Elliptic Integrals and Elliptic Functions

Theta functions

$\S$10.1 Definition of θ-function

Recall: everywhere holomorphic elliptic function = constant.

$\Leftarrow$ The condition "doubly periodic" was too strong.

Let us consider weaker condition of "*quasi-periodicity*".

Remark & notations:

Up to now, we used $(\Omega_1, \Omega_2) \in \mathbb{C}^2$ as periods: $f(u+\Omega_i) = f(u)$ $(i = 1, 2)$.

Renormalise the variable $u \mapsto u/\Omega_1$. $\Longrightarrow$ periods $= (1, \Omega_2/\Omega_1)$.

May assume $\mathrm{Im}\,\Omega_2/\Omega_1 > 0$. (If not, use $-\Omega_2/\Omega_1$ instead.)

Hereafter, periods $= (1, \tau)$, $\mathrm{Im}\,\tau > 0$.

The period lattice $\Gamma = \mathbb{Z} + \mathbb{Z}\tau$.

Let us find an entire function with (multiplicative) quasi-periodicity:

$$f(u+1) = f(u), \qquad f(u+\tau) = e^{au+b} f(u).$$

- The parameter a cannot be arbitrary: Compute $f(u+1+\tau)$ in two ways:

$$\begin{aligned} f(u+1+\tau) &= f(u+\tau) = e^{au+b} f(u) \\ &= e^{a(u+1)+b} f(u+1) = e^{au+a+b} f(u). \end{aligned}$$

$\Longrightarrow e^a = 1$ (if f is not 0), i.e., $a = 2\pi i k$ ($k \in \mathbb{Z}$).

- Periodicity $f(u+1) = f(u) \Rightarrow$ Fourier expansion: $f(u) = \displaystyle\sum_{n \in \mathbb{Z}} a_n e^{2\pi i n u}$.

Quick proof for an entire function:

$g(v) := f\left(\dfrac{\log v}{2\pi i}\right)$: well-defined on $\mathbb{C} \setminus \{0\}$ (ambiguity of $\dfrac{\log v}{2\pi i} \in \mathbb{Z}$).

Laurent expansion $g(v) = \sum a_n v^n \Rightarrow f(u) = \sum a_n e^{2\pi i n u}$.

- Quasi-periodicity $f(u+\tau) = e^{2\pi iku+b} f(u) \Rightarrow$ recursion relation for a_n.

$$f(u+\tau) = \sum a_n e^{2\pi in\tau} e^{2\pi inu},$$
$$e^{2\pi iku+b} f(u) = \sum a_n e^{2\pi iku+b} e^{2\pi inu} = \sum e^b a_n \, e^{2\pi i(n+k)u}$$

$\Longrightarrow a_n = e^{-2\pi in\tau+b} a_{n-k}$.

<u>Exercise</u>:

Show that if $k = 0$, then $f(u) = \alpha e^{2\pi inu}$ ($\exists \; \alpha \in \mathbb{C}$, $n \in \mathbb{Z}$).

- <u>Case $k > 0$</u>:

Fix $n = km + n_0$ ($0 \leqq n_0 < k$).

$$a_n = e^{-2\pi in\tau+b} \, a_{n-k} = e^{-2\pi in\tau+b} e^{-2\pi i(n-k)\tau+b} \, a_{n-k} = \cdots$$
$$= e^{-2\pi i\left(n+(n-k)+\cdots+(n_0+k)\right)+mb} \, a_{n_0} = e^{-\pi im(m+1)k-2\pi imn_0+mb} \, a_{n_0}.$$

Recall: $g(v) = \sum a_n v^n$ is holomorphic on $\mathbb{C} \setminus \{0\}$.

$$a_n = \frac{1}{2\pi i} \oint_{|v|=R} \frac{g(v)}{v^{n+1}}\, dv \implies |a_n| \leqq \frac{M}{R^n} \xrightarrow{n\to\infty} 0 \quad (M := \max_{|v|=R} |g(v)|).$$

On the other hand,

$$|a_{km+n_0}| = \left| a_{n_0} e^{-2\pi i m n_0 + mb} \right| e^{\pi m(m+1)k \operatorname{Im}\tau} \sim (\text{const.}) \times q^{m^2} \ (q = e^{\pi k \operatorname{Im}\tau} > 1).$$

This diverges when $m \to \infty$: Contradiction! $\implies \nexists\, f(u)$.

- Case $k = -1$:

$$a_n = a_0 e^{\pi i n(n-1)\tau - nb}, \text{ i.e., } f(u) = a_0 \sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n\left(u - \frac{b}{2\pi i} - \frac{\tau}{2}\right)}.$$

Definition: $\theta(u,\tau) := \displaystyle\sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n u}$: θ-function.

$$\implies f(u) = a_0\, \theta\!\left(u - \tfrac{b}{2\pi i} - \tfrac{\tau}{2}, \tau\right).$$

<u>Lemma (Convergence of θ)</u>:

The series $\sum_{n\in\mathbb{Z}} e^{\pi i n^2 \tau + 2\pi i n u}$ converges absolutely and uniformly on

$$\{(u,\tau) \mid |\operatorname{Im} u| \leqq C,\ \operatorname{Im}\tau \geqq \varepsilon\} \qquad (\forall C > 0,\ \varepsilon > 0).$$

<u>Proof</u>:

$$\left| e^{\pi i n^2 \tau + 2\pi i n u} \right| = e^{-\pi n^2 \operatorname{Im}\tau - 2\pi n \operatorname{Im} u} \leqq e^{-\pi n^2 \varepsilon} e^{2\pi |n| C}.$$

$e^{-\pi n^2 \varepsilon} \searrow 0$ much faster than $e^{2\pi|n|C} \nearrow +\infty$.

$\Longrightarrow \sum e^{-\pi n^2 \varepsilon} e^{2\pi|n|C}$ converges.

$\Longrightarrow \sum e^{\pi i n^2 \tau + 2\pi i n u}$ converges absolutely and uniformly. $\square$

By Weierstrass's theorem, the θ-function $\theta(u,\tau)$ is

- entire in u,
- holomorphic in τ on $\mathbb{H} := \{\tau \mid \operatorname{Im}\tau > 0\}$.

$$\theta(u+1,\tau)=\theta(u,\tau), \qquad \theta(u+\tau,\tau)=e^{-\pi i\tau-2\pi iu}\theta(u,\tau).$$

In general,

$$\theta(u+m+n\tau,\tau)=e^{-\pi in^2\tau-2\pi inu}\,\theta(u,\tau) \qquad (m,n\in\mathbb{Z}).$$

Exercise:

Show that for $k\in\mathbb{Z}_{>0}$ the space of entire functions satisfying

$$f(u+1)=f(u), \qquad f(u+\tau)=e^{-2\pi iku+b}f(u)$$

is of dimension k. Construct a basis of this space, using θ-functions.

We need variants of the θ-function: *θ-functions with characteristics*.

$a, b \in \mathbb{R}$: characteristics (Usually $a, b \in \mathbb{Q}$, most often $a, b \in \{0, \frac{1}{2}\}$)

$$\theta_{a,b}(u) = \theta_{a,b}(u, \tau) := \sum_{n \in \mathbb{Z}} e^{\pi i (n+a)^2 \tau + 2\pi i (n+a)(u+b)}.$$

Easily checked:

- $\theta_{a,b}(u, \tau)$: entire in u, holomorphic in τ on $\mathbb{H}$.
- $\theta_{00}(u) = \theta(u)$.
- $\theta_{a,b+b'}(u) = \theta_{a,b}(u + b')$.
- $\theta_{a+a',b}(u) = e^{\pi i a'^2 \tau + 2\pi i a'(u+b)} \theta_{a,b}(u + a'\tau)$.
- $\theta_{a+p,b+q}(u) = e^{2\pi i a q} \theta_{a,b}(u)$ for $p, q \in \mathbb{Z}$.

Hereafter only $\theta_{a,b}(u)$ with $a, b \in \{0, \frac{1}{2}\}$ are used.

$\Longrightarrow$ Shorthand notations: $\theta_{\varepsilon_1 \varepsilon_2}(u, \tau) := \theta_{\varepsilon_1/2, \varepsilon_2/2}(u, \tau)$ $(\varepsilon_1, \varepsilon_2 \in \{0, 1\})$.

Remark:

This is Mumford's notations in "Tata Lectures on Theta".

Correspondence with more common notations:

$\theta_1(u) = -\theta_{11}(u)$, $\theta_2(u) = \theta_{10}(u)$, $\theta_3(u) = \theta_{00}(u)$, $\theta_4(u) = \theta_{01}(u)$.

Relations with $\theta(u, \tau)$:

$$\theta_{00}(u) = \theta(u), \qquad \theta_{01}(u) = \theta(u + \tfrac{1}{2}),$$

$$\theta_{10}(u) = e^{\pi i \tau/4 + \pi i u}\theta(u + \tfrac{\tau}{2}), \quad \theta_{11}(u) = e^{\pi i \tau/4 + \pi i (u+1/2)}\theta(u + \tfrac{1+\tau}{2}).$$

§10.2 Properties of θ-functions

- Quasi-periodicity and parity.

For $k, l \in \{0, 1\} = \mathbb{Z}/2\mathbb{Z}$ (i.e., "$1 + 1 = 0$"), the following are easily checked:

- $\theta_{kl}(u + \frac{1}{2}) = (-1)^{kl}\theta_{k,l+1}(u)$.
- $\theta_{kl}(u + \frac{\tau}{2}) = (-i)^l e^{-\pi i\tau/4-\pi iu}\theta_{k+1,l}(u)$.
- $\theta_{kl}(u)$: even if $(k, l) \neq (1, 1)$, odd if $(k, l) = (1, 1)$.

- Zeros.

Lemma: $\theta_{kl}(u)$ has only one zero in each period parallelogram.

Proof:

$\theta_{kl}(u) =$ (non-zero function)$\times\theta(u + \text{shift})$.

$\Longrightarrow$ sufficient to show the lemma for $\theta(u)$.

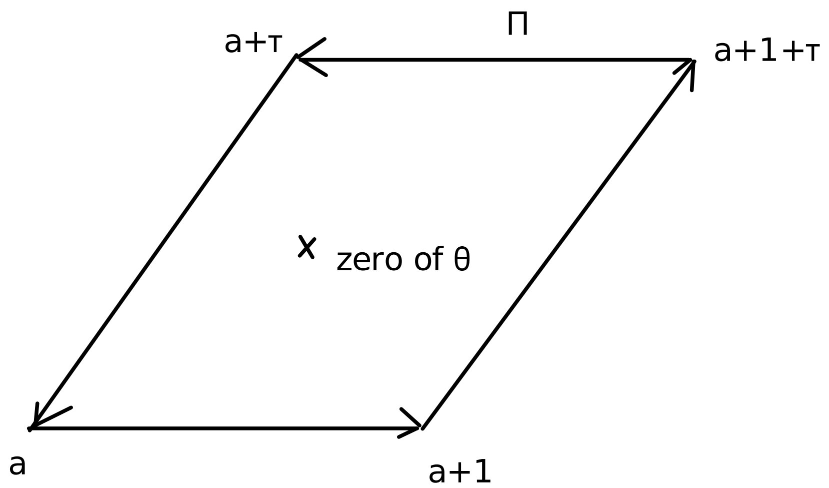
Differentiate the transformation rule:

$$\log\theta(u+1) = \log\theta(u), \quad \log\theta(u+\tau) = \log\theta(u) - \pi i\tau - 2\pi i u.$$

$$\Longrightarrow \frac{d}{du}\log\theta(u+1) = \frac{d}{du}\log\theta(u), \quad \frac{d}{du}\log\theta(u+\tau) = \frac{d}{du}\log\theta(u) - 2\pi i.$$

Recall the argument principle: $\Pi =$ a period parallelogram (cf. Figure),

$$\sharp \text{ of zeros of } \theta(u) \text{ in } \Pi = \frac{1}{2\pi i}\oint_{\partial\Pi} \frac{d}{du}\log\theta(u)\,du.$$



$$
\begin{aligned}
\oint_{\partial\Pi} \frac{d}{du}\log\theta(u)\,du &= \int_a^{a+1} + \int_{a+1}^{a+1+\tau} + \int_{a+1+\tau}^{a+\tau} + \int_{a+\tau}^{a} \\
&= \int_a^{a+1} \left(\frac{d}{du}\log\theta(u) - \frac{d}{du}\log\theta(u+\tau) \right) du \\
&\quad + \int_{a+\tau}^{a} \left(\frac{d}{du}\log\theta(u) - \frac{d}{du}\log\theta(u+1) \right) du \\
&= \int_a^{a+1} (2\pi i)\,du = 2\pi i.
\end{aligned}
$$

$\Longrightarrow$ ($\sharp$ of zeros of $\theta(u)$ in Π) $= 1$. $\square$

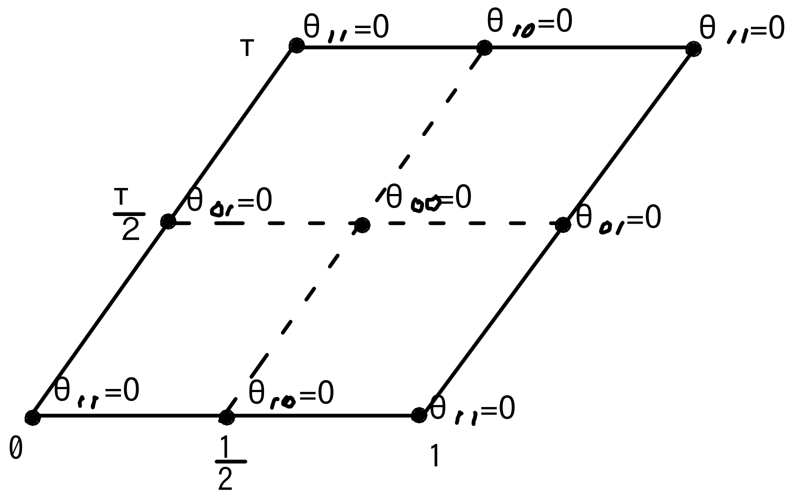
$\theta_{11}(u)$: odd $\Rightarrow \theta_{11}(0) = 0$. $\overset{\text{Lemma}}{\Longrightarrow}$ $\{$zeros of $\theta_{11}(u)\} = \Gamma$.

Because $\theta_{10}(u) = -\theta_{11}(u + \frac{1}{2})$, $\theta_{01}(u) = (\text{non-zero}) \times \theta_{11}(u + \frac{\tau}{2})$, $\theta_{00}(u) = \theta_{01}(u + \frac{1}{2})$,

$$\theta_{00}(u) = 0 \Leftrightarrow u \in \Gamma + \frac{1+\tau}{2}, \qquad \theta_{01}(u) = 0 \Leftrightarrow u \in \Gamma + \frac{\tau}{2},$$

$$\theta_{10}(u) = 0 \Leftrightarrow u \in \Gamma + \frac{1}{2}, \qquad \theta_{11}(u) = 0 \Leftrightarrow u \in \Gamma.$$

(Figure: zeros of θ's.)



- Jacobi's θ-relations = Analogue of addition formulae.

Theorem

Let $A = \frac{1}{2}\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ 1 & -1 & 1 & -1 \\ 1 & -1 & -1 & 1 \end{pmatrix}$ and $\vec{y} = A\vec{x}$, where $\vec{x} := \begin{pmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{pmatrix}$, $\vec{y} := \begin{pmatrix} y_1 \\ y_2 \\ y_3 \\ y_4 \end{pmatrix}$.
Then,

$$\text{(J0)} \quad \prod_{j=1}^{4} \theta_{00}(x_j) + \prod_{j=1}^{4} \theta_{01}(x_j) + \prod_{j=1}^{4} \theta_{10}(x_j) + \prod_{j=1}^{4} \theta_{11}(x_j) = 2\prod_{j=1}^{4} \theta_{00}(y_j).$$

Remark on the name:

- In [Mumford: Tata Lectures on Theta I]: "Riemann's relation".
- In [Whittaker and Watson: A Course in Modern Analysis] Jacobi's work is cited.

Proof:

Notations: $(\vec{a},\vec{b}) = a_1b_1 + \cdots + a_4b_4$ for $\vec{a},\vec{b} \in \mathbb{C}^4$, $\prod = \prod_{j=1}^{4}$, $\sum = \sum_{\vec{m}\in\mathbb{Z}^4}$.

By the definition of θ's,

$$\prod \theta_{00}(x_j) = \sum \exp\left(\pi i\tau(\vec{m},\vec{m}) + 2\pi i(\vec{m},\vec{x})\right),$$
$$\prod \theta_{01}(x_j) = \sum \exp\left(\pi i\tau(\vec{m},\vec{m}) + 2\pi i(\vec{m},\vec{x}) + \pi i(m_1 + \cdots + m_4)\right),$$
$$\prod \theta_{10}(x_j) = \sum \exp\left(\pi i\tau(\vec{m}',\vec{m}') + 2\pi i(\vec{m}',\vec{x})\right),$$
$$\prod \theta_{11}(x_j) = \sum \exp\left(\pi i\tau(\vec{m}',\vec{m}') + 2\pi i(\vec{m}',\vec{x}) + \pi i(m_1' + \cdots + m_4')\right),$$

where $\vec{m}' = \left(m_i + \frac{1}{2}\right)_{i=1,\ldots,4}$.

When they are summed up,

- $m_1 + \cdots + m_4$ or $m_1' + \cdots + m_4'$: odd $\Rightarrow$ The summands cancel.
- $m_1 + \cdots + m_4$ or $m_1' + \cdots + m_4'$: even $\Rightarrow$ The summands are doubled.

$$\Longrightarrow \quad \text{The LHS of (J0)} = 2{\sum}' \exp\left(\pi i\tau(\vec{m},\vec{m}) + 2\pi i(\vec{m},\vec{x})\right),$$

where ${\sum}'$ = the sum over $\vec{m} \in \frac{1}{2}\mathbb{Z}$, satisfying either (i) or (ii):

(i) $\forall m_i \in \mathbb{Z}$ and $m_1 + \cdots + m_4 \in 2\mathbb{Z}$ ($\Leftarrow$ θ_{00}, θ_{01}).

(ii) $\forall m_i \in \frac{1}{2} + \mathbb{Z}$ and $m_1 + \cdots + m_4 \in 2\mathbb{Z}$ ($\Leftarrow$ θ_{10}, θ_{11}).

A: orthogonal, i.e., ${}^tA\,A = \mathrm{Id}_4$. $\Rightarrow$ For $\vec{n} := A\vec{m}$,

- $(\vec{m},\vec{m}) = (\vec{n},\vec{n})$, $(\vec{m},\vec{x}) = (\vec{n},\vec{y})$.
- $\vec{m}$ satisfies (i) or (ii) $\Longleftrightarrow \vec{n} \in \mathbb{Z}^4$.

$$\begin{aligned}{\sum}' \exp\left(\pi i\tau(\vec{m},\vec{m}) + 2\pi i(\vec{m},\vec{x})\right) &= \sum_{\vec{n}\in\mathbb{Z}^4} \exp\left(\pi i\tau(\vec{n},\vec{n}) + 2\pi i(\vec{n},\vec{y})\right) \\ &= \prod_{j=1}^{4} \theta_{00}(y_j). \quad \square\end{aligned}$$

$\exists$ More than twenty variants. We need the following.

Corollary:

(J1) $\prod \theta_{00}(x_j) - \prod \theta_{01}(x_j) - \prod \theta_{10}(x_j) + \prod \theta_{11}(x_j) = 2 \prod \theta_{11}(y_j)$.

(J2) $\prod \theta_{00}(x_j) + \prod \theta_{01}(x_j) - \prod \theta_{10}(x_j) - \prod \theta_{11}(x_j) = 2 \prod \theta_{01}(y_j)$.

(J3)

$$
\begin{aligned}
&\theta_{00}(x_1)\theta_{01}(x_2)\theta_{10}(x_3)\theta_{11}(x_4) + \theta_{01}(x_1)\theta_{00}(x_2)\theta_{11}(x_3)\theta_{10}(x_4) \\
&+\theta_{10}(x_1)\theta_{11}(x_2)\theta_{00}(x_3)\theta_{01}(x_4) + \theta_{11}(x_1)\theta_{10}(x_2)\theta_{01}(x_3)\theta_{00}(x_4) \\
&= 2\theta_{11}(y_1)\theta_{10}(y_2)\theta_{01}(y_3)\theta_{00}(y_4).
\end{aligned}
$$

Proof:

Shift arguments in (J0):

$$x_1 \mapsto x_1 + 1 + \tau \Longrightarrow \begin{cases} \theta_{00}(x_1 + 1 + \tau) = e^{-\pi i \tau - 2\pi i x_1} \theta_{00}(x_1), \text{ etc.} \\ y_j \mapsto y_j + \frac{1+\tau}{2}, \end{cases}$$

$$\Longrightarrow \text{(J1)}.$$

$$x_1 \mapsto x_1 + 1 \qquad \Longrightarrow \text{(J2)}.$$

$$\left.\begin{array}{ll} x_1 \mapsto x_1, & x_2 \mapsto x_2 + \frac{1}{2}, \\ x_3 \mapsto x_3 + \frac{\tau}{2}, & x_4 \mapsto x_4 + \frac{1+\tau}{2} \end{array}\right\} \Longrightarrow \text{(J3)}$$

$\square$

Notations: $\theta_{kl} := \theta_{kl}(0)$. (Note: $\theta_{11} = 0$.)

Corollary: (Addition formulae; $\exists$ Many variants.)

(A1)
$$\begin{aligned}\theta_{00}(x+u)\theta_{00}(x-u)\theta_{00}^2 &= \theta_{00}(x)^2\theta_{00}(u)^2 + \theta_{11}(x)^2\theta_{11}(u)^2 \\ &= \theta_{01}(x)^2\theta_{01}(u)^2 + \theta_{10}(x)^2\theta_{10}(u)^2.\end{aligned}$$

(A2)
$$\theta_{01}(x+u)\theta_{01}(x-u)\theta_{01}^2 = \theta_{01}(x)^2\theta_{01}(u)^2 - \theta_{11}(x)^2\theta_{11}(u)^2.$$

(A3)
$$\theta_{11}(x+u)\theta_{01}(x-u)\theta_{10}\theta_{00} = \theta_{00}(x)\theta_{10}(x)\theta_{01}(u)\theta_{11}(u)+\theta_{01}(x)\theta_{11}(x)\theta_{00}(u)\theta_{10}(u).$$

Proof:

Specialisation of x_j's:

$x_1 = x_2 = x$, $x_3 = x_4 = u \Longrightarrow y_1 = x+u$, $y_2 = x-u$, $y_3 = y_4 = 0$.

(J1) $\overset{\text{specialisation}}{\Longrightarrow}$

$$\theta_{00}(x)^2\theta_{00}(u)^2 - \theta_{01}(x)^2\theta_{01}(u)^2 - \theta_{10}(x)^2\theta_{10}(u)^2 + \theta_{11}(x)^2\theta_{11}(u)^2$$
$$= 2\theta_{11}(x+u)\theta_{11}(x-u)\theta_{11}^2 = 0$$

$\Longrightarrow$ second equation in (A1).

(J0) $\overset{\text{specialisation}}{\Longrightarrow}$

$$\theta_{00}(x)^2\theta_{00}(u)^2 + \theta_{01}(x)^2\theta_{01}(u)^2 + \theta_{10}(x)^2\theta_{10}(u)^2 + \theta_{11}(x)^2\theta_{11}(u)^2$$
$$= 2\theta_{00}(x+u)\theta_{00}(x-u)\theta_{00}^2.$$

$$\begin{cases} \text{LHS} = 2(\theta_{00}(x)^2\theta_{00}(u)^2 + \theta_{11}(x)^2\theta_{11}(u)^2) & \text{(second eq. in (A1))} \\ \text{RHS} = 2 \times \text{LHS of (A1).} \end{cases}$$

$\Longrightarrow$ first equation in (A1).

Similarly, (J2) $\overset{\text{specialisation}}{\Longrightarrow}$ (A2), (J3) $\overset{\text{specialisation}}{\Longrightarrow}$ (A3). $\square$

• Heat equation.

The Fourier series defining θ's converge uniformly on compact sets.

$\Longrightarrow$ May differentiate termwise.

$$\frac{\partial^2}{\partial u^2} e^{\pi i(n+a)^2\tau + 2\pi i(n+a)(u+b)} = -4\pi^2 (n+a)^2\, e^{\pi i(n+a)^2\tau + 2\pi i(n+a)(u+b)},$$
$$\frac{\partial}{\partial \tau} e^{\pi i(n+a)^2\tau + 2\pi i(n+a)(u+b)} = \pi i\, (n+a)^2\, e^{\pi i(n+a)^2\tau + 2\pi i(n+a)(u+b)}.$$

$$\Longrightarrow \quad \frac{\partial}{\partial \tau}\theta_{kl}(u,\tau) = \frac{1}{4\pi i}\frac{\partial^2}{\partial u^2}\theta_{kl}(u,\tau).$$

For $t > 0$, $x \in \mathbb{R}$, this is the *heat equation*:

$$\frac{\partial}{\partial t}\theta_{kl}(x,it) = \frac{1}{4\pi}\frac{\partial^2}{\partial x^2}\theta_{kl}(x,it).$$

• Jacobi's derivative formula.

Notations: $\theta_{kl} = \theta_{kl}(0,\tau)$ as before, $\theta'_{11} := \left.\dfrac{\partial}{\partial u}\right|_{u=0} \theta_{11}(u,\tau)$.

Theorem:

$$\theta'_{11} = -\pi \theta_{00}\theta_{01}\theta_{10}.$$

Proof:

(J3) with $x_1 = x$, $x_2 = x_3 = x_4 = 0$:

$$\theta_{11}(x)\,\theta_{10}\,\theta_{01}\,\theta_{00} = 2\,\theta_{11}\left(\frac{x}{2}\right)\theta_{10}\left(\frac{x}{2}\right)\theta_{01}\left(\frac{x}{2}\right)\theta_{00}\left(\frac{x}{2}\right).$$

Substitute the Taylor expansion:

$$\theta_{kl}(x) = \theta_{kl} + \frac{\theta''_{kl}}{2}x^2 + O(x^4) \qquad ((k,l)\neq(1,1)),$$
$$\theta_{11}(x) = \theta'_{11}x + \frac{\theta'''_{11}}{6}x^3 + O(x^5).$$

(Recall: $\theta_{kl}(x)$ $((k,l)\neq(1,1))$: even, $\theta_{11}(x)$: odd.)

Coefficients of x^3:

$$\frac{1}{6}\theta'''_{11}\theta_{10}\theta_{01}\theta_{00} = \frac{1}{24}\theta'''_{11}\theta_{10}\theta_{01}\theta_{00} + \frac{1}{8}\theta'_{11}(\theta''_{10}\theta_{01}\theta_{00} + \theta_{10}\theta''_{01}\theta_{00} + \theta_{10}\theta_{01}\theta''_{00}).$$

$$\Longrightarrow \frac{\theta'''_{11}}{\theta'_{11}} - \frac{\theta''_{00}}{\theta_{00}} - \frac{\theta''_{01}}{\theta_{01}} - \frac{\theta''_{10}}{\theta_{10}} = 0.$$

The heat equation $\Rightarrow \theta''_{kl} = 4\pi i \dfrac{\partial}{\partial\tau}\theta_{kl}$ ($(k,l) \neq (1,1)$), $\theta'''_{11} = 4\pi i \dfrac{\partial}{\partial\tau}\theta'_{11}$.

$$\Longrightarrow 0 = \frac{\frac{\partial}{\partial\tau}\theta'_{11}}{\theta'_{11}} - \frac{\frac{\partial}{\partial\tau}\theta_{00}}{\theta_{00}} - \frac{\frac{\partial}{\partial\tau}\theta_{01}}{\theta_{01}} - \frac{\frac{\partial}{\partial\tau}\theta_{10}}{\theta_{10}} = \frac{\partial}{\partial\tau}\log\frac{\theta'_{11}}{\theta_{00}\,\theta_{01}\,\theta_{10}}.$$

$$\Longrightarrow \frac{\theta'_{11}}{\theta_{00}\,\theta_{01}\,\theta_{10}} = \text{constant in } \tau.$$

The constant $=$ the value at $\tau \to \infty$, or $q = e^{\pi i\tau} \to 0$.

Expand θ_{kl}'s and θ'_{11} in q.

$$
\begin{aligned}
\theta_{00} &= \sum e^{\pi i n^2 \tau} && = 1 + O(q), \\
\theta_{01} &= \sum e^{\pi i n^2 \tau + \pi i n} && = 1 + O(q), \\
\theta_{10} &= \sum e^{\pi i (n+\frac{1}{2})^2 \tau} && = 2q^{1/4} + O(q), \\
\theta'_{11} &= \sum \pi i (2n+1) e^{\pi i (n+\frac{1}{2})^2 \tau + \pi i (n+\frac{1}{2})} && = -2\pi q^{1/4} + O(q).
\end{aligned}
$$

$$
\frac{\theta'_{11}}{\theta_{00}\,\theta_{01}\,\theta_{10}} = \lim_{q\to 0} \frac{-2\pi q^{1/4} + O(q)}{2q^{1/4} + O(q)} = -\pi.
$$

$\square$

Remark:

θ-functions appear in

- algebraic geometry,
- number theory (especially $\theta_{ab}(0)$),
- representation theory (as characters of ∞-dim. representations),
- mathematical physics,
- etc.

Exercise:

$a_i, b_i, c \in \mathbb{C}$ $(i = 1, \dots, N)$, $\sum a_i = \sum b_i$

$\Longrightarrow f(u) = c\,\dfrac{\theta_{11}(u - a_1) \cdots \theta_{11}(u - a_N)}{\theta_{11}(u - b_1) \cdots \theta_{11}(u - b_N)}$: an elliptic function.

Any elliptic function with periods 1 and τ has this form.