Exercises on topology 17.01.2020

1. Prove that (a) $\mathbb{RP}^n \cong S^n/\operatorname{O}(1) = S^n/\{\pm 1\}$;
(b) $\mathbb{CP}^n \cong S^{2n+1}/\operatorname{U}(1) = S^{2n+1}/S^1$;
(c) The space of quaternionic 1-dimensional subspaces in $\mathbb{H}^{n+1}$ (quaternionic projective space) $\mathbb{HP}^n \cong S^{4n+3}/\operatorname{USp}(1) = S^{4n+3}/S^3$ (recall that $\operatorname{USp}(n)$ is the subgroup of quaternionic-unitary transformations in $\operatorname{GL}(n, \mathbb{H})$).

2. Prove that (a) $\operatorname{Gr}(k, n, \mathbb{R}) \cong \operatorname{O}(n)/(\operatorname{O}(k) \times \operatorname{O}(n-k))$;
(b) The space of *oriented* k-subspaces in $\mathbb{R}^n$ (Grassmannian of oriented subspaces) $\operatorname{Gr}_+(k, n, \mathbb{R}) \cong \operatorname{SO}(n)/(\operatorname{SO}(k) \times \operatorname{SO}(n-k))$;
(c) $\operatorname{Gr}(k, n, \mathbb{C}) \cong \operatorname{U}(n)/(\operatorname{U}(k) \times \operatorname{U}(n-k))$;
(d) $\operatorname{Gr}(k, n, \mathbb{H}) \cong \operatorname{USp}(n)/(\operatorname{USp}(k) \times \operatorname{USp}(n-k))$.

3. Prove that (a) $\operatorname{Gr}_+(2, 4, \mathbb{R}) \cong S^2 \times S^2$; (b) $\operatorname{SO}(4) \cong S^3 \times \operatorname{SO}(3)$.

4. Let $p\colon S^n \to \mathbb{RP}^n$ be the natural projection. Prove that
(a) the cone $\operatorname{Cone}(p) \cong \mathbb{RP}^{n+1}$;
(b) The smash product $S^m \# S^n \cong S^{m+n}$;
(c) The suspension $\Sigma X \cong X \# S^1$.

5. Prove that (a) the infinite-dimensional sphere S^∞ is contractible;
(b) The cone $\operatorname{Cone} X$ is contractible for any X.
(c) The space of based paths $E(X, x)$ (subspace of $C(I, X)$) is contractible for any $X \ni x$.

Exercises on topology 24.01.2020

1. Prove that (a) $\pi_n(X, x) = \pi_0(\Omega^n X)$, where $\Omega^n X = \Omega\Omega \ldots \Omega X$ (the iterated loop space based at x);
(b) $\pi_n(X, x) = \pi_k(\Omega^{n-k} X, x)$.

2. A space Y is called an H-space (H for H. Hopf) if we are given maps $\mu\colon Y \times Y \to Y$ and $\nu\colon Y \to Y$ such that:
$\mu \circ (\operatorname{Id}_Y \times \mu) \sim \mu \circ (\mu \times \operatorname{Id}_Y)\colon Y \times Y \times Y \to Y$ (homotopy associativity);
$\mu \circ j_1 \sim \operatorname{Id}_Y \sim \mu \circ j_2\colon Y \to Y$ where $j_1(y) = (y_0, y)$, $j_2(y) = (y, y_0)$ (homotopy unit $y_0 \in Y$);
$\mu \circ (\operatorname{Id}_Y \times \nu) \sim \varepsilon \sim \mu \circ (\nu \times \operatorname{Id}_Y)\colon Y \to Y$, where $\varepsilon(y) = y_0$ (homotopy inverse).
Prove that (a) $X \to \operatorname{Ho}_b(X, Y)$ (homotopy classes of based maps) is a functor into the category of groups if and only if Y is an H-space;
(b) $X \to \operatorname{Ho}_b(X, Y)$ is a functor into the category of abelian groups if and only if Y is homotopically commutative.

3. Dually, a space Y is called an H'-space if we are given maps $\Delta\colon Y \to Y \vee Y$ (coproduct) and $\nu\colon Y \to Y$ such that:
$(\operatorname{Id}_Y \vee \Delta) \circ \Delta \sim (\Delta \vee \operatorname{Id}_Y) \circ \Delta\colon Y \to Y \vee Y \vee Y$ (homotopy coassociativity);

$\varpi_1 \circ \Delta \sim \mathrm{Id}_Y \sim \varpi_2 \circ \Delta\colon Y \to Y$ where ϖ_1, ϖ_2 are the contractions of $y_0 \vee Y,\ Y \vee y_0$ (homotopy counit);

$(\mathrm{Id}_Y \vee \nu) \circ \Delta \sim \varepsilon \sim (\nu \vee \mathrm{Id}_Y) \circ \Delta\colon Y \to Y$ (homotopy coinverse).

Prove that (a) $X \to \mathrm{Ho}_b(Y, X)$ (homotopy classes of based maps) is a functor into the category of groups if and only if Y is an H'-space;

(b) $X \to \mathrm{Ho}_b(Y, X)$ is a functor into the category of abelian groups if and only if Y is homotopically cocommutative.

4. Let X be a topological space. Prove that

(a) ΩX is an H-space; (b) $\Omega\Omega X$ is homotopically commutative;

(c) ΣX is an H'-space; (d) $\Sigma\Sigma X$ is homotopically cocommutative.

5. Given $f\colon A \to X$ and $g\colon A \to Y$ we define the *coproduct*

$$X \sqcup_A Y := (X \sqcup Y)/(f(a) \sim g(a))$$

and maps

$$i\colon X \to X \sqcup Y \to X \sqcup_A Y,\ j\colon Y \to X \sqcup Y \to X \sqcup_A Y.$$

The corresponding square is called *cocartesian*:

$$\begin{tikzcd} A \arrow[r, "g"'] \arrow[d, "f"] & Y \arrow[d, "j"'] \\ X \arrow[r, "i"] & X \sqcup_A Y \end{tikzcd}$$

Prove that (a) $\mathrm{Cyl}(f\colon X \to Y) = (X \times I) \sqcup_X Y$;

(b) $C(X \sqcup_A Y, Z) = C(X, Z) \times_{C(A,Z)} C(Y, Z)$;

(c) If g is a cofibration, then i is a cofibration.

Exercises on topology 31.01.2020

1. Let $f\colon A \to X$ be a continuous map. Prove that

(a) f is a cofibration if and only if $f \times \mathrm{Id}_Y\colon A \times Y \to X \times Y$ is a cofibration;

(b) The composition of cofibrations is a cofibration;

(c) f is a cofibration if and only if the canonical map $i\colon\ \mathrm{Cyl}(f) \to X \times I$ is retractable, that is $\exists\ r : X \times I \to \mathrm{Cyl}(f)$ such that $r \circ i = \mathrm{Id}_{\mathrm{Cyl}(f)}$.

2. Let $i\colon A \to X$ be a cofibration. Prove that (a) i is injective;

(b) If $i(A)$ is closed in X, then the topology of A is induced from X (such a pair (X, A) is called a *Borsuk pair*).

3. Let $A \subset X$ be a Borsuk pair, $\widetilde{A} := \mathrm{Cyl}(A \to X),\ i\colon \widetilde{A} \to X \times I$ is the canonical map and $r\colon X \times I \to \widetilde{A}$ is a retraction of problem 1(c). Set $i(r(x,t)) =: (\xi(x,t), \tau(x,t))$ and define $\varphi(x) := \max_{t \in I}(t - \tau(x,t))$. Prove that

(a) $\varphi\colon X \to I$ is continuous and $\{x \in X : \varphi(x) = 0\} = A$;

(b) $\overline{H}(x,t,s) := (\xi(x,(1-s)t),(1-s)\tau(x,t)+st)$ is a homotopy $(X\times I)\times I \to (X\times I)$ between $i\circ r$ and $\mathrm{Id}_{X\times I}$;

(c) The extension of homotopy property holds true for the cofibration $\widetilde{A}\to X\times I$ (set $H(x,t,s) := \overline{H}(x,t,s/\varphi(x))$ for $s\le\varphi(x)$ and $(x,t)\notin\widetilde{A}$, otherwise $H(x,t,s) := (x,t)$.).

4. Prove that if (X,A) is a Borsuk pair and X is locally compact, then the restriction morphism $C(X,Y)\to C(A,Y)$ is a fibration for any Y.

5. Using the fibrations $S^\infty\to\mathbb{RP}^\infty$ and $S^\infty\to\mathbb{CP}^\infty$ find all the homotopy groups of (a) $\mathbb{RP}^\infty$; (b) $\mathbb{CP}^\infty$.

Exercises on topology 07.02.2020

1. Recall that a homotopy between two morphisms $f^\bullet,g^\bullet\colon A^\bullet\to B^\bullet$ of complexes is a collection of maps $h^i\colon A^i\to B^{i-1}$ such that $h^i\circ d_A + d_B h^{i-1} = f^i - g^i$. Prove that
(a) the relation of being homotopic is an equivalence relation on $\mathrm{Hom}(A^\bullet,B^\bullet)$;
(b) A contractible complex $C^\bullet$ (i.e. such that $\mathrm{Id}_{C^\bullet}$ is homotopic to 0) is acyclic;
(c) The set of morphisms in $\mathrm{Hom}(A^\bullet,A^\bullet)$ homotopic to 0 forms a two-sided ideal.

2. Prove that an acyclic complex of free abelian groups is contractible.

3. Let $(C^\bullet,\delta)$ be a complex of abelian groups. We define a new complex $(D^\bullet,d)$ by $D^i := \mathrm{Hom}(C^{-i},\mathbb{Z})$, $df(c) := f(\delta c)$. Prove that
(a) This defines a functor from the category of complexes of abelian groups to itself, and this functor takes homotopical morphisms to homotopical morphisms;
(b) If $C^\bullet$ is contractible, then $D^\bullet$ is contractible as well;
(c) If $C^\bullet$ is acyclic and consists of free abelian groups, then $D^\bullet$ is also acyclic and consists of free abelian grops;
(d) Give an example of an acyclic complex $C^\bullet$ such that $D^\bullet$ is not acyclic.

4. Let $f\colon K^\bullet\to L^\bullet$ be a morphism of complexes. Prove that
(a) $\mathrm{Cone}(f)^i := L^i\oplus K^{i+1}$, $d^i_{\mathrm{Cone}(f)}(l^i,k^{i+1}) := (d^i_L(l_i)+f^{i+1}(k^{i+1}), -d^{i+1}_K(k^{i+1}))$ is a complex;
(b) $\pi\colon L^\bullet\to\mathrm{Cone}(f)^\bullet$, $l^i\mapsto(l^i,0)$, is a morphism of complexes;
(c) $\delta\colon\ \mathrm{Cone}(f)^\bullet\to K^{\bullet+1}$, $(l^i,k^{i+1})\mapsto k^{i+1}$, is a morphism of complexes (here the differential in the shifted complex $K^{\bullet+1} =: K[1]^\bullet$ is multiplied by -1 by definition);
(d) The sequence

$$\ldots\to H^n(K^\bullet)\xrightarrow{f}H^n(L^\bullet)\xrightarrow{\pi}H^n(\mathrm{Cone}(f)^\bullet)\xrightarrow{\delta}H^{n+1}(K^\bullet)\xrightarrow{f}H^{n+1}(L^\bullet)\to\ldots$$

is exact.

5. Let $f\colon K^\bullet\to L^\bullet$ be a morphism of bounded complexes of free abelian groups inducing an isomorphism on cohomology. Prove that
(a) $\mathrm{Cone}(f)^\bullet$ is a bounded acyclic complex of free abelian groups;

(b) $\mathrm{Cone}(f)^\bullet$ is contractible;

(c) f is a homotopical equivalence.

Exercises on topology 14.02.2020

1. For complexes $A^\bullet, B^\bullet$ we define $\mathrm{Hom}^i(A^\bullet, B^\bullet) := \bigoplus_{n\in\mathbb{Z}} \mathrm{Hom}(A^n, B^{n+i})$.

(a) Define a differential $d\colon \mathrm{Hom}^i(A^\bullet, B^\bullet) \to \mathrm{Hom}^{i+1}(A^\bullet, B^\bullet)$;

(b) Thus the category of complexes acquires a structure of differential graded category (dg-category). Prove that $H^0(\mathrm{Hom}^\bullet(A^\bullet, B^\bullet))$ is the usual space of morphisms from $A^\bullet$ to $B^\bullet$ up to homotopy.

2. Prove that (a) the quotient space of a CW-complex X modulo a CW-subcomplex A is a CW-complex again;

(b) The suspension of a (pointed) CW-complex is a CW-complex again;

(c) If X, Y are CW-complexes, $A \subset Y$ is a CW-subcomplex, and $\varphi\colon A \to X$ is a cellular map, then $X \cup_\varphi Y$ is a CW-complex again.

(d) The cylinder and the cone of a cellular map are CW-complexes;

(e) The bouquet of CW-complexes is a CW-complex;

(f) The bouquet of n-spheres is the n-skeleton of the product of these spheres.

3. Let X, Y be CW-complexes. We define a topology on $X \times Y$ by the axiom (W) and denote the resulting space by $X \times_W Y$. Prove that

(a) The tautological map $X \times_W Y \to X \times Y$ is continuous;

(b) If X or Y are locally finite, then $X \times_W Y \simeq X \times Y$;

(c) If X and Y are locally countable, then $X \times_W Y \simeq X \times Y$.

4. Construct a cellular decomposition of (a) S^∞; (b) D^∞.

5. Consider an embedding $\mathbb{RP}^k \hookrightarrow \mathbb{RP}^n,\ (x_0, \ldots, x_k) \mapsto (x_0, \ldots, x_k, 0, \ldots, 0)$.

(a) Prove that $\mathbb{RP}^n = \bigcup_{k=0}^n (\mathbb{RP}^k \setminus \mathbb{RP}^{k-1})$ is a cellular decomposition;

(b) Construct a cellular decomposition of $\mathbb{CP}^n$ and of $\mathbb{HP}^n$.

Exercises on topology 21.02.2020

1. Let $k \geq \alpha_1 \geq \alpha_2 \geq \ldots \geq \alpha_{n-k} \geq 0$ be a collection of nonnegative integers. We define $e_\alpha \subset \mathrm{Gr}(k, n, \mathbb{R})$ as $e_\alpha := \{U \subset \mathbb{R}^n : \dim(U \cap \mathbb{R}^m) = m - j \text{ for any } k - \alpha_j + j \leq m < k - \alpha_{j+1} + j + 1\}$ (where we set $\alpha_0 = k,\ \alpha_{n-k+1} = 0$, and $\mathbb{R}^1 \subset \mathbb{R}^2 \subset \ldots \subset \mathbb{R}^{n-1} \subset \mathbb{R}^n$ is a fixed complete flag). Prove that

(a) $e_\alpha \simeq \mathbb{R}^{\alpha_1 + \ldots + \alpha_{n-k}}$;

(b) $\mathrm{Gr}(k, n, \mathbb{R}) = \bigcup_\alpha e_\alpha$ is a cellular decomposition.

2. Let $\tau \in S_n$ be a permutation. Define

$$e_\tau = \big\{U_1 \subset \ldots \subset U_{n-1} \subset \mathbb{R}^n : \dim(U_i \cap \mathbb{R}^j) = \sharp\{p \leq i : \tau_p \leq j\}\ \big\} :$$

a subset of the space of complete flags in $\mathbb{R}^n$. Prove that

(a) $e_\tau \simeq \mathbb{R}^{\ell(\tau)}$, where $\ell(\tau)$ is the length (the number of disorders, i.e. $\sharp\{1 \le i < j \le n : \tau_i > \tau_j\}$) of τ;

(b) $F\ell(\mathbb{R}^n) = \bigcup_{\tau \in S_n} e_\tau$ is a cellular decomposition of the space of complete flags in $\mathbb{R}^n$.

3. Prove that (a) the 2-dimensional sphere with g handles has a cellular decomposition with a unique 2-cell and $2g$ 1-cells;

(b) the real projective plane with g handles has a cellular decomposition with a unique 2-cell and $2g+1$ 1-cells;

(c) the Klein bottle with g handles has a cellular decomposition with a unique 2-cell and $2g+2$ 1-cells.

4. A map $f\colon X \to Y$ is called a *weak homotopy equivalence* if $f_*\colon \pi_n(X) \to \pi_n(Y)$ is an isomorphism for any n. Prove that if $f\colon X \to Y$ is a weak homotopy equivalence, then

(a) For any CW-complex Z the morphism $f_*\colon \mathrm{Ho}(Z,X) \to \mathrm{Ho}(Z,Y)$ (homotopy classes of maps) is a bijection;

(b) If X and Y are CW-complexes, then f is a homotopy equivalence.

5. Prove that (a) for any Y there is a CW-complex X and a weak homotopy equivalence $f\colon X \to Y$ (it is called a *cellular approximation* of Y);

(b) If $g\colon Z \to Y$ is another weak homotopy equivalence, there is a unique up to homotopy map $h\colon X \to Z$ such that $g \circ h \sim f$;

(c) A cellular approximation is unique up to a homotopy equivalence.

Exercises on topology 28.02.2020

1. Prove that a simply-connected CW-complex is contractible if it has trivial

(a) homotopy groups; (b) homology groups; (c) cohomology groups.

2. Prove that (a) any two $K(\pi,n)$ spaces are weakly homotopy equivalent;

(b) The homotopy groups of S^2 and $S^3 \times \mathbb{CP}^\infty$ are the same, but they are not homotopy equivalent;

(c) The homotopy groups of $S^m \times \mathbb{RP}^n$ and $S^n \times \mathbb{RP}^m$ are the same, but they are not homotopy equivalent for $1 < m \ne n > 1$.

3. Prove that (a) $\mathbb{CP}^\infty = K(\mathbb{Z},2)$; (b) $\mathbb{RP}^\infty = K(\mathbb{Z}/2\mathbb{Z},1)$;

(c) The lense space $L_m^\infty := S^\infty/\mu_m$ (the m-th roots of unity group $\mu_m \subset S^1$ acts freely on S^∞) is $K(\mathbb{Z}/m\mathbb{Z},1)$;

(d) $\Omega K(\pi,n) = K(\pi,n-1)$; in particular, $\Omega\mathbb{CP}^\infty$ is homotopy equivalent to S^1.

4. (a) Prove that any smooth compact surface is either S^2, or $\mathbb{RP}^2$, or $K(\pi,1)$;

(b) Compute the fundamental group of any smooth compact surface.

5. Prove that the fundamental group $\pi_1(X)$ acts trivially on the Hurewicz homomorphism $h\colon \pi_n(X,x) \to H_n(X,\mathbb{Z})$, i.e. for $\alpha \in \pi_1(X,x)$, we have $h \circ \alpha_\# = h$.

Exercises on topology 06.03.2020

1. Compute the groups (a) $H_\bullet(\mathbb{RP}^n, \mathbb{Z}/\ell\mathbb{Z})$; (b) $H^\bullet(\mathbb{RP}^n, \mathbb{Z}/\ell\mathbb{Z})$;
(c) Compute the ring $H^\bullet(\mathbb{RP}^n, \mathbb{Z}/\ell\mathbb{Z})$. Here ℓ is a nonnegative integer (in particular, it may happen $\ell = 0$).

2. Let X be a CW-complex, and let $f, g\colon \operatorname{sk}_n X \to Y$ be continuous maps to a space Y coinciding on $\operatorname{sk}_{n-1} X$. We consider a cochain $c_f \in \mathcal{C}^{n+1}(X, \pi_n(Y))$ (in the complex computing cohomology of X with coefficients in the abelian group $\pi_n(Y)$) given by

$$c_f(\alpha) := f \circ \chi_\alpha|_{S^n}\colon S^n = \partial D^{n+1} \to Y,$$

and $\delta_{f,g} \in \mathcal{C}^n(X, \pi_n(Y))$ given by

$$\delta_{f,g}(\beta) := (f \circ \chi_\beta) \cup_{S^{n-1}} (g \circ \chi_\beta)\colon S^n = D^n \cup_{S^{n-1}} D^n \to Y.$$

Prove that (a) The differential $dc_f = 0$; (b) $d\delta_{f,g} = c_g - c_f$;
(c) Any cochain $\delta \in \mathcal{C}^n(X, \pi_n(Y))$ can be realized as $\delta_{f,g}$ for any f and an appropriate g.

3. Prove that the class of c_f in $H^{n+1}(X, \pi_n(Y))$ vanishes iff f can be extended to $\operatorname{sk}_{n+1} X$ after certain correction on $\operatorname{sk}_n X \setminus \operatorname{sk}_{n-1} X$.

4. Prove that if f, g are defined on $\operatorname{sk}_{n+1} X$, then $d\delta_{f,g} = 0$, and the class of $\delta_{f,g}$ in $H^n(X, \pi_n(Y))$ vanishes iff $f|_{\operatorname{sk}_n X}$ and $g|_{\operatorname{sk}_n X}$ can be connected by a homotopy constant on $\operatorname{sk}_{n-1} X$.

5. Let π be an abelian group. (a) Prove that $H_n(K(\pi, n), \mathbb{Z}) = \pi$ and $H^n(K(\pi, n), \pi) = \operatorname{Hom}(\pi, \pi)$;
(b) Let X be a CW-complex. To a map $f\colon X \to K(\pi, n)$ we associate the class $f^* \operatorname{Id}_\pi \in H^n(X, \pi)$, where $\operatorname{Id}_\pi \in H^n(K(\pi, n), \pi)$ is the fundamental class. Prove that $f \mapsto f^* \operatorname{Id}_\pi$ is a bijection $\operatorname{Ho}(X, K(\pi, n)) \cong H^n(X, \pi)$.

Exercises on topology 13.03.2020

The first 5 problems comprise the notes for the class of March 6th, so it is not really necessary to submit their solutions.

1. An *exact pair* consists of two objects D, E of an abelian category (say, modules over a ring R) and three morphisms i, j, k forming an exact sequence

$$\ldots \to D \xrightarrow{i} D \xrightarrow{j} E \xrightarrow{k} D \xrightarrow{i} D \to \ldots$$

In particular, $0 = (jk)^2\colon E \to E$, so that we obtain the homology $H(E, jk) = \operatorname{Ker}(jk)/\operatorname{Im}(jk)$. The *derived pair*

$$\ldots \to D' \xrightarrow{i'} D' \xrightarrow{j'} E' \xrightarrow{k'} D' \xrightarrow{i'} D' \to \ldots$$

is defined as follows: $D' = \operatorname{Im}(i)$, $E' = H(E, jk)$, i', j', k' are induced by i, j, k:
i' is the restriction of i to $\operatorname{Im}(i) \subset D$;

$j'(i(x))$ is the class of $j(x)$ in $H(E, jk),\ x \in D$;
k'(of the class of y) $= k(y),\ y \in E,\ jk(y) = 0$.

Prove that the derived pair of an exact pair is exact. Hence we have a sequence of exact pairs $P_r = (D_r, E_r, i_r, j_r, k_r)$: for $r = 1$ this is the initial pair $P_1 = (D, E, i, j, k)$, and for $r \geq 1$ the pair P_{r+1} is the derived pair of P_r.

2. Assume that an exact pair P_1 is bigraded, i.e. D, E are bigraded, $D = \bigoplus D^{p,q},\ E = \bigoplus E^{p,q}$, and the morphisms i, j, k have bidegrees $(-1, 1), (0, 0), (1, 0)$ respectively. Prove that the derived pairs P_r are bigraded as well, and the morphisms i_r, j_r, k_r have bidegrees $(-1, 1)$,
$(r-1, -r+1), (1, 0)$ respectively. Hence $d_r = j_r k_r$ is a differential in E_r of bigegree $(r, -r+1)$, and its cohomology H_r is isomorphic to E_{r+1} as a bigraded object. Thus, (E_r^{pq}, d_r) form a *spectral sequence.*

3. Under the assumptions of the previous problem, consider a diagram

$$\begin{tikzcd}
 & & & & & \\
\cdots \arrow[r] & E^{p-2,q} \arrow[r, "k"] & D^{p-1,q} \arrow[u, "i"] \arrow[r, "j"] & E^{p-1,q} \arrow[r, "k"] & D^{p,q} \arrow[u, "i"] \arrow[r, "j"] & \cdots \\
\cdots \arrow[r] & E^{p-1,q-1} \arrow[r, "k"] & D^{p,q-1} \arrow[u, "i"] \arrow[r, "j"] & E^{p,q-1} \arrow[r, "k"] & D^{p+1,q-1} \arrow[u, "i"] \arrow[r, "j"] & \cdots \\
 & & \arrow[u, "i"] & & \arrow[u, "i"] &
\end{tikzcd}$$

Here any sequence consisting of one vertical step i, two horizontal steps j, k, a new vertical step i and so on, is an exact sequence. With this description, $E_r^{p,q}$ is a subquotient of $E^{p,q}$, obtained by factoring $k^{-1}(\mathrm{Im}(i^{r-1}))$ modulo $j(\mathrm{Ker}(i^{r-1}))$ (where k^{-1} is the full preimage with respect to k). Describe the limit $E_\infty^{p,q}$ as $r \to \infty$.

4. Let $F^p K^\bullet$ be a decreasing filtration of a complex $K^\bullet$.

a) Using the long exact sequences of cohomology arising from the exact triples of complexes

$$0 \to F^{p+1}K^\bullet \to F^p K^\bullet \to F^p K^\bullet / F^{p+1} K^\bullet \to 0,$$

construct a bigraded exact pair with

$$D^{p,q} = H^{p+q}(F^p K^\bullet),\ E^{p,q} = H^{p+q}(F^p K^\bullet / F^{p+1} K^\bullet)$$

and morphisms i, j, k of bidegrees $(-1, 1), (0, 0), (1, 0)$ respectively. The spectral sequence of this exact pair is the *spectral sequence of the filtered complex* $F^\bullet K^\bullet$.

b) Check that $E_r^{p,q} = Z_r^{p,q} / (Z_{r-1}^{p+1,q-1} + dZ_{r-1}^{p-r+1,q+r-2})$, where $Z_r^{p,q} = d^{-1}(F^{p+r} K^{p+q+1}) \cap F^p K^{p+q}$, and $d_r^{p,q} \colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ is induced by the differential d of the complex $K^\bullet$.

5. Assume that in the setup of the previous problem the filtration $F^\bullet$ on every K^n is *finite* and *exhaustive*, i.e. $\exists\ p_+(n), p_-(n)$ such that $F^{p_+(n)} K^n = K^n,\ F^{p_-(n)} K^n = 0$. Prove that for

$$r \geq r_0(p, q) := \max(p_+(p+q+1) - p_-(p+q) + 1, p_+(p+q) - p_-(p+q-1) + 1)$$

we have $d_r^{p,q} = 0$, $d_r^{p-r,q+r-1} = 0$. Hence, in the limit $E_\infty^{p,q} = Z_r^{p,q}/Z_{r-1}^{p+1,q-1}$ for $r \geq r_0(p,q)$, and $Z_r^{p,q} = \mathrm{Ker}(d|_{F^pK^{p+q}})$, $Z_{r-1}^{p+1,q-1} = \mathrm{Ker}(d|_{F^{p+1}K^{p+q}})$ for $r > r_0(p,q)$.

We define a filtration $F^\bullet$ on $H^n := H^n(K^\bullet)$ by $F^pH^n =$ the image of $H^n(F^pK^\bullet)$ under the natural morphism $F^pK^\bullet \to K^\bullet$. Then for $n = p+q$ we have $E_\infty^{p,q} = F^pH^n/F^{p+1}H^n$, i.e. $\mathrm{gr}_F H^n = \bigoplus_{p+q=n} E_\infty^{p,q}$, i.e. *the spectral sequence converges to the cohomology* $H^\bullet(K^\bullet)$.

6. a) Consider the *canonical* filtration of a complex

$$(F^pK^\bullet)^n = K^n \text{ for } n < -p,\ (F^pK^\bullet)^n = \mathrm{Ker}(d_n) \text{ for } n = -p,\ (F^pK^\bullet)^n = 0 \text{ for } n > -p.$$

Prove that the corresponding filtration on $H^n(K^\bullet)$ is trivial: $F^pH^n = H^n$ for $n \leq -p$, $F^pH^n = 0$ for $n > -p$. Prove that $E_1^{p,q} = H^{-p}(K^\bullet)$ for $q = -2p$, $E_1^{p,q} = 0$ for $q \neq -2p$, and $d_1^{p,q} = 0$ for all p, q. Thus, $d_r = 0$ for all $r \geq 1$, and $E_r^{p,q} = E_1^{p,q}$.

b) Consider the *bête*, i.e. *"stupid"* filtration of a complex $(G^pK^\bullet)^n = 0$ for $n < p$, $(G^pK^\bullet)^n = K^n$ for $n \geq p$. Prove that the corresponding filtration on $H^n(K^\bullet)$ is trivial: $G^pH^n = H^n$ for $p \leq n$, $G^pH^n = 0$ for $p > n$. Prove that $E_r^{p,q} = 0$ for $q \neq 0$, $E_r^{p,q} = K^p$ for $q = 0, r = 1$, $E_r^{p,q} = H^p(K^\bullet)$ for $q = 0, 2 \leq r \leq \infty$. Finally, $d_r^{p,q}\colon E_r^{p,q} \to E_r^{p+r,q-r+1}$ coincides with $d_p\colon K^p \to K^{p+1}$ for $q = 0, r = 1$, and vanishes in the remaining cases.

7. A *bicomplex* $(L^{\bullet\bullet}, d', d'')$ is a collection of differentials $d'_{ij}\colon L^{ij} \to L^{i+1,j}$, $d''_{ij}\colon L^{ij} \to L^{i,j+1}$ with relations $(d')^2 = 0$, $(d'')^2 = 0$, $d'd'' + d''d' = 0$. The corresponding *total complex* $K^\bullet = \mathrm{Tot}(L^{\bullet\bullet})$ is defined as $K^n = \mathrm{Tot}^n := \oplus_{i+j=n} L^{ij}$, $d := d' + d''$. Consider a decreasing filtration $F'_p(K^n) := \bigoplus_{i+j=n}^{i \geq p} L^{ij}$. The corresponding spectral sequence is denoted $'E_r^{pq}$. Prove that $'E_2^{pq} = H^p_{d'}(H^{\bullet,q}_{d''}(L^{\bullet\bullet}))$.

Exercises on topology 20.03.2020

1. Recall the lense space L_m^∞ of problem 3(c) of February 28th. Prove that

$$H^0(L_m^\infty, \mathbb{Z}) = \mathbb{Z},\ H^{2n+1}(L_m^\infty, \mathbb{Z}) = 0,\ H^{2n}(L_m^\infty, \mathbb{Z}) = \mathbb{Z}/m\mathbb{Z} \text{ for } n > 0.$$

2. Compute the cohomology $H^\bullet(\mathrm{USp}(n), \mathbb{Z})$.

3. Prove that for any fibration $E \to B$ with fiber F the Euler characteristics (for rational cohomology) satisfy the relation $\chi(E) = \chi(B)\chi(F)$.

4. Let $p\colon E \to B$ be a fibration with fiber S^n and a simply connected base B. Construct the *Gysin* exact sequence

$$\ldots \to H^k(B) \xrightarrow{p^*} H^k(E) \to H^{k-n}(B) \xrightarrow{d_{n+1}} H^{k+1}(B) \xrightarrow{p^*} H^{k+1}(E) \to \ldots$$

5. Let $p\colon E \to S^n$ be a fibration with fiber $F \overset{i}{\hookrightarrow} E$, and let $n \geq 2$. Construct the *Wang* exact sequence

$$\ldots \to H^k(E) \xrightarrow{i^*} H^k(F) \xrightarrow{d_n} H^{k-n+1}(F) \to H^{k+1}(E) \xrightarrow{i^*} H^{k+1}(F) \to \ldots$$

Exercises on topology 21.03.2020

The *Postnikov tower* of a topological space X is a sequence

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & X|_4 & \longrightarrow & X|_3 & \longrightarrow & X|_2 & \longrightarrow & X|_1 = X \\
 & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
 & K(\pi_4(X),4) & & K(\pi_3(X),3) & & K(\pi_2(X),2) & & K(\pi_1(X),1)
\end{array}$$

where all the vertical maps are Serre fibrations inducing isomorphisms of the lowest homotopy groups, and $X|_{n+1}$ is the homotopy fiber of $X|_n \to K(\pi_n(X), n)$. In particular, $\pi_{<n}(X|_n) = 0$, and $\pi_{\geq n}(X|_n) \cong \pi_{\geq n}(X)$. The Postnikov tower is Eckmann-Hilton dual to the filtration $\mathrm{sk}_1 X \subset \mathrm{sk}_2 X \subset \mathrm{sk}_3 X \subset \ldots$

1. Prove that a) $X|_2$ is the universal cover of X;
b) The homotopy fiber of $X|_{n+1} \to X|_n$ is $K(\pi_n(X), n-1)$.

2. Prove that $H^\bullet(K(\mathbb{Z},3),\mathbb{Z}) = \mathbb{Z} \oplus 0 \oplus 0 \oplus \mathbb{Z} \oplus 0 \oplus 0 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \ldots$ (we only care up to degree 6).

3. a) Prove that $H^\bullet(S^3|_4) = \mathbb{Z} \oplus 0 \oplus 0 \oplus 0 \oplus 0 \oplus \mathbb{Z}/2\mathbb{Z} \oplus \ldots$ (we only care up to degree 5).
b) By the universal coefficients formula and Hurewicz isomorphism deduce that $\pi_4(S^3) = \pi_4(S^3|_4) = H_4(S^3|_4) \cong \mathbb{Z}/2\mathbb{Z}$. Note that by the Freudenthal suspension theorem it follows that $\pi_{n+1}(S^n) = \mathbb{Z}/2\mathbb{Z}$ for $n > 2$.

4. Prove that $H^\bullet(K(\mathbb{Z},n),\mathbb{Q})$ is a free algebra with one generator in degree n (i.e. it is the algebra of polynomials if n is even, while for odd n it is the exterior algebra isomorphic to $H^\bullet(S^n,\mathbb{Q})$).

5. Prove that if $\pi_1(X) = 0$, and $H^\bullet(X,\mathbb{Q})$ is a free supercommutative graded algebra with generators x_s in degrees d_s, then the rank of $\pi_i(X) \otimes_{\mathbb{Z}} \mathbb{Q}$ equals $\sharp\{s : d_s = i\}$ (Theorem of H. Cartan and J.-P. Serre). More invariantly, recall that the rational *homology* $H_\bullet(X,\mathbb{Q})$ carries a structure of a coalgebra, and the rational Hurewicz homomorphism $h \otimes \mathbb{Q}$ is an isomorphism $\pi_\bullet(X) \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\sim} \operatorname{Prim} H_\bullet(X,\mathbb{Q})$, where $\operatorname{Prim} H_\bullet(X,\mathbb{Q}) = \{a \in H_\bullet(X,\mathbb{Q}) : \Delta(a) = a \otimes 1 + 1 \otimes a\}$. In particular, $\pi_\bullet(\mathrm{SU}(n))$ up to torsion is $\mathbb{Z}$ in degrees $3, 5, \ldots, 2n-1$ (in fact, there is no torsion).

Hint: prove that the homotopy fiber of the classifying morphism $X \to \prod_s K(\mathbb{Z}, d_s)$ has trivial rational cohomology, homology and homotopy groups.

Exercises on topology 27.03.2020

1. Compute $\pi_2(S^1 \vee S^2)$.

2. Suppose a finite group Γ acts freely on a topological space Y, and $X = Y/\Gamma$ (in other words, Y is a Γ-torsor over X). Prove that
a) $H^\bullet(X,\mathbb{Q}) = H^\bullet(Y,\mathbb{Q})^\Gamma$ (the Γ-invariants);

b) The Euler characteristics $\chi(Y) = |\Gamma| \cdot \chi(X)$ if both X and Y are finite CW-complexes.

3. Prove that a) the reduced cohomology of the suspension ΣX is equal to the shifted reduced cohomology of X: $H^\bullet(\Sigma X, \mathrm{pt}) = H^\bullet(X, \mathrm{pt})[-1]$, i.e. $H^{n+1}(\Sigma X, \mathrm{pt}) = H^n(X, \mathrm{pt})$;
b) The multiplication in $H^\bullet(\Sigma X)$ is trivial.

4. Let $i\colon Y \hookrightarrow X$ be a Borsuk pair (i.e. a cofibration) (e.g. a CW-subcomplex). Let $\delta\colon H^\bullet(Y,\pi) \to H^{\bullet+1}(X/Y,\pi)$ be the connecting homomorphism in the long exact sequence of the pair $Y \subset X$. Let f denote the composition $\mathrm{Cone}(i) = X \cup_i \mathrm{Cone}(Y) \to \big(X \cup_i \mathrm{Cone}(Y)\big)/X = \Sigma Y$. Prove the anticommutativity of the diagram

$$\begin{array}{ccc} H^\bullet(Y,\pi) & \xrightarrow{\delta} & H^{\bullet+1}(X/Y,\pi) \\ \Sigma\big\downarrow\wr & & \Big\| \\ H^{\bullet+1}(\Sigma Y,\pi) & \xrightarrow{f^*} & H^{\bullet+1}(\mathrm{Cone}(i),\pi). \end{array}$$

5. For a fibration $F \to \mathcal{E} \xrightarrow{p} B$ prove that the transgression $\tau\colon E_m^{0,m-1} \to E_m^{m,0}$ coincides with the composition

$$E_m^{0,m-1} \hookrightarrow H^{m-1}(F) \xrightarrow{\delta} H^m(\mathcal{E}, F) \overset{(p^*)^{-1}}{\dashrightarrow} H^m(B, \mathrm{pt}) = H^m(B) \twoheadrightarrow E_m^{m,0},$$

i.e. $(p^*)^{-1}$ is well defined on (δ of) the image of $E_m^{0,m-1}$ in $H^{m-1}(F)$ modulo the kernel of the projection $H^m(B) \twoheadrightarrow E_m^{m,0}$, where δ is the connecting homomorphism in the long exact sequence of the pair $F \subset \mathcal{E}$.

Exercises on topology 03.04.2020

1. Prove that a) The Stiefel variety $\mathrm{St}(k,\infty,\mathbb{C})$ is contractible;
b) $\mathrm{Gr}(k,\infty,\mathbb{C}) \sim B\,\mathrm{GL}(k,\mathbb{C})$;
c) $F\ell(\mathbb{C}^k) = \mathrm{U}(k)/(S^1)^k$.

2. Since the suspension functor Σ is the left adjoint functor to the loops functor Ω, and $\Omega K(\pi, n+1) = K(\pi, n)$, we get $f_n\colon \Sigma K(\pi,n) \to K(\pi, n+1)$. Prove that the induced map $f_n^*\colon H^{n+k+1}(K(\pi,n+1),\pi) \to H^{n+k}(K(\pi,n),\pi)$ is an isomorphism for $n > k$.

3. Prove that Sq^1 is the Bockstein homomorphism.

4. Construct the natural transformation $\phi\colon D_2 \circ [-1] \to [-1] \circ D_2$.

5. Let X be a simplicial (i.e. triangulated) space; we denote by $\mathscr{C}^\bullet(X,\mathbb{F}_2)$ its cochain complex spanned by functionals on standard simplices. For a simplex σ and $i \le j$ we denote by $\Gamma_i^j\sigma$ its $(j-i)$-dimensional face with vertices $i, i+1, \dots, j-1, j$. Prove that
a) For a cohomology class in $H^n(X,\mathbb{F}_2)$ represented by a cocycle $\alpha \in \mathscr{C}^n(X,\mathbb{F}_2)$, the Steenrod square $Sq^i(\alpha)$ is represented by a cocycle $\gamma \in \mathscr{C}^{n+i}(X,\mathbb{F}_2)$ such that $\gamma(\sigma) = \alpha(\Gamma_0^n\sigma)\alpha(\Gamma_i^{n+i}\sigma)$;

b) The same formula holds true in the singular cochain complex $C^\bullet(X, \mathbb{F}_2)$ of an arbitrary space X.

Exercises on topology 10.04.2020

1. Prove the existence of higher homotopies $\Delta^{(q)}$ of [Notes, §2.1].

2. a) Prove that the configuration space $\overset{\circ}{\mathbb{C}}{}^{(n)}$ of unordered n-tuples of distinct complex numbers is of type $K(\pi, 1)$.

Its fundamental group $\pi = B_n$ is the braid group with generators $\sigma_1, \dots, \sigma_{n-1}$ and relations $\sigma_i\sigma_{i+1}\sigma_i = \sigma_{i+1}\sigma_i\sigma_{i+1},\ \sigma_i\sigma_j = \sigma_j\sigma_i$ for $|j-i| > 1$.

b) Define these generators and prove these relations.

The fundamental group $\pi_1(\overset{\circ}{\mathbb{C}}{}^n)$ of the configuration space of *ordered* n-tuples of distinct complex numbers is the pure braid group PB_n: the kernel of the projection $B_n \to S_n$.

3. Prove that a) the cohomology ring of the lense space

$$H^\bullet(L_p^\infty, \mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(\tau) \otimes \mathbb{F}_p[t],\ \deg\tau = 1,\ \deg t = 2;$$

b) The Bockstein homomorphism $\beta_p(\tau) = t$.

4. Recall that for a discrete group Γ, the cohomology $H^\bullet(\Gamma, \pi)$ with coefficients in abelian group π is defined as $H^\bullet(K(\Gamma, 1), \pi)$. In particular, $H^\bullet(\mathbb{Z}/p\mathbb{Z}, \mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(\tau) \otimes \mathbb{F}_p[t]$. Let $\Gamma = \mathbb{F}_p^\times \ltimes \mathbb{F}_p$ be the group of affine transformations of the line over $\mathbb{F}_p$. Prove that

a) $H^\bullet(\Gamma, \mathbb{F}_p) = \Lambda_{\mathbb{F}_p}(t^{p-2}\tau) \otimes \mathbb{F}_p[t^{p-1}]$;

b) The restriction homomorphism $H^\bullet(S_p, \mathbb{F}_p) \to H^\bullet(\Gamma, \mathbb{F}_p)$ is an isomorphism.

5. Prove that the differential operators

$$\Delta_{n,s},\ n \in \mathbb{N},\ s = 0, 1,\ \deg\Delta_{n,s} = -2n - s,\ \Delta_{0,0} = \mathrm{Id}$$

of [Notes, §4.1], form a topological basis of $\Lambda[\![t, \tau]\!]$-module $\mathrm{End}(\Lambda[\![t, \tau]\!])$, and the following relations hold:

$$\Delta_{n,0}\Delta_{m,0} = \binom{n+m}{n}\Delta_{n+m,0},\ \Delta_{n,0}t^m = \sum\binom{m}{k}t^{m-k}\Delta_{n-k,0},$$

$$\Delta_{n,0}\Delta_{0,1} = \Delta_{n,1},\ \Delta_{0,1}t^m = t^m\Delta_{0,1},\ \Delta_{n,0}\tau = \tau\Delta_{n,0},\ \Delta_{0,1}\tau = 1 - \tau\Delta_{0,1}.$$

Exercises on topology 17.04.2020

1. Prove that any endomorphism of the formal additive group $\widehat{\mathbb{G}}_a$ with coordinate t over a field of characteristic p is of the form $t \mapsto \sum_{i=0}^\infty \xi_i t^{p^i}$.

Hence the endomorphism ring of the formal additive group over $\overline{\mathbb{F}}_p$ is the ring $\overline{\mathbb{F}}_p\{\{\mathrm{Fr}\}\}$ with relations $\mathrm{Fr}\cdot x = x^p \cdot \mathrm{Fr}$.

2. Prove that a) the Thom isomorphisms for a cartesian diagram

$$\begin{array}{ccc} \mathcal{V}' & \xrightarrow{f} & \mathcal{V} \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

satisfy $\Phi' f^* = f^* \Phi$;

b) For vector bundles $\mathcal{V}_1 \to B_1$, $\mathcal{V}_2 \to B_2$ and their direct product $\mathcal{V}_1 \times \mathcal{V}_2 \to B_1 \times B_2$ we have $\Phi_{\mathcal{V}_1 \times \mathcal{V}_2}(x_1 \otimes x_2) = \Phi_{\mathcal{V}_1}(x_1) \otimes \Phi_{\mathcal{V}_2}(x_2)$.

3. Prove that a) the cotangent bundle $T^*\mathbb{RP}^n$ and the trivial line bundle $\mathbb{R}$ sum up to the direct sum of $n+1$ copies of the tautological line bundle $\mathcal{S}_1 = \mathcal{O}(-1)$ over $\mathbb{RP}^n$: $T^*\mathbb{RP}^n \oplus \mathbb{R} \simeq \mathcal{S}_1^{\oplus n+1}$;

b) The total Stiefel-Whitney class $w(\mathbb{RP}^n) := w(T\mathbb{RP}^n) = (1+\tau)^{n+1}$.

4. Prove that a) the tangent bundle of $\mathbb{RP}^r$ is nontrivial if $r \neq 2^n - 1$;

b) If $\mathbb{RP}^{2^n}$ admits an immersion (= locally in $\mathbb{RP}^n$ closed embedding) into $\mathbb{R}^{2^n+k}$, then $k \geq 2^n - 1$. *Hint:* $T\mathbb{RP}^{2^n} \oplus \mathcal{N}_{\mathbb{RP}^{2^n}/\mathbb{R}^{2^n+k}}$ (the normal vector bundle) is the restriction $T\mathbb{R}^{2^n+k}|_{\mathbb{RP}^{2^n}}$, i.e. a trivial vector bundle.

5. Prove that if $\mathbb{RP}^r$ can be immersed into $\mathbb{R}^{r+1}$, then $r = 2^n - 1$ or $r = 2^n - 2$.

Exercises on topology 24.04.2020

1. Recall the Postnikov tower of Exercises of 21.03.20. Prove that

a) $\mathrm{Gr}(n, \infty, \mathbb{R})|_2 = \mathrm{Gr}_+(n, \infty, \mathbb{R}) \simeq B\,\mathrm{SO}(n, \mathbb{R})$ (notation of Exercise 2 of 17.01.20);

b) $\mathrm{Gr}(n, \infty, \mathbb{R})|_3 \simeq B\,\mathrm{Spin}(n, \mathbb{R})$, where $\mathrm{Spin}(n, \mathbb{R})$ is the two-fold (universal) cover of $\mathrm{SO}(n, \mathbb{R})$.

2. Recall the setup of Exercise 1 of 21.02.2020. Prove that all the incidence indices (the matrix coefficients of the differential of the cellular complex) are even, and hence the cells form a basis of $H^\bullet(\mathrm{Gr}(k, n, \mathbb{R}), \mathbb{F}_2)$, and in particular, $\dim H^r(\mathrm{Gr}(k, n, \mathbb{R}), \mathbb{F}_2)$ is the number of Young diagrams of total area r inscribed into the $k \times (n-k)$ rectangle.

3. Prove that the algebra $H^\bullet(\mathrm{Gr}(k, n, \mathbb{R}), \mathbb{F}_2)$ is generated by the Stiefel-Whitney classes $w_1(\mathcal{S}_k), \ldots, w_k(\mathcal{S}_k), w_1(\mathcal{Q}_{n-k}), \ldots, w_{n-k}(\mathcal{Q}_{n-k})$ (where $\mathcal{Q}_{n-k}$ is the universal quotient bundle of rank $n-k$) with the unique relation $w(\mathcal{S}_k) w(\mathcal{Q}_{n-k}) = 1$.

4. Let $\mathcal{V}, \mathcal{V}'$ be vector bundles over B of ranks n, n'. Prove that the total Stiefel-Whitney class $w(\mathcal{V} \otimes \mathcal{V}') = P(w_1(\mathcal{V}), \ldots, w_n(\mathcal{V}), w_1(\mathcal{V}'), \ldots, w_{n'}(\mathcal{V}'))$, where the polynomial P of $n+n'$ variables is defined by the property $P(e_1, \ldots, e_n, e'_1, \ldots, e'_{n'}) = \prod_{i=1}^n \prod_{j=1}^{n'} (1 + t_i + t'_j)$, and e_i (resp. e'_j) are the elementary symmetric polynomials of t (resp. of t').

5. a) Prove the Wu formula $Sq^k(w_m(\mathcal{V})) = \sum_{i=0}^k \binom{m-k+i-1}{i} w_{k-i}(\mathcal{V}) w_{m+i}(\mathcal{V})$;

b) Prove that if $w(\mathcal{V}) \neq 1$, then the minimal r such that $w_r(\mathcal{V}) \neq 0$ is necessarily of the form $r = 2^n$.

Exercises on topology 01.05.2020

1. A real vector bundle $\mathcal{V} \to B$ of rank n is *orientable* if $\Lambda^n\mathcal{V} \setminus B$ has two connected components. In other words, the corresponding $\mathrm{GL}(n,\mathbb{R})$-torsor $\mathcal{E} \to B$ can be reduced to a $\mathrm{GL}_+(n,\mathbb{R})$-torsor $\mathcal{E}_+$, that is $\mathcal{E} = \mathcal{E}_+ \overset{\mathrm{GL}_+(n,\mathbb{R})}{\times} \mathrm{GL}(n,\mathbb{R})$. Alternatively, choosing a metric, the corresponding $\mathrm{O}(n,\mathbb{R})$-torsor $\mathcal{F} \to B$ can be reduced to a $\mathrm{SO}(n,\mathbb{R})$-torsor $\mathcal{F}_+$, that is $\mathcal{F} = \mathcal{F}_+ \overset{\mathrm{SO}(n,\mathbb{R})}{\times} \mathrm{O}(n,\mathbb{R})$. Prove that

a) $\mathcal{V}$ is orientable if and only if $w_1(\mathcal{V}) = 0$.

b) An orientable vector bundle $\mathcal{V}$ has $w_2(\mathcal{V}) = 0$ if and only if the corresponding $\mathrm{SO}(n,\mathbb{R})$-torsor $\mathcal{F}_+$ can be reduced to a $\mathrm{Spin}(n,\mathbb{R})$-torsor $\widetilde{\mathcal{F}}_+$, i.e. $\mathcal{F}_+ = \widetilde{\mathcal{F}}_+ \overset{\mathrm{Spin}(n,\mathbb{R})}{\times} \mathrm{SO}(n,\mathbb{R})$.

2. Construct an isomorphism $\mathrm{St}(2,3,\mathbb{R}) \simeq \mathbb{RP}^3$.

3. Recall the setup of Exercise 1 of 21.02.2020. Consider the cell $e_{\alpha^{(i)}} \subset \mathrm{Gr}(k,n,\mathbb{R})$ corresponding to $\alpha_1^{(i)} = i, \alpha_2^{(i)} = 0, \ldots$ (i.e. the corresponding partition is $(i, 0, \ldots)$). Prove that $\langle w_i(\mathcal{S}_k), e_\beta \rangle = \delta_{\alpha^{(i)}\beta}$.

4. Given a rank n vector bundle $\mathcal{V} \to B$ pick a metric and consider the corresponding sphere bundle $S(\mathcal{V}) \xrightarrow{p} B$. The pullback $p^*\mathcal{V}$ is equipped with a canonical section s and its orthogonal complement $s^\perp \subset p^*\mathcal{V}$: a vector bundle of rank $n-1$. Prove that

a) For $i < n$, $p^*w_i(\mathcal{V}) = w_i(s^\perp)$;

b) For $i < n$, $p^*\colon H^i(B, \mathbb{F}_2) \to H^i(S(\mathcal{V}), \mathbb{F}_2)$ is injective.

This allows for an alternative definition of the Stiefel-Whitney classes by descending induction if we only know the definition of the top class $w_n(\mathcal{V}) = \Phi^{-1} Sq^n(\mathsf{t}_\mathcal{V}) = \Phi^{-1}\mathsf{t}_\mathcal{V}^2$. In particular, there is no need to use Steenrod squares. This is a typical application of the *splitting principle*.

5. a) Prove that the tangent bundle $T\,\mathrm{Gr}(k,n,\mathbb{R}) \cong \mathcal{H}om(\mathcal{S}_k, \mathcal{Q}_{n-k})$;

b) Given a smooth k-dimensional submanifold $M \subset \mathbb{R}^n$ with normal bundle $\mathcal{N}_{M/\mathbb{R}^n}$, we have the Gauß map $\gamma\colon M \to \mathrm{Gr}(k,n,\mathbb{R})$. Prove that its differential $d\gamma$ defines a section of the vector bundle $\mathcal{H}om(TM, \mathcal{H}om(TM, \mathcal{N}_{M/\mathbb{R}^n})) = \mathcal{H}om(TM \otimes TM, \mathcal{N}_{M/\mathbb{R}^n})$ (called the *second fundamental form* of $M \subset \mathbb{R}^n$).

Exercises on topology 08.05.2020

1. Prove that a) The total Stiefel-Whitney class $w(TM)$ of a compact r-dimensional manifold M is invertible in the cohomology ring $H^\bullet(M, \mathbb{F}_2)$. We denote the homogeneous components of the inverse class by $\sum \bar{w}_i(TM) = w(TM)^{-1}$;

b) $\langle Sq^{\mathrm{inv}}(\xi), [M]\rangle = \langle w(TM)^{-1} \cdot \xi, [M]\rangle$ for any $\xi \in H^\bullet(M, \mathbb{F}_2)$;
c) $\bar{w}_r = 0$ and $\bar{w}_{r-1} = 0$ unless $r = 2^n$.

2. Recall the cap product $\frown\colon H^k(M, \mathbb{F}_2) \otimes H_j(M, \mathbb{F}_2) \to H_{j-k}(M, \mathbb{F}_2)$. We define the Steenrod squares in *homology* $Sq^i\colon H_j(M, \mathbb{F}_2) \to H_{j-i}(M, \mathbb{F}_2)$ by the requirement $\langle \xi, Sq^i(\eta)\rangle = \langle (Sq^{\mathrm{inv}})^i(\xi), \eta\rangle$. Prove that
a) $Sq(\xi \frown \eta) = Sq(\xi) \frown Sq(\eta)$;
b) $Sq(D[\Delta_M]/\eta) = Sq(D[\Delta_M])/Sq(\eta)$;
c) $Sq[M] = \bar{w}(TM) \frown [M]$;
d) $Sq^{\mathrm{inv}}[M] = v^M \frown [M]$, where $v^M \in H^\bullet(M, \mathbb{F}_2)$ is defined in [Notes, §7.1.2].

3. Let $(r_1, r_2, \ldots, r_n) \in \mathbb{N}^n$ be a sequence s.t. $r_1 + 2r_2 + \ldots + nr_n = n$ (i.e. (i^{r_i}) is a partition of n). For a smooth n-dimensional manifold M the Stiefel-Whitney number $w_1^{r_1} \cdots w_n^{r_n}[M] \in \mathbb{F}_2$ is the value of $w_1(TM)^{r_1} \cdots w_n(TM)^{r_n}$ on the fundamental homological class of M. Prove that if M is the boundary of a smooth $n+1$-dimensional manifold with boundary, then all the Stiefel-Whitney numbers of M vanish.

4. Prove that $H^\bullet(\mathrm{Gr}(k, \infty, \mathbb{C}), \mathbb{Z}) = \mathbb{Z}[c_1(\mathcal{S}_k), \ldots, c_k(\mathcal{S}_k)]$.

5. For a complex rank n vector bundle $\mathcal{V} \to B$ we denote by $\mathcal{V}_\mathbb{R} \to B$ the real rank $2n$ vector bundle obtained by restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$. Prove that
a) For the complex conjugate bundle $\overline{\mathcal{V}}$ we have $c_k(\mathcal{V}) = (-1)^k c_k(\overline{\mathcal{V}})$;
b) The Pontriagin class $\sum_{i\geq 0}(-1)^i p_i(\mathcal{V}_\mathbb{R}) = c(\mathcal{V}) \cdot \sum_{i\geq 0}(-1)^i c_i(\mathcal{V})$.

Exercises on topology 15.05.2020

1. Prove that for a complex vector bundle $\mathcal{V} \to B$, $w(\mathcal{V}_\mathbb{R}) = c(\mathcal{V}) \pmod 2$.

2. For a real vector bundle $\mathcal{V} \to B$ prove that $w_{2i}^2(\mathcal{V}) = p_i(\mathcal{V}) \pmod 2$.

3. Prove that a) for a real vector bundle $\mathcal{V} \to B$,

$$Sq^1(w_{2i}(\mathcal{V}) \cdot w_{2i+1}(\mathcal{V})) = c_{2i+1}(\mathcal{V} \otimes_\mathbb{R} \mathbb{C}) \pmod 2;$$

b) For an oriented rank $2k+1$ vector bundle $\mathcal{V} \to B$, $Sq^1 w_{2k}(\mathcal{V}) = \mathsf{eu}_{2k+1}(\mathcal{V}) \pmod 2$.

4. Prove that a) $\pi_r(\mathrm{St}(k, n, \mathbb{C})) = 0$ for $r \leq 2(n-k)$;
b) $\pi_{2n-2k+1}(\mathrm{St}(k, n, \mathbb{C})) = \mathbb{Z}$.

5. Prove that the first obstruction $o_{n-k+1}(\mathcal{V})$ to the extension of k-tuple of linearly independent sections of a complex rank n vector bundle $\mathcal{V} \to B$ from $\mathrm{sk}_{2n-2k+1} B$ to $\mathrm{sk}_{2n-2k+2} B$ coincides with the Chern class $c_{n-k+1}(\mathcal{V})$.

Exercises on topology 22.05.2020

1. Let $\mathcal{V} \to B$ be a rank n complex vector bundle. We define its *Chern character* $\mathrm{ch}(\mathcal{V})$ as $n + \sum_{i=1}^{\infty} m^c_{(i)}(\mathcal{V})/i! \in H^\bullet(B, \mathbb{Q})$. Prove that

a) For a line bundle $\mathcal{L} \to B,\ \mathrm{ch}(\mathcal{L}) = \exp(c_1(\mathcal{L}))$;

b) $\mathrm{ch}(\mathcal{V} \oplus \mathcal{V}') = \mathrm{ch}(\mathcal{V}) + \mathrm{ch}(\mathcal{V}')$;

c) The Chern character of any complex vector bundle is uniquely characterized by the properties a,b);

d) $\mathrm{ch}(\mathcal{V} \otimes \mathcal{V}') = \mathrm{ch}(\mathcal{V}) \cdot \mathrm{ch}(\mathcal{V}')$.

2. Prove that $m^p_\lambda(\mathcal{V}_\mathbb{R}) = m^c_{2\lambda}(\mathcal{V})$, where $2\lambda := (2\lambda_1, \ldots, 2\lambda_\ell)$.

3. Consider a smooth compact complex variety X of dimension $n+1$ containing a smooth compact complex subvariety Y of dimension n. We denote by $L = D[Y] \in H^2(X, \mathbb{Z})$ the (Poincaré dual) fundamental class of $Y \subset X$. Prove that the total Chern class of the complex tangent bundle $c(TY) = c(TX) \cdot (1 + L)^{-1}|_Y$.

4. Let $Y \subset \mathbb{CP}^{n+1}$ be a smooth hypersurface of degree d (the zero level of a homogeneous degree d polynomial in $n+2$ variables).

a) Compute the total Chern class $c(TY)$;

b) Prove that the characteristic number $m^c_{(n)}[Y] = d(n + 2 - d^n)$.

5. Let Y be a smooth hypersurface of bidegree $(1, 1)$ in $\mathbb{CP}^n \times \mathbb{CP}^m$, where $n, m \geq 2$.

a) Prove that $m^c_{(n+m-1)}[Y] = -\binom{n+m}{n}$;

b) Prove that for any $n \geq 1$ there is a smooth compact complex variety Y of dimension n such that $m^c_{(n)}[Y] = p$ if $n + 1$ is a power of a prime number p, or $m^c_{(n)}[Y] = 1$ if n is not a prime power.

Exercises on topology 29.05.2020

1. Let X be a connected finite CW-complex such that $\pi_{<k}(X, x) = 0$ for some $k \geq 2$. Using [Notes, Theorem 1.3], prove that the Hurewicz homomorphism $\pi_r(X, x) \to H_r(X, \mathbb{Z})$ is an isomorphism modulo torsion for $r < 2k - 1$.

2. Let M be a real smooth compact oriented $4n$-dimensional manifold equal to the boundary of an oriented manifold N. Prove that

a) The long exact sequence

$$... \to H^i(N, M; \mathbb{R}) \to H^i(N, \mathbb{R}) \to H^i(M, \mathbb{R}) \to H^{i+1}(N, M; \mathbb{R}) \to \ldots$$

is Poincaré self-dual "symmetrically with the center in $H^{2n}(M, \mathbb{R})$";

b) The (nondegenerate) Poincaré pairing on $H^{2n}(M, \mathbb{R})$ has signature 0 (i.e. the number of positive squares equals the number of negative squares).

3. Prove that the stable complex cobordism ring $\Omega^\mathbb{C}_\bullet \otimes_\mathbb{Z} \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^1, \mathbb{CP}^2, \mathbb{CP}^3, \ldots]$.

4. Similarly to Exercise 5 of 22.05.2020, using hypersurfaces in $\mathbb{RP}^n \times \mathbb{RP}^m$ prove the existence of smooth manifolds Y^k of real dimension k such that $m^w_{(k)}[Y^k] \neq 0$ if $k+1 \neq 2^i$ (monomial Stiefel-Whitney numbers).

5. Prove that a) Y^k is orientable if k is odd;

b) The products $Y^{\lambda_1} \times \ldots \times Y^{\lambda_\ell}$ for $\lambda \in \mathfrak{P}(n)$ such that $\lambda_i + 1$ is not a power of 2 for any i, are linearly independent over $\mathbb{F}_2$ in the group $\mathscr{N}_n$ of cobordisms of *not necessarily oriented* n-dimensional manifolds. (In this group $M + M = 0$, evidently.)

Exercises on topology 05.06.2020

1. For a characteristic power series $Q(x) = 1 + q_1x + q_2x^2 + \ldots$ we write the corresponding multiplicative Hirzebruch sequence as $K_i(p_1, \ldots, p_i; q_1, \ldots, q_i)$ (certain polynomials in $2i$ variables with integral coefficients). Prove the following symmetry property: $K_i(p_1, \ldots, p_i; q_1, \ldots, q_i) = K_i(q_1, \ldots, q_i; p_1, \ldots, p_i)$.

2. Prove that $Td(\mathbb{CP}^n) = 1$ for any n.

3. Prove that the coefficient of p_n in the multiplicative Hirzebruch sequence $L_n(p_1, \ldots, p_n)$ of the L-genus equals $(-1)^{n-1}2^{2n}(2^{2n-1} - 1)B_{2n}/(2n)!$.

4. Let M be a compact manifold of dimension n. Prove that

a) If $n < 2k - 1$, then $\mathrm{Ho}(M, S^k) \simeq \mathrm{Ho}(\Sigma M, \Sigma S^k = S^{k+1}) \simeq \mathrm{Ho}(M, \Omega S^{k+1})$ is a group (*cohomotopy group* $\pi^k(M)$);

b) The generator $s \in H^k(S^k, \mathbb{Z})$ defines the classifying map $\sigma\colon S^k \to K(\mathbb{Z}, k)$. The composition with $\sigma\colon \pi^k(M) \to \mathrm{Ho}(M, K(\mathbb{Z}, k)) = H^k(M, \mathbb{Z})$ induces an isomorphism modulo torsion for $n < 2k - 1$.

5. Let $f\colon M^n \to S^{n-4i}$ be a smooth map from a smooth oriented compact n-dimensional manifold to a sphere, and let $y \in S^{n-4i}$ be a regular value of f, so that $M^{4i} := f^{-1}(y)$ is a smooth compact oriented manifold of dimension $4i$. Prove that

a) $\langle L_i(TM^n) \cdot f^*s, [M^n] \rangle = \mathrm{sign}(M^{4i})$, where s is the generator of $H^{n-4i}(S^{n-4i}, \mathbb{Z})$;

b) If $8i < n - 1$, then the cohomological class $L_i(TM^n) \in H^{4i}(M^n, \mathbb{Q})$ is uniquely characterized by the property a).