

Elliptic Functions

Weierstrass \wp -function

Recall that

1. an elliptic function $f(u)$ is holomorphic \implies constant.
2. \nexists an elliptic function of order 1.

\implies The simplest non-trivial elliptic function has $\left\{ \begin{array}{l} \text{one double pole,} \\ \text{or} \\ \text{two simple poles,} \end{array} \right.$

in a period parallelogram; $\wp(u)$ is the former, $\text{sn}(u)$ is the latter.

We have defined $\wp(u)$ as

$$\text{the inverse function of } u(z) = \int_{\infty}^z \frac{dx}{\sqrt{4x^3 - g_2x - g_3}}.$$

Here we construct it as a doubly periodic function by a series.

§9.1 Construction of Weierstrass \wp -function

Notations:

- $\Omega_1, \Omega_2 \in \mathbb{C}$: linearly independent over \mathbb{R} .
- $\Gamma := \mathbb{Z}\Omega_1 + \mathbb{Z}\Omega_2$.

Goal: Construct a “simple” elliptic function with double poles at Γ .

An elliptic function $f(u)$ with poles of order n at Γ is expanded as:

$$f(u) = \frac{c}{(u - m_1\Omega_1 - m_2\Omega_2)^n} + \dots$$

at $u = m_1\Omega_1 + m_2\Omega_2 \in \Gamma$.

\implies The simplest candidate of elliptic functions with poles of order n at Γ :

$$f_n(u) := \sum_{m_1, m_2 \in \mathbb{Z}} \frac{1}{(u - m_1\Omega_1 - m_2\Omega_2)^n}.$$

Theorem: Assume $n \geq 3$.

- The series $f_n(u)$ converges absolutely and uniformly on any compact set in $\mathbb{C} \setminus \Gamma$.
- $f_n(u)$ is an elliptic function with poles of order n at Γ .
- $f_n(u)$: even when n is even, odd when n is odd.

Proof:

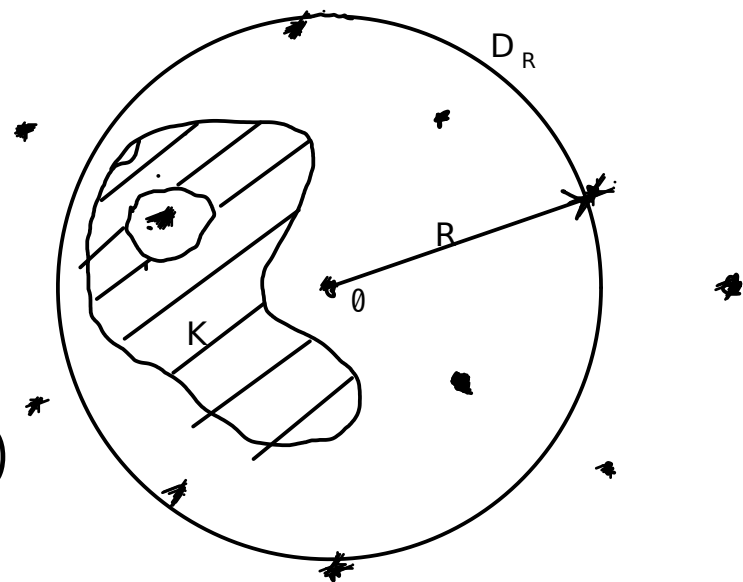
$K \subset \mathbb{C} \setminus \Gamma$: compact.

$D_R := \{z \in \mathbb{C} \mid |z| \leq R\}$: a closed disk. (cf. Figure.)

R : sufficiently large so that $K \subset D_R$.

Enough to show: $f_{n,R}(u) := \sum_{\substack{\Omega \in \Gamma, \\ \Omega \notin D_{2R}}} \frac{1}{(u - \Omega)^n}$ converges absolutely

and uniformly on D_R . ($f_n = f_{n,R} +$ (finite terms).)



$$\frac{1}{(u - \Omega)^n} = \frac{1}{\Omega^n} \frac{1}{\left(\frac{u}{\Omega} - 1\right)^n}.$$

Lemma:

1) $\exists M > 0$ s.t. $\frac{1}{\left(\frac{u}{\Omega} - 1\right)^n} < M$ for $u \in D_R, \Omega \in \Gamma \setminus D_{2R}$.

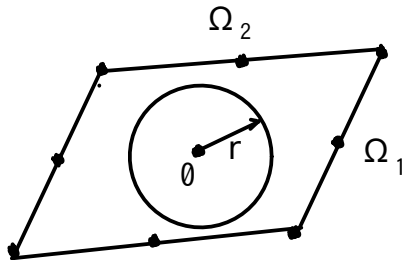
2) $\sum_{\Omega \in \Gamma, \Omega \neq 0} \frac{1}{|\Omega|^n}$ converges for $n \geq 3$.

Weierstrass's M -test $\implies f_{n,R}$ converges absolutely and uniformly on D_R .

Proof of 2):

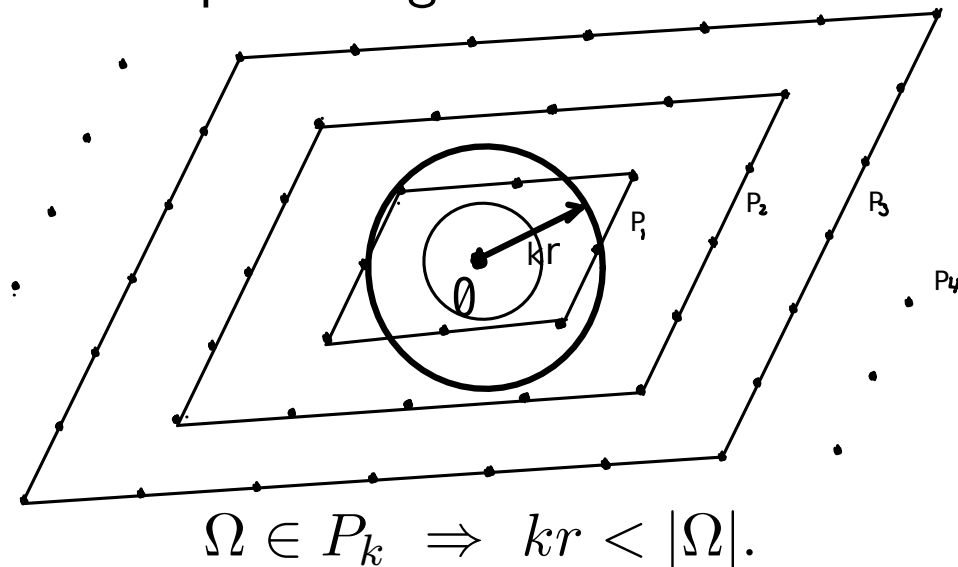
r : radius of a disk with centre $0 \subsetneq$ parallelogram with vertices $\pm\Omega_1 \pm \Omega_2$.

(Figure).



$P_k: \Gamma \cap$ (boundary of the parallelogram with vertices $\pm k\Omega_1 \pm k\Omega_2$).

(Figure)



\Rightarrow

$$\sum_{\Omega \in \Gamma, \Omega \neq 0} \frac{1}{|\Omega|^n} = \sum_{k=1}^{\infty} \sum_{\Omega \in P_k} \frac{1}{|\Omega|^n} < \sum_{k=1}^{\infty} 8k \frac{1}{k^n r^n} = \frac{8}{r^n} \sum_{k=1}^{\infty} \frac{1}{k^{n-1}},$$

which converges when $n \geq 3$.

□ Lemma 2)

The second statement of the theorem \iff the first:

- each summand in f_n is holomorphic in $\mathbb{C} \setminus \Gamma$
 $\implies f_n$ is holomorphic in $\mathbb{C} \setminus \Gamma$. (Weierstrass' theorem).
- $\frac{1}{(u - \Omega)^n}$ has a pole of order n at $\Omega \in \Gamma$.

The third statement:

$$f_n(-u) = \sum_{\Omega \in \Gamma} \frac{1}{(-u - \Omega)^n} = \sum_{\Omega' (= -\Omega) \in \Gamma} \frac{(-1)^n}{(u - \Omega')^n} = (-1)^n f_n(u).$$

□Theorem

However, $\sum_{\Omega \in \Gamma, \Omega \neq 0} \frac{1}{|\Omega|^2}$ diverges! \implies Theorem is not true for $n = 2$.

Need “correction” to each summand.

Theorem: The series

$$\wp(u) := \frac{1}{u^2} + \sum_{\Omega \in \Gamma, \Omega \neq 0} \left(\frac{1}{(u - \Omega)^2} - \frac{1}{\Omega^2} \right)$$

- converges absolutely and uniformly on any compact set in $\mathbb{C} \setminus \Gamma$.
- gives an even elliptic function with poles of order 2 at Γ .

Namely,

$\wp(u)$ is an elliptic function of order 2: *Weierstrass's \wp -function*.

Proof:

We know: $f_3(u) = \sum_{\Omega \in \Gamma} \frac{1}{(u - \Omega)^3}$ is an elliptic function.

Idea: Integrate f_3 to get \wp !

Integrate $f_3(u)$ without the first term $\frac{1}{u^3}$ from 0:

$$\begin{aligned}
 \int_0^u \left(f_3(v) - \frac{1}{v^3} \right) dv &= \int_0^u \left(\sum_{\Omega \in \Gamma \setminus \{0\}} \frac{1}{(v - \Omega)^3} \right) dv \\
 &= \sum_{\Omega \in \Gamma \setminus \{0\}} \int_0^u \frac{1}{(v - \Omega)^3} dv \quad (\Leftarrow \text{uniform convergence}) \\
 &= -\frac{1}{2} \sum_{\Omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(u - \Omega)^2} - \frac{1}{\Omega^2} \right).
 \end{aligned}$$

(Absolutely and uniformly convergent on a compact set $\subset ((\mathbb{C} \setminus \Gamma) \cup \{0\})$.)

\implies

- $\wp(u) = \frac{1}{u^2} - 2 \int_0^u \left(f_3(v) - \frac{1}{v^3} \right) dv$: meromorphic with poles at Γ .
- $\wp'(u) = -2f_3(u)$.

- Evenness:

$$\begin{aligned}
 \wp(-u) &= \frac{1}{(-u)^2} + \sum_{\Omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(-u - \Omega)^2} - \frac{1}{\Omega^2} \right) \\
 &= \frac{1}{u^2} + \sum_{\Omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(u + \Omega)^2} - \frac{1}{\Omega^2} \right) \\
 &= \frac{1}{u^2} + \sum_{\Omega' \in \Gamma \setminus \{0\}} \left(\frac{1}{(u - \Omega')^2} - \frac{1}{\Omega'^2} \right) = \wp(u), \quad (\Omega' = -\Omega).
 \end{aligned}$$

- Periodicity:

$f_3(u)$: elliptic function

$$\implies \wp'(u + \Omega_1) = \wp'(u), \quad \wp'(u + \Omega_2) = \wp'(u).$$

$$\implies \exists C_1, C_2: \wp(u + \Omega_1) = \wp(u) + C_1, \quad \wp(u + \Omega_2) = \wp(u) + C_2.$$

Setting $u = -\frac{\Omega_i}{2}$, $C_i = \wp\left(\frac{\Omega_i}{2}\right) - \wp\left(-\frac{\Omega_i}{2}\right) \stackrel{\wp: \text{even}}{=} 0.$

□

- Other properties of $\wp(u)$.

Laurent expansion at $u = 0$:

$$\begin{aligned}\wp(u) &= \frac{1}{u^2} + \sum_{\Omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(u - \Omega)^2} - \frac{1}{\Omega^2} \right) \\ &= \frac{1}{u^2} + c_0 + c_2 u^2 + \cdots + c_{2n} u^{2n} + \cdots.\end{aligned}$$

$$\begin{aligned}c_{2n} &= \frac{1}{(2n)!} \left. \frac{d^{2n}}{du^{2n}} \right|_{u=0} \left(\sum_{\Omega \in \Gamma \setminus \{0\}} \left(\frac{1}{(u - \Omega)^2} - \frac{1}{\Omega^2} \right) \right) \\ &= \begin{cases} 0 & (n = 0), \\ (2n + 1) \sum_{\Omega \in \Gamma \setminus \{0\}} \frac{1}{\Omega^{2n+2}} & (n \neq 0). \end{cases}\end{aligned}$$

By convention: $g_2 := 20c_2 = 60 \sum \frac{1}{\Omega^4}$, $g_3 := 28c_4 = 140 \sum \frac{1}{\Omega^6}$.

With these notations,

$$\wp(u) = \frac{1}{u^2} + \frac{g_2}{20}u^2 + \frac{g_3}{28}u^4 + O(u^6),$$
$$\wp'(u) = -\frac{2}{u^3} + \frac{g_2}{10}u + \frac{g_3}{7}u^3 + O(u^5).$$

Hence,

$$\wp'(u)^2 = \frac{4}{u^6} - \frac{2g_2}{5} \frac{1}{u^2} - \frac{4g_3}{7} + O(u),$$
$$-4\wp(u)^3 = -\frac{4}{u^6} - \frac{3g_2}{5} \frac{1}{u^2} - \frac{3g_3}{7} + O(u),$$
$$g_2 \wp(u) = g_2 \frac{1}{u^2} + O(u^2).$$

Summing up,

$$\wp'(u)^2 - 4\wp(u)^3 + g_2 \wp(u) = -g_3 + O(u).$$

$\wp'(u)^2 - 4\wp(u) + g_2 \wp(u)$: elliptic function with possible poles at Γ .

$-g_3 + O(u)$: no pole at 0.

$\implies \wp'(u)^2 - 4\wp(u) + g_2 \wp(u)$: elliptic function without poles = constant.

Namely, $\wp'(u)^2 - 4\wp(u) + g_2 \wp(u) = -g_3$, or,

$$\wp'(u)^2 = 4\wp(u)^3 - g_2 \wp(u) - g_3.$$

This gives the equivalence of definitions:

$$\frac{d\wp}{du} = \sqrt{4\wp^3 - g_2 \wp - g_3}, \text{ i.e., } du = \frac{d\wp}{\sqrt{4\wp^3 - g_2 \wp - g_3}}.$$

Integrate from $u = 0$ ($\leftrightarrow \wp(u) = \infty$) to u ($\leftrightarrow \wp(u)$):

$$u = \int_{\infty}^{\wp(u)} \frac{dz}{\sqrt{4z^3 - g_2 z - g_3}}.$$

$\implies \wp(u)$ is the inverse function of the elliptic integral!

In fact,

$$W : \mathbb{C}/\Gamma \ni u \mapsto (\wp(u), \wp'(u)) \in \bar{\mathcal{R}}$$

is the inverse of the Abel-Jacobi map AJ .

$\bar{\mathcal{R}}$: the elliptic curve = compactification of $\{(z, w) \mid w^2 = 4z^3 - g_2 z - g_3\}$.

Exercise: Prove the bijectivity of W as follows:

(i) Show that W is *holomorphic* even at $u = 0$ as a map to $\bar{\mathcal{R}}$.

(ii) Show that $\wp'(\Omega_i/2) = 0$ for $i = 1, 2, 3$ ($\Omega_3 = \Omega_1 + \Omega_2$).

(iii) Show the bijectivity.

(Hint: $\wp(u)$ is of order 2, i.e., takes any value $\in \mathbb{P}^1$ twice on \mathbb{C}/Γ .)

Exercise: Prove that any elliptic function $f(u)$ with period Γ is expressed as follows:

$$f(u) = R_1(\wp(u)) + R_2(\wp(u)) \wp'(u), \quad R_1, R_2 : \text{rational functions.}$$

§9.2 Addition formulae of the \wp -function

Elliptic curve $\cong \mathbb{C}/\Gamma$ has an additive group structure:

$$u_1 \bmod \Gamma + u_2 \bmod \Gamma = u_1 + u_2 \bmod \Gamma.$$

\implies addition formulae of elliptic functions.

Theorem (Addition formula of \wp).

If $u_1 + u_2 + u_3 = 0$ (or $\equiv 0 \pmod{\Gamma}$),

$$\begin{vmatrix} \wp'(u_1) & \wp(u_1) & 1 \\ \wp'(u_2) & \wp(u_2) & 1 \\ \wp'(u_3) & \wp(u_3) & 1 \end{vmatrix} = 0.$$

(Note: $\wp'(u_3) = -\wp'(u_1 + u_2)$, $\wp(u_3) = \wp(u_1 + u_2)$.)

Proof:

May assume $\wp(u_1) \neq \wp(u_2)$.

(\implies analytically continued to all values afterwards.)

(a, b) : a solution of

$$a\wp(u_1) + b = \wp'(u_1),$$

$$a\wp(u_2) + b = \wp'(u_2).$$

Explicit formulae (*not* used in the proof, used in the exercise):

$$a = \frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)}, \quad b = \frac{\wp(u_1)\wp'(u_2) - \wp'(u_1)\wp(u_2)}{\wp(u_1) - \wp(u_2)}.$$

$f(u) := \wp'(u) - a\wp(u) - b$: an elliptic function of the third order, because

- linear combination of elliptic functions.
- a third order pole (that of $\wp'(u)$) at $u = 0$.

$\implies \exists$ three points, at which $f = 0$.

We know two of them: $f(u_1) = f(u_2) = 0$. Let us call the third one u_0 .

By the general theorem for elliptic functions:

$$u_0 + u_1 + u_2 \equiv (\text{sum of poles}) = 0 \pmod{\Gamma}.$$

$\implies u_0 \equiv u_3 \pmod{\Gamma}$, i.e., $f(u_3) = 0$.

$$f(u_1) = f(u_2) = f(u_3) = 0 \Leftrightarrow \begin{pmatrix} \wp'(u_1) & \wp(u_1) & 1 \\ \wp'(u_2) & \wp(u_2) & 1 \\ \wp'(u_3) & \wp(u_3) & 1 \end{pmatrix} \begin{pmatrix} 1 \\ -a \\ -b \end{pmatrix} = 0.$$

$\begin{pmatrix} 1 \\ -a \\ -b \end{pmatrix} \neq 0 \implies$ The matrix is degenerate, i.e., $\det = 0$. □

Corollary:

$$\wp(u_1 + u_2) = -\wp(u_1) - \wp(u_2) + \frac{1}{4} \left(\frac{\wp'(u_1) - \wp'(u_2)}{\wp(u_1) - \wp(u_2)} \right)^2.$$

Proof: Exercise.

Hint: u_1, u_2 and u_3 satisfy

$$\wp'(u)^2 = 4\wp(u)^3 - g_2\wp(u) - g_3, \quad \wp'(u) = a\wp(u) + b.$$

$\implies \wp(u_1), \wp(u_2)$ and $\wp(u_3)$ satisfy a cubic equation.

Remark: The addition formula has a geometric interpretation.

(cf. Exercise.)