u)$: *Weierstrass's \wp function*.

$AJ(\infty) = 0 \implies \wp(0) = \infty$, i.e., $u = 0$ is a pole.

- $\varphi(z) = (1 - z^2)(1 - k^2z^2)$. ($k \in \mathbb{C}$, $k \neq 0, \pm 1$)

Fix the base point of the Abel-Jacobi map to 0:

$$AJ(z) = \int_0^z \frac{dz}{w} = \int_0^z \frac{dz}{\sqrt{(1 - z^2)(1 - k^2z^2)}}.$$

$\text{sn}(u) := \text{pr} \circ AJ^{-1} \circ \pi(u)$: *Jacobi's sn function*.

A natural generalisation of the previously defined sn over \mathbb{R} .

• $\wp(u), \operatorname{sn}(u)$: periodic with periods Ω_A, Ω_B . $\Leftarrow \pi : \mathbb{C} \rightarrow \mathbb{C}/\Gamma$.

• For $\varphi(z) = (1 - z^2)(1 - k^2 z^2)$, we have computed

$$\Omega_A = 4K(k), \quad \Omega_B = 2iK'(k).$$

\implies Periods of $\operatorname{sn}(u)$: $4K(k), 2iK'(k)$.

Consistent with the previous definition for $\operatorname{sn}(x), x \in \mathbb{R}$.

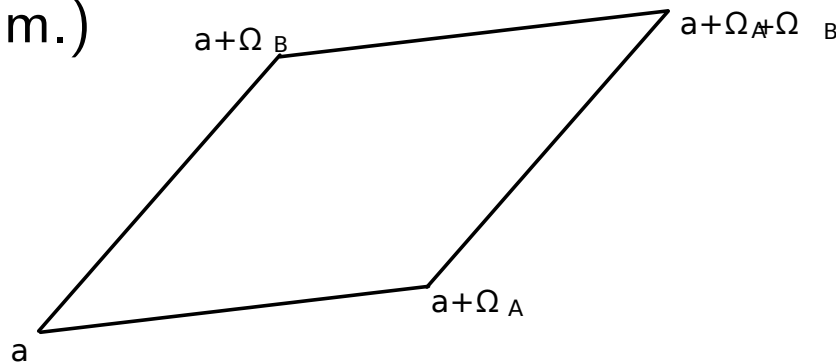
• We construct $\wp(u)$ and $\operatorname{sn}(u)$ by different methods later.

§8.2 General Properties of Elliptic Functions

$f(u)$: an elliptic function on \mathbb{C} with periods Ω_A and Ω_B .

We call a parallelogram spanned by Ω_A & Ω_B a *period parallelogram*.

(Figure of a period parallelogram.)



Theorem (Liouville)

If an elliptic function $f(u)$ is entire, then $f(u)$ is constant.

Proof: f : doubly periodic. $\implies f(\mathbb{C}) = f(\text{period parallelogram})$.

f : continuous & a period parallelogram is bounded. $\implies f(\mathbb{C})$: bounded.

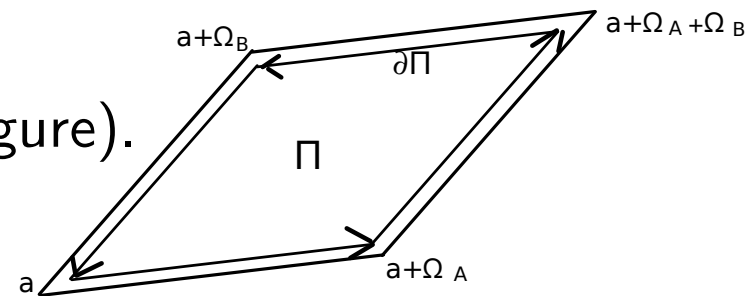
Liouville's theorem (Complex analysis!) $\implies f$: constant. □

Theorem (Liouville)

The sum of residues of $f(u)$ at poles in one period parallelogram is zero.

Proof:

Π : a period parallelogram (poles of $f \notin \partial\Pi$; cf. Figure).



$$\int_{\partial\Pi} f(u) du = 2\pi i(\text{the sum of residues in } \Pi).$$

On the other hand,

$$\int_{\partial\Pi} f(u) du = \left(\int_a^{a+\Omega_A} + \int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} + \int_{a+\Omega_A+\Omega_B}^{a+\Omega_B} + \int_{a+\Omega_B}^a \right) f(u) du.$$

By the periodicity,

$$\int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} f(u) du = \int_a^{a+\Omega_B} f(u) du = - \int_{a+\Omega_B}^a f(u) du,$$
$$\int_{a+\Omega_A+\Omega_B}^{a+\Omega_B} f(u) du = \int_{a+\Omega_A}^a f(u) du = - \int_a^{a+\Omega_A} f(u) du.$$

Summing up, $2\pi i$ (the sum of residues in Π) = $\int_{\partial\Pi} f(u) du = 0$. □

Corollary:

\nexists an elliptic function with only one simple pole in a period parallelogram.

Proof:

Otherwise, the sum of residue = the residue at the simple pole $\neq 0$. □

Remark: We have already proved the same fact in the proof of the Abel-Jacobi theorem ($\nexists F(z)\omega_1$).

Definition:

order of $f = \text{ord } f := \#$ poles with multiplicity in a period parallelogram.

Corollary means “There is no elliptic function of order 1.”

For the next theorem, we need an obvious lemma:

Lemma: $f(u)$: an elliptic function $\implies f'(u)$: an elliptic function.

($f(u + \Omega_A) = f(u + \Omega_B) = f(u) \implies f'(u + \Omega_A) = f'(u + \Omega_B) = f'(u)$)

Theorem:

For any $a \in \mathbb{C}$ and Π : a period parallelogram ($f \neq \infty, a$ on $\partial\Pi$).

$\#$ of $\{u \in \Pi \mid f(u) = a\}$ with multiplicities = $\text{ord } f$.

Proof:

$f(u) - a$: an elliptic function of order $\text{ord } f$. \implies May assume $a = 0$.

$$\begin{aligned} & \#\{\text{zeroes of } f(u) \text{ in } \Pi\} - \#\{\text{poles of } f(u) \text{ in } \Pi\} \\ &= \frac{1}{2\pi i} \oint_{\partial\Pi} \frac{f'(u)}{f(u)} du \quad (\text{argument principle}) \\ &= 0. \quad \left(\begin{array}{l} \text{lemma } \implies f'/f : \text{ elliptic function;} \\ \oint_{\partial\Pi} (\text{elliptic function}) du = 0. \end{array} \right) \end{aligned}$$

□

Theorem: $N := \text{ord } f$, $a \in \mathbb{C}$.

$\alpha_1, \dots, \alpha_N$: points in Π , $f(\alpha_i) = a$ (with multiplicities).

β_1, \dots, β_N : poles of $f(u)$ in Π (with multiplicities).

$$\implies \alpha_1 + \dots + \alpha_N \equiv \beta_1 + \dots + \beta_N \pmod{\mathbb{Z}\Omega_A + \mathbb{Z}\Omega_B}.$$

Proof:

Again, we may assume $a = 0$.

Recall the generalised argument principle in complex analysis:

D : a domain, f : meromorphic, φ : holomorphic in a nbd of \bar{D}

$$\implies \frac{1}{2\pi i} \oint_{\partial D} \varphi(u) \frac{f'(u)}{f(u)} du = \sum_{\alpha \in D: f(\alpha)=0} \varphi(\alpha) - \sum_{\beta \in D: \text{pole of } f} \varphi(\beta).$$

Apply it to $D = \Pi$, $\varphi(u) = u$:

$$\frac{1}{2\pi i} \oint_{\partial \Pi} u \frac{f'(u)}{f(u)} du = \sum_{j=1}^N \alpha_j - \sum_{j=1}^N \beta_j.$$

NOTE: The integrand is *NOT* an elliptic function because of u !

Let us compute

$$\oint_{\partial\Pi} u \frac{f'(u)}{f(u)} du = \left(\int_a^{a+\Omega_A} + \int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} + \int_{a+\Omega_A+\Omega_B}^{a+\Omega_B} + \int_{a+\Omega_B}^a \right) u \frac{f'(u)}{f(u)} du.$$

The second term is

$$\begin{aligned} \int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} u \frac{f'(u)}{f(u)} du &= \int_a^{a+\Omega_B} (u + \Omega_A) \frac{f'(u + \Omega_A)}{f(u + \Omega_A)} du \\ &= - \int_{a+\Omega_B}^a u \frac{f'(u)}{f(u)} du - \Omega_A \int_{a+\Omega_B}^a \frac{f'(u)}{f(u)} du. \end{aligned}$$

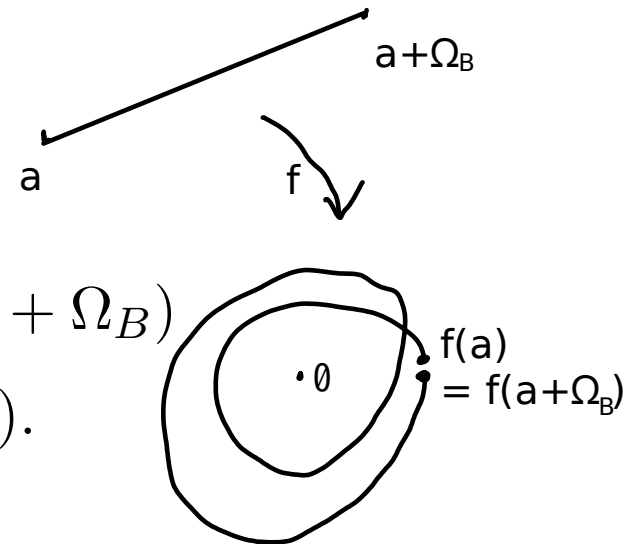
Recall the proof of the “argument principle” on the winding number:

$$\int_{a+\Omega_B}^a \frac{f'(u)}{f(u)} du = \int_{a+\Omega_B}^a d \log f(u) = \log f(a) - \log f(a + \Omega_B).$$

Note that $f(a) = f(a + \Omega_B)$, BUT $\log f(a) \neq \log f(a + \Omega_B)$,

because of multivaluedness of \log .

$$\begin{aligned}\log f(a) &= \log |f(a)| + i \arg f(a), \\ \log f(a + \Omega_B) &= \log |f(a + \Omega_B)| + i \arg f(a + \Omega_B) \\ &= \log |f(a)| + i \arg f(a + \Omega_B).\end{aligned}$$



The argument is determined only up to $2\pi\mathbb{Z}$.

$$\begin{aligned}\log f(a + \Omega_B) - \log f(a) &= i(\arg f(a + \Omega_B) - \arg f(a)) \\ &= 2\pi in. \quad (\exists n \in \mathbb{Z})\end{aligned}$$

$$\implies \int_{a+\Omega_B}^a \frac{f'(u)}{f(u)} du = -2\pi in.$$

$$\implies \int_{a+\Omega_A}^{a+\Omega_A+\Omega_B} u \frac{f'(u)}{f(u)} du = - \int_{a+\Omega_B}^a u \frac{f'(u)}{f(u)} du + 2\pi in \Omega_A.$$

Similarly,

$$\begin{aligned}
\int_{a+\Omega_A+\Omega_B}^{a+\Omega_B} u \frac{f'(u)}{f(u)} du &= \int_{a+\Omega_A}^a (u + \Omega_B) \frac{f'(u + \Omega_B)}{f(u + \Omega_B)} du \\
&= - \int_a^{a+\Omega_A} u \frac{f'(u)}{f(u)} du - \Omega_B \int_a^{a+\Omega_A} \frac{f'(u)}{f(u)} du. \\
&= - \int_a^{a+\Omega_A} u \frac{f'(u)}{f(u)} du + 2\pi i m \Omega_B. \quad (\exists m \in \mathbb{Z})
\end{aligned}$$

Summing up,

$$\sum_{j=1}^N \alpha_j - \sum_{j=1}^N \beta_j = \frac{1}{2\pi i} \oint_{\partial\Pi} u \frac{f'(u)}{f(u)} du = n\Omega_A + m\Omega_B.$$

□