1. March 27th

1.1. $H^q(K(\pi,n),\mathbb{Z})$ are finitely generated: clear for $n=1$, then by induction in n from the s.s. of

$$\mathrm{pt}\xrightarrow{K(\pi,n-1)}K(\pi,n):$$

the first non-finitely generated group in the first row cannot be cancelled by anything.

1.2. If $\pi_1(X)=0$ and $H^\bullet(X)$ are finitely generated (resp. finite), then so are $\pi_\bullet(X)$ (Counterexample: $X=S^1\vee S^2$.) Indeed, if both $H^\bullet(F)$ and $H^\bullet(B)$ are fin.gen. (resp. finite), then so is $H^\bullet(E)$: clear from s.s. Now apply to

$$X|_3\xrightarrow{K(H_2(X),1)}X|_2=X,\ \ X|_4\xrightarrow{K(H_3(X_3),2)}X|_3,\ \ldots$$

get fin.gen. (resp. finiteness) of $H^\bullet(X|_n)$, hence of $H_\bullet(X|_n)$, hence of $\pi_n(X|_n)$.

1.3. **Theorem.** $\pi_\bullet(S^{2n+1})\otimes\mathbb{Q}$ is 1-dimensional in degree $2n+1$, while $\pi_\bullet(S^{2n})\otimes\mathbb{Q}$ is 1-dimensional in degrees $2n$ and $4n-1$.

Proof: The $E_2\otimes\mathbb{Q}$ of s.s. of

$$S^{2n+1}|_{2n+2}\xrightarrow{K(\mathbb{Z},2n)}S^{2n+1}|_{2n+1}=S^{2n+1}$$

has columns $0,2n+1$ and rows $0,2n,4n,\ldots$ Figure We have $d_{2n+1}x^k=kx^{k-1}d_{2n+1}x=kx^{k-1}s$, hence $E_\infty=\mathbb{Q}[0]$, hence $H^\bullet(S^{2n+1}|_{2n+2})$ are finite, hence $\pi_\bullet(S^{2n+1}|_{2n+2})$ are finite, hence $\pi_{>2n+1}(S^{2n+1})$ are finite.

In the even case the $E_2\otimes\mathbb{Q}$ of s.s. of

$$S^{2n}|_{2n+1}\xrightarrow{K(\mathbb{Z},2n-1)}S^{2n}|_{2n}=S^{2n}$$

has columns $0,2n$ and rows $0,2n-1$. Figure Hence $H^\bullet(S^{2n}|_{2n+1},\mathbb{Q})=H^\bullet(S^{4n-1},\mathbb{Q})$, hence $H_{2n+1}(S^{2n}|_{2n+1})=\pi_{2n+1}(S^{2n}|_{2n+1})=\pi_{2n+1}(S^{2n})$ is finite. Now from the fibration

$$S^{2n}|_{2n+2}\xrightarrow{K(\pi_{2n+1}(S^{2n}),2n)}S^{2n}|_{2n+1}$$

we know that the cohomology of the fiber are finite, hence $H^\bullet(S^{2n}|_{2n+2},\mathbb{Q})=H^\bullet(S^{2n}|_{2n+1},\mathbb{Q})$ for $n>1$. Therefore, $\pi_{2n+2}(S^{2n}|_{2n+2})=\pi_{2n+2}(S^{2n})$ is finite, hence $H^\bullet(S^{2n}|_{2n+3},\mathbb{Q})=H^\bullet(S^{2n}|_{2n+2},\mathbb{Q})\Rightarrow\ldots\Rightarrow H^\bullet(S^{2n}|_{4n-1},\mathbb{Q})=H^\bullet(S^{2n}|_{4n-2},\mathbb{Q})=\ldots=H^\bullet(S^{2n}|_{2n+1},\mathbb{Q})=H^\bullet(S^{4n-1},\mathbb{Q})$, and $H_{4n-1}(S^{2n}|_{4n-1},\mathbb{Q})=\pi_{4n-1}(S^{2n}|_{4n-1})\otimes\mathbb{Q}=\pi_{4n-1}(S^{2n})\otimes\mathbb{Q}=\mathbb{Q}$.

Finally, the s.s. of

$$S^{2n}|_{4n}\xrightarrow{K(\pi_{4n-1}(S^{2n}),4n-2)}S^{2n}|_{4n-1}$$

implies $H^\bullet(S^{2n}|_{4n},\mathbb{Q})=\mathbb{Q}[0]$, hence $H_{\geq 4n}(S^{2n}|_{4n})$ are finite, hence $\pi_{\geq 4n}(S^{2n}|_{4n})=\pi_{\geq 4n}(S^{2n})$ are finite. $\square$

1.4. **Theorem.** (A. Borel, 1953) Consider a fibration $F \to E \to B$ with $\pi_1(B) = 0$ and contractible E. Then $H^\bullet(B, \mathbb{Q}) = \mathbb{Q}[x_1, \dots, x_k]$,
$\deg x_i = 2r_i$ iff $H^\bullet(F, \mathbb{Q}) = \Lambda_{\mathbb{Q}}(y_1, \dots, y_k)$, $\deg y_i = 2r_i - 1 = \deg x_i - 1$.

Follows from the Eilenberg-Moore s.s. (1962) with

$$E_2^{p,q} = \mathrm{Tor}_{p,\deg=q}^{H^\bullet(B)}(H^\bullet(X), H^\bullet(Y)) \implies H^\bullet(X \times_B Y)$$

applied to $X = E$, $Y = \mathrm{pt}$.

This is the s.s. of the bicomplex $\mathrm{Bar} \otimes_{C^\bullet(B)} C^\bullet(Y)$ quasiisomorphic to $C^\bullet(X \times_B Y)$. Here $C^\bullet(?)$ is the singular cochain complex, and Bar is the bar-resolution of $C^\bullet(X)$ as of $C^\bullet(B)$-dg-module.

In particular, $d_{2r_i} y_i = x_i$, while all the previous differentials annihilate y_i. Such a homomorphism from the subgroup of $H^n(F)$ annihilated by $d_{\leq n}$ to the quotient of $H^{n+1}(B)$ modulo the image of $d_{\leq n}$ ("partially defined multivalued homomorphism $H^n(F) \to H^{n+1}(B)$") is called the *transgression* τ.

1.5. **Classifying spaces.** If a topological group G acts freely on a contractible space EG, the quotient space $BG := EG/G$ is called the *classifying space* of G. For instance, $K(\pi, 1) = B\pi$, and $\mathbb{CP}^\infty = BS^1$. Let $\mathrm{St}(k, n, \mathbb{C})$ be the space of orthonormal collections of k vectors in $\mathbb{C}^n$ equipped with a positive definite hermitian product (the *Stiefel* variety). We have an embedding $\mathrm{St}(k, n, \mathbb{C}) \hookrightarrow \mathrm{St}(k, n+1, \mathbb{C})$, and $\mathrm{St}(k, \infty, \mathbb{C}) := \lim\limits_{n\to\infty} \mathrm{St}(k, n, \mathbb{C})$. It is contractible, and $\mathrm{St}(k, \infty, \mathbb{C})/\,\mathrm{U}(k) = \mathrm{Gr}(k, \infty, \mathbb{C}) := \lim\limits_{n\to\infty} \mathrm{Gr}(k, n, \mathbb{C})$. Hence $\mathrm{Gr}(k, \infty, \mathbb{C}) = B\mathrm{U}(k)$.

Given a subgroup $H \subset G$, $EG/H = BH$, and we get a fibration $G/H \to BH \to BG$. Take $G = \mathrm{U}(k) \supset T = (S^1)^k = H$. Then $G/H = F\ell(\mathbb{C}^k)$ is the space of complete flags in $\mathbb{C}^k$, and $BH = (\mathbb{CP}^\infty)^k$, so that $H^\bullet(BH, \mathbb{Q}) = \mathbb{Q}[z_1, \dots, z_k]$, $\deg z_i = 2$. The normalizer $N(k)$ of T in $\mathrm{U}(k)$ is the semidirect product of T and the symmetric group S_k, and we consider the composition $BT \to BN(k) \to B\mathrm{U}(k)$. The left arrow is an S_k-torsor, so that $H^\bullet(BN(k), \mathbb{Q}) = H^\bullet(BT, \mathbb{Q})^{S_k} \cong \mathbb{Q}[x_1, x_2, \dots, x_k]$, $\deg x_i = 2i$. The right arrow is a fibration with fiber $F\ell(\mathbb{C}^k)/S_k$, and $H^\bullet(F\ell(\mathbb{C}^k)/S_k, \mathbb{Q}) = H^\bullet(F\ell(\mathbb{C}^k), \mathbb{Q})^{S_k}$. We know that $\dim H^\bullet(F\ell(\mathbb{C}^k), \mathbb{Q}) = \chi(F\ell(\mathbb{C}^k)) = k!$ by the Bruhat decomposition (= Schubert cells). It follows that $H^\bullet(F\ell(\mathbb{C}^k)/S_k, \mathbb{Q}) = \mathbb{Q}[0]$, and hence $H^\bullet(B\mathrm{U}(k), \mathbb{Q}) = H^\bullet(BN(k), \mathbb{Q}) = \mathbb{Q}[x_1, x_2, \dots, x_k]$.

Finally, from the Borel Theorem we obtain yet another way to calculate $H^\bullet(\mathrm{U}(k), \mathbb{Q})$. It works for any compact Lie group. For instance, for $\mathrm{USp}(k)$ the quotient of the normalizer of the maximal torus by this torus (the Weyl group) is the semidirect product $W = S_k \ltimes \mathbb{F}_2^k$ (where $\mathbb{F}_2^k$ acts by changing the signs of z_i's), hence $H^\bullet(BT, \mathbb{Q})^W \cong \mathbb{Q}[x'_1, x'_2, \dots, x'_k]$, $\deg x'_i = 4i$.

1.6. **Cohomological operations.** They are the natural transformations $\mathcal{O}(n, q, \pi)$ from $H^n(?, \pi)$ to $H^q(?, \pi)$. By Yoneda, $\mathcal{O}(n, q, \pi) = H^q(K(\pi, n), \pi)$.

1.6.1. *Example: Bockstein homomorphism.* A short exact sequence $0 \to \mathbb{Z}/\ell\mathbb{Z} \to \mathbb{Z}/\ell^2\mathbb{Z} \to \mathbb{Z}/\ell\mathbb{Z} \to 0$ gives rise to a long exact sequence

$$\ldots \to H^n(X, \mathbb{Z}/\ell\mathbb{Z}) \to H^n(X, \mathbb{Z}/\ell^2\mathbb{Z}) \to H^n(X, \mathbb{Z}/\ell\mathbb{Z}) \xrightarrow{b_n} H^{n+1}(X, \mathbb{Z}/\ell\mathbb{Z}) \to \ldots$$

In fact, b_n is a generator of $H^{n+1}(K(\mathbb{Z}/\ell\mathbb{Z}, n), \mathbb{Z}/\ell\mathbb{Z})$.

1.6.2. *Stability.* Since the suspension Σ is left adjoint to the loops Ω, and $\Omega K(\pi, n+1) = K(\pi, n)$, we get $\Sigma \colon H^n(X, \pi) \xrightarrow{\sim} H^{n+1}(\Sigma X, \pi)$. *Stable operations* $\mathcal{O}_s(k, \pi)$ are those that commute with Σ and raise the cohomological degree by k, i.e. $\mathcal{O}_s(k, \pi) \subset \mathcal{O}(n, n+k, \pi)$. They are also compatible with l.s.e.

$$\ldots \to H^n(X, \pi) \xrightarrow{i^*} H^n(Y, \pi) \xrightarrow{\delta} H^{n+1}(X/Y, \pi) \to H^{n+1}(X) \to \ldots$$

for a Borsuk pair (cofibration) $Y \xrightarrow{i} X$. We only need to check that they commute with δ. Since

$$X/Y \sim \mathrm{Cone}(i)/\mathrm{Cone}(Y) \sim \mathrm{Cone}(i) \xrightarrow{f} \big(X \cup_i \mathrm{Cone}(Y)\big)/X = \Sigma Y,$$

the desired commutativity follows from Exercise 4. Now from Exercise 5 we see that the stable operations commute with transgression in s.s.: if $\alpha \in H^n(F, \pi) = E_2^{0,n}$ is transgressive, i.e. $d_2\alpha = \ldots = d_n\alpha = 0$, and φ is a stable operation in $\mathcal{O}_s(n, n+k, \pi)$, then $\varphi(\alpha) \in H^{n+k}(F, \pi)$ is also transgressive. Moreover, if

$$\tau(\alpha) := d_{n+1}\alpha \in E_{n+1}^{n+1,0} = H^{n+1}(B, \pi)/(+\operatorname{Im} d_{\leq n})$$

contains $\beta \in H^{n+1}(B, \pi)$, then $\tau(\varphi(\alpha)) \ni \varphi(\beta)$.

1.6.3. *Relation with cohomology of Eilenberg-MacLane spaces.* Since the suspension Σ is left adjoint to the loops Ω, and $\Omega K(\pi, n+1) = K(\pi, n)$, we get $f_n \colon \Sigma K(\pi, n) \to K(\pi, n+1)$, and hence

$$\ldots \to H^{n+k}(K(\pi, n), \pi) \xrightarrow{f_{n-1}^*} H^{n-1+k}(K(\pi, n-1), \pi) \to \ldots \to H^{k+1}(K(\pi, 1), \pi),$$

and $\mathcal{O}_s(k, \pi) = \varprojlim$ of this sequence. In fact, this sequence stabilizes for $n > k$.

Finally, $\bigoplus_k \mathcal{O}_s(k, \pi)$ forms a graded *noncommutative* ring A_π with respect to composition. If $\pi = \mathbb{F}_p$, it is called the *Steenrod algebra* A_p (over $\mathbb{F}_p$).

1.7. **Steenrod squares.** They are generators of A_2 (so that $p = 2$), e.g. $Sq^1 = b_2$ is the Bockstein homomorphism.

Construction: let $e_n \in H^n(K(\mathbb{F}_2, n), \mathbb{F}_2)$ be the fundamental class. We set $Sq^n e_n := e_n^2$ (hence the name). Let $n > 1$. Consider the s.s. of

$$\mathrm{pt} \xrightarrow{K(\mathbb{F}_2, n-1)} K(\mathbb{F}_2, n).$$

Figure Everything in the 0th column below e_{n-1}^2 is transgressive, and e_{n-1}^2 is transgressive as well. Indeed, for $d_n \colon E_n^{0,2n-2} \to E_n^{0,n-1}$ we have $d_n(e_{n-1}^2) = 2e_{n-1} \cdot d_n e_{n-1} = 0$. Hence

$$\tau e_{n-1}^2 := d_{2n-1} e_{n-1}^2 =: h \in E_{2n-1}^{2n-1,0} = E_2^{2n-1,0} = H^{2n-1}(K(\mathbb{F}_2, n), \mathbb{F}_2)$$

does not vanish, and we set $Sq^{n-1}(e_n) := h$.

Now let $n > 2$. Then $Sq^{n-2}(e_{n-1}) \in E_2^{0,2n-3}$ is transgressive, and

$$Sq^{n-2}e_n := \tau Sq^{n-2}(e_{n-1}) \in H^{2n-2}(K(\mathbb{F}_2, n), \mathbb{F}_2),$$

and so on. Also, $Sq^{>n}e_n := 0$. It remains to check the stability property $f_{n-1}^* Sq^k e_n = Sq^k e_{n-1}$.

First we assume $k \leq n-1$. Since f_{n-1}^* is the composition of what

$$\Sigma K(\mathbb{F}_2, n-1) = \Sigma\Omega K(\mathbb{F}_2, n) \to K(\mathbb{F}_2, n)$$

induces in cohomology and of the suspension isomorphism, f_{n-1}^* is inverse to the transgression in the s.s. of

$$\text{pt} \xrightarrow{K(\mathbb{F}_2, n-1)} K(\mathbb{F}_2, n).$$

But $\tau(Sq^k e_{n-1}) = Sq^k e_n$ for $k \leq n-1$ by construction, hence $f_{n-1}^* Sq^k e_n = Sq^k e_{n-1}$.

Now $k = n$; then $Sq^k e_{n-1} = Sq^{k+1}e_{n-1} = \ldots = 0$ by construction, and we have to check $f_{k-1}^* Sq^k e_k = f_{k-1}^* e_k^2 = 0$. Recall that f_{k-1}^* is the composition of

$$H^\bullet(K(\mathbb{F}_2, k), \mathbb{F}_2) \to H^\bullet(\Sigma K(\mathbb{F}_2, k-1), \mathbb{F}_2) \to H^{\bullet-1}(K(\mathbb{F}_2, k-1), \mathbb{F}_2).$$

The left map is a ring homomorphism taking $e_k \mapsto \Sigma e_{k-1}$, and hence $e_k^2 \mapsto (\Sigma e_{k-1})^2 = 0$ by the triviality of multiplication in the cohomology of suspension (Exercise 3).

1.7.1. *H. Cartan formula.* $Sq^k(\alpha\beta) = \sum_{l+j=k} Sq^l\alpha \cdot Sq^j\beta$.

2. April 3rd

2.1. **Commutativity of multiplication in cohomology.** Recall the cellular structure of $S^\infty = ES_2$ with two cells $e_\pm^n$ in each dimension n. It is acted upon by the antipodal automorphism $T\colon e_\pm^n \mapsto e_\mp^n$. The corresponding cellular chain complex $\mathcal{C}_\bullet(S^\infty, \mathbb{F}_2)$ is acted upon by the symmetric group S_2 with generator T, and forms a free resolution $\mathcal{P}^\bullet := \mathcal{C}_{-\bullet}(S^\infty, \mathbb{F}_2)$ of the trivial S_2-module $\mathbb{F}_2$.

Given a complex V of $\mathbb{F}_2$-vector spaces, a *symmetric multiplication* on V is a morphism of complexes $V \otimes V \otimes \mathcal{P}^\bullet \to V$ that factors through the coinvariants $D_2(V) := (V \otimes V \otimes \mathcal{P}^\bullet)_{S_2}$ where T acts on $V \otimes V$ by permutation of factors. For a CW-complex X we construct a symmetric multiplication on the cellular cochain complex $\mathcal{C}^\bullet(X, \mathbb{F}_2)$.

We fix a cellular approximation $\widetilde{\Delta}\colon X \to X \times X$ of the diagonal $\Delta\colon X \hookrightarrow X \times X$, and get $\Delta^{(0)} := (\widetilde{\Delta})^*\colon \mathcal{C}^\bullet(X \times X, \mathbb{F}_2) \to \mathcal{C}^\bullet(X, \mathbb{F}_2)$. We have $T \circ \widetilde{\Delta} \sim \widetilde{\Delta}$, hence $\Delta^{(0)}T \sim \Delta^{(0)}$, i.e. there is a homotopy $\Delta^{(1)}\colon \mathcal{C}^\bullet(X \times X, \mathbb{F}_2) \to \mathcal{C}^{\bullet-1}(X, \mathbb{F}_2)$ such that (note that $+ = -$ in $\mathbb{F}_2$) $\Delta^{(1)}d + d\Delta^{(1)} = \Delta^{(0)}T + \Delta^{(0)}$.

Now there is a homotopy $\Delta^{(2)}\colon \mathcal{C}^\bullet(X \times X, \mathbb{F}_2) \to \mathcal{C}^{\bullet-2}(X, \mathbb{F}_2)$ such that $\Delta^{(2)}d + d\Delta^{(2)} = \Delta^{(1)}T + \Delta^{(1)}$, and so on: $\Delta^{(q)}\colon \mathcal{C}^\bullet(X \times X, \mathbb{F}_2) \to \mathcal{C}^{\bullet-q}(X, \mathbb{F}_2)$ such that $\Delta^{(q)}d + d\Delta^{(q)} = \Delta^{(q-1)}T + \Delta^{(q-1)}$. This defines the desired symmetric multiplication

$$\mathbf{m}\colon \mathcal{C}^\bullet(X) \otimes \mathcal{C}^\bullet(X) \otimes \mathcal{P}^\bullet = \mathcal{C}^\bullet(X \times X) \otimes \mathcal{P}^\bullet \to \mathcal{C}^\bullet(X),$$

$$? \otimes e_+^q \mapsto \Delta^{(q)}(?),\ ? \otimes e_-^q \mapsto \Delta^{(q)}T.$$

2.2. **Another construction of Steenrod squares.** Let $c_n \in H^n(\mathbb{F}_2[-n])$ be the unique nonzero element. We have

$$D_2(\mathbb{F}_2[-n]) = \big((\mathbb{F}_2[-n]) \otimes (\mathbb{F}_2[-n]) \otimes \mathcal{P}^\bullet\big)_{S_2} = (\mathbb{F}_2[-2n]) \otimes (\mathcal{P}^\bullet_{S_2})$$

The latter factor is the cellular chain complex of $\mathbb{RP}^\infty$ that has a unique nonzero homology class x_m in (homological) degree $m \in \mathbb{N}$.

For a complex V, a class $v \in H^nV$ determines a homotopy class of maps $\eta\colon \mathbb{F}_2[-n] \to V$. For $k \leq n$, we set

$$\overline{Sq}^k(v) := \text{the image of } \; c_{2n} \otimes x_{n-k} \in c_{2n} \otimes H_{n-k}(\mathbb{RP}^\infty, \mathbb{F}_2) \cong H^{n+k}D_2(\mathbb{F}_2[-n])$$

under the induced map

$$D_2(\mathbb{F}_2[-n]) \xrightarrow{D_2(\eta)} D_2(V),$$

and we set $\overline{Sq}^k(v) := 0$ for $k > n$. If V is equipped with a symmetric multiplication $D_2(V) \to V$, we set $Sq^k(v) :=$ the image of $\overline{Sq}^k(v)$ under the induced map $H^{n+k}D_2(V) \to H^{n+k}V$. When $V = \mathcal{C}^\bullet(X, \mathbb{F}_2)$, we obtain $Sq^k\colon H^n(X, \mathbb{F}_2) \to H^{n+k}(X, \mathbb{F}_2)$.

2.3. **Additivity.** For $v, v' \in H^nV$ and any k, we have

$$\overline{Sq}^k(v + v') = \overline{Sq}^k(v) + \overline{Sq}^k(v') \in H^{n+k}D_2(V).$$

In particular, if V is equipped with a symmetric multiplication, we have $Sq^k(v + v') = Sq^k(v) + Sq^k(v') \in H^{n+k}V$.

Indeed, for $k > n$ we have all zeroes, and for $k = n$

$$\overline{Sq}^n(v + v') = (v + v')^2 = \overline{Sq}^n(v) + \overline{Sq}^n(v') + (vv' + v'v).$$

But the multiplication $V \otimes V \to D_2(V)$ is commutative up to homotopy, so that $vv' + v'v = 2vv' = 0$.

If $k < n$, by functoriality it suffices to consider the universal case $V = \mathbb{F}_2[-n] \oplus \mathbb{F}_2[-n]$. It reduces to the evident trivial case $v = 0$ or $v' = 0$ by the following result. Let $v_i,\ i \in I$, be an ordered basis of $H^\bullet V$. Then $\{v_iv_j,\ i < j\} \cup \{\overline{Sq}^r(v_i),\ r \leq \deg v_i\}$ is a basis of $H^\bullet D_2(V)$. This in turn follows by induction from

$$D_2(V \oplus W) \simeq D_2(V) \oplus (V \otimes W) \oplus D_2(W).$$

2.4. **Stability.** There is a canonical map $\phi\colon D_2(W[-1]) \to D_2(W)[-1]$. For $W = \mathbb{F}_2[-n]$ it sends $c_{2n+2} \otimes x_m \mapsto c_{2n} \otimes x_{m-1}$ in notation of beginning of §2.2. The following diagram commutes:

$$\begin{array}{ccc} H^\bullet(W[-1]) & = & H^{\bullet-1}W \\ \big\downarrow{\scriptstyle \overline{Sq}^k} & & \big\downarrow{\scriptstyle \overline{Sq}^k} \\ H^{\bullet+k}(D_2(W[-1])) & \xrightarrow{\phi} & H^{\bullet+k-1}D_2(W) \end{array}$$

Proof: Set $V = W[-1]$. Fix $v \in H^nV$, and set $w =$ the corresponding class in $H^{n-1}W$. By functoriality, we may assume $V = \mathbb{F}_2[-n],\ W = \mathbb{F}_2[1-n]$. For $k \geq n,\ H^{n+k-1}D_2(W) = 0,$

so we consider $k < n$. Then $H^{n+k-1}D_2(W)$ (resp. $H^{n+k}D_2(V)$) is generated by $\overline{Sq}^k(w)$ (resp. $\overline{Sq}^k(v)$). It suffices to show that $\phi\colon H^m D_2(V) \xrightarrow{\sim} H^{m-1}D_2(W)$ for $m < 2n$.

Let U be the exact complex $\ldots \to 0 \to \mathbb{F}_2 w \xrightarrow{\sim} \mathbb{F}_2 v \to 0 \to \ldots$, so that $V \to U \to W$ is a distinguished triangle (an exact triple of complexes). From the sequence $V^{\otimes 2} \to U^{\otimes 2} \to W^{\otimes 2}$, the complex $W^{\otimes 2}[-1]$ is homotopic to $\ldots \to 0 \to \mathbb{F}_2 vw \oplus \mathbb{F}_2 wv \to \mathbb{F}_2 v^2 \to 0 \to \ldots$ So we get a distinguished triangle $V^{\otimes 2} \to W^{\otimes 2}[-1] \to \mathbb{F}_2^2[1-2n]$ of complexes equipped with an S_2-action. The (homotopy) coinvariants takes distinguished triangles to distinguished triangles, so we get a distinguished triangle $D_2(V) \to D_2(W)[-1] \to \mathbb{F}_2[1-2n]$. The associated long exact sequence gives rise to $H^m D_2(V) \xrightarrow{\sim} H^{m-1}D_2(W)$ for $m < 2n$. $\square$

2.5. **Stability of Steenrod squares.** If a complex V carries a symmetric multiplication $D_2(V) \to V$, then $V[-1]$ inherits a symmetric multiplication given by $D_2(V[-1]) \xrightarrow{\phi} D_2(V)[-1] \to V[-1]$, so we get a commutative diagram

$$\begin{array}{ccc} H^{\bullet+1}D_2(V[-1]) & \longrightarrow & H^{\bullet+1}(V[-1]) \\ \phi\big\downarrow & & \big\downarrow\wr \\ H^{\bullet}D_2(V) & \longrightarrow & H^{\bullet}V. \end{array}$$

Hence the canonical isomorphism $H^\bullet V \cong H^{\bullet+1}(V[-1])$ commutes with the Steenrod squares Sq^k. In particular, if X is a pointed topological space, then the canonical isomorphism $H^\bullet(X, \mathbb{F}_2) \cong H^{\bullet+1}(\Sigma X, \mathbb{F}_2)$ commutes with the action of Steenrod squares Sq^k.

2.5.1. *Example.* Let $v \in H^n_{\mathrm{red}}(S^n, \mathbb{F}_2)$ be the fundamental class. Then $Sq^k(v) = v$ if $k = 0$, and 0 otherwise. By stability this reduces to $n = 0$.

For arbitrary X, $v \in H^n(X, \mathbb{F}_2)$, we have $Sq^k(v) = v$ if $k = 0$, and 0 if $k < 0$. Indeed, since $H^n(X, \mathbb{F}_2)$ is representable by $K(\mathbb{F}_2, n)$, it suffices to consider the universal case $X = K(\mathbb{F}_2, n)$, $v = e_n$ is the fundamental class. Consider the classifying map $f\colon S^n \to K(\mathbb{F}_2, n)$. The induced map $H^{n+k}(K(\mathbb{F}_2, n), \mathbb{F}_2) \to H^{n+k}(S^n, \mathbb{F}_2)$ is bijective for $k \le 0$, so we are done by the previous example.

Note that for an arbitrary complex V the negative Steenrod squares may happen to be nontrivial.

3. April 10th

3.1. **Cartan formula.** For complexes V, W we have isomorphisms

$$D_2(V) \otimes D_2(W) = (V^{\otimes 2} \otimes \mathcal{P}^\bullet)_{S_2} \otimes (W^{\otimes 2} \otimes \mathcal{P}^\bullet)_{S_2} = \big((V \otimes W)^{\otimes 2} \otimes (\mathcal{P}^\bullet)^{\otimes 2}\big)_{S_2 \times S_2},$$

$$D_2(V \otimes W) = \big((V \otimes W)^{\otimes 2} \otimes \mathcal{P}^\bullet\big)_{S_2}.$$

A canonical map $\big((V \otimes W)^{\otimes 2} \otimes \mathcal{P}^\bullet\big)_{S_2} \to \big((V \otimes W)^{\otimes 2} \otimes (\mathcal{P}^\bullet)^{\otimes 2}\big)_{S_2 \times S_2}$ given by the diagonal embedding of S_2 into $S_2 \times S_2$ induces

$$\psi\colon D_2(V \otimes W) \to D_2(V) \otimes D_2(W).$$

3.1.1. *Proposition.* Let $v \in H^m V,\ w \in H^n W$, so that $v \otimes w \in H^{m+n}(V \otimes W)$. Then for any $k \in \mathbb{Z}$ we have

$$\psi \overline{Sq}^k(v \otimes w) = \sum_{l+j=k} \overline{Sq}^l(v) \otimes \overline{Sq}^j(w) \in H^{m+n+k}(D_2(V) \otimes D_2(W)).$$

(The sum is actually finite.)

Proof: If $k > m+n$ both sides vanish. If $k = m+n-i$, we rewrite

$$\psi \overline{Sq}^{m+n-i}(v \otimes w) = \sum_{i'+i''=i} \overline{Sq}^{m-i'}(v) \otimes \overline{Sq}^{n-i''}(w),\ \ i', i'' \in \mathbb{N}.$$

By functoriality we may assume $V = \mathbb{F}_2[-m],\ W = \mathbb{F}_2[-n]$. Then in notation of beginning of §2.2, $H^\bullet D_2(V) \cong \mathbb{F}_2 c_{2m} \otimes x_{2m-\bullet}$,

$$H^\bullet D_2(W) \cong \mathbb{F}_2 c_{2n} \otimes x_{2n-\bullet},\ H^\bullet D_2(V \otimes W) \cong \mathbb{F}_2 c_{2m+2n} \otimes x_{2m+2n-\bullet},\ \text{and}$$

$$\overline{Sq}^{m+n-i}(v \otimes w) = c_{2m+2n} \otimes x_i,\ \overline{Sq}^{m-i'}(v) = c_{2m} \otimes x_{i'},\ \overline{Sq}^{n-i''}(w) = c_{2n} \otimes x_{i''}.$$

The map ψ corresponds to the coproduct

$$\Delta\colon H_\bullet(\mathbb{RP}^\infty, \mathbb{F}_2) \to H_\bullet(\mathbb{RP}^\infty, \mathbb{F}_2) \otimes H_\bullet(\mathbb{RP}^\infty, \mathbb{F}_2).$$

Since $H_\bullet(\mathbb{RP}^\infty, \mathbb{F}_2) = \mathbb{F}_2[t]$, $\deg t = 1$, and x_j is dual to t^j, the coproduct is $\Delta(x_i) = \sum_{i'+i''=i} x_{i'} \otimes x_{i''}$. Finally,

$$\overline{Sq}^{m+n-i}(v \otimes w) = c_{2m+2n} \otimes x_i \mapsto \sum_{i'+i''=i} (c_{2m} \otimes x_{i'}) \otimes (c_{2n} \otimes x_{i''}) = \overline{Sq}^{m-i'}(v) \otimes \overline{Sq}^{n-i''}(w).$$

3.1.2. *Very symmetric multiplication.* Generalizing the case of 2 factors, we may consider $D_n(V) := (V^{\otimes n} \otimes \mathcal{P}_n^\bullet)_{S_n}$, where $\mathcal{P}_n^\bullet$ is a free resolution of the trivial S_n-module $\mathbb{F}_2$. There is a canonical map $\varphi\colon D_m(D_n(V)) \to D_{mn}(V)$ since the LHS is represented by

$$\big((V^{\otimes n} \otimes \mathcal{P}_n^\bullet)_{S_n}^{\otimes m} \otimes \mathcal{P}_m^\bullet\big)_{S_m} \sim (V^{\otimes mn} \otimes Q^\bullet)_{S_m \ltimes S_n^m},$$

where $Q^\bullet$ is a free resolution of the trivial $S_m \wr S_n$-module $\mathbb{F}_2$. The RHS is $(V^{\otimes mn} \otimes \mathcal{P}_{mn}^\bullet)_{S_{mn}}$, and φ is induced by the embedding $S_m \wr S_n \hookrightarrow S_{mn}$.

A symmetric multiplication $\mathbf{m}\colon D_2(V) \to V$ is called *very symmetric* if there is a map $\mathbf{m}'\colon D_4(V) \to V$ such that the following diagram commutes:

$$\begin{array}{ccc} D_2(D_2(V)) & \xrightarrow{D_2(\mathbf{m})} & D_2(V) \\ \varphi\big\downarrow & & \mathbf{m}\big\downarrow \\ D_4(V) & \xrightarrow{\mathbf{m}'} & V. \end{array}$$

(The cellular cochain complex $V = \mathcal{C}^\bullet(X, \mathbb{F}_2)$ has a very symmetric multiplication (as any E_∞-algebra), see e.g. arXiv:0106024, §2).

Then the following diagram commutes up to homotopy:

$$\begin{array}{ccccccc} D_2(V\otimes V) & \longrightarrow & D_2(D_2(V)) & \xrightarrow{D_2(\mathbf{m})} & D_2(V) & \xrightarrow{\mathbf{m}} & V \\ {\scriptstyle\psi}\big\downarrow & & & & & & \big\| \\ D_2(V)\otimes D_2(V) & \xrightarrow{\mathbf{m}\otimes\mathbf{m}} & V\otimes V & \longrightarrow & D_2(V) & \xrightarrow{\mathbf{m}} & V. \end{array}$$

So passing to cohomology and applying the above Proposition, we obtain the Cartan formula 1.7.1. In other words, the *total Steenrod square* $Sq(x) := \sum_{n\geq 0} Sq^n(x)$ is a multiplicative operation.

3.1.3. *Corollary.* Recall $H^\bullet(\mathbb{RP}^\infty, \mathbb{F}_2) = \mathbb{F}_2[\tau]$. We have $Sq^k(\tau^n) = \binom{n}{k}\tau^{n+k}$.

Proof: $\deg\tau = 1 \Rightarrow Sq^n(\tau) = 0$ for $n > 1$, and $Sq^1(\tau) = \tau^2$. Hence the total Steenrod square $Sq(\tau) = Sq^0(\tau) + Sq^1(\tau) = \tau + \tau^2$. By multiplicativity, $Sq(\tau^n) = (\tau + \tau^2)^n = \sum_{0\leq k\leq n}\binom{n}{k}\tau^{n+k}$. □

3.2. **Odd primes.** We consider the complexes of $\mathbb{F}_p$-vector spaces. Contrary to 3.1.2 we denote by $\mathcal{P}_p^\bullet$ a free resolution of the trivial S_p-module $\mathbb{F}_p$ (as opposed to $\mathbb{F}_2$). We define $D_p(V) := (V^{\otimes p}\otimes\mathcal{P}_p^\bullet)_{S_p}$. A symmetric multiplication on a complex V is a morphism of complexes $D_p(V) \to V$. The cochain complex of a topological space $C^\bullet(X, \mathbb{F}_p)$ can be equipped with a symmetric multiplication.

Similarly to 2.2, any homology class in $H_r(S_p, \mathbb{F}_p)$ defines a cohomological operation $H^n(V) \to H^{pn-r}V$. If $n = 2m$, and $r = 2(p-1)(m-i)$, the homological class dual to $t^{(p-1)(m-i)} \in H^{2(p-1)(m-i)}(S_p, \mathbb{F}_p)$ (see Exercise 4) defines an operation $P^i = St^{2(p-1)i}$ increasing cohomological degree by $2(p-1)i$. In particular, P^m on $H^{2m}V$ is nothing but raising to p-th power. These operations are extended to odd cohomology by stability (or else one can use the homological classes dual to $t^{(p-1)j}t^{p-2}\tau$ to handle the odd degrees). For the cochain complexes of topological spaces, these operations together with the Bockstein homomorphism β_p generate the Steenrod algebra A_p. The Cartan formula holds true: $P^k(xy) = \sum_{l+j=k} P^l(x)P^j(y)$. Also, $\beta_p(xy) = \beta_p(x)y + (-1)^{\deg x}x\beta_p(y)$.

For the unification of notation, in case $p = 2$ we set $P^n := Sq^{2n}$. In fact, already $\beta_p, P^1, P^p, P^{p^2}, P^{p^3}, \ldots$ generate A_p.

3.3. **Comultiplication in the Steenrod algebra.** Since the cohomology $H^k(X, \mathbb{F}_p)$ is represented by the Eilenberg-Maclane space $K(\mathbb{F}_p, k)$, the supercommutative multiplication on cohomology gives rise to multiplication $K(\mathbb{F}_p, l)\times K(\mathbb{F}_p, j) \to K(\mathbb{F}_p, l+j)$. This in turn induces a coproduct on the cohomology of Eilenberg-Maclane spaces, and gives rise to a supercocommutative Hopf algebra structure on A_p. For an operation $\theta\in A_p$ the coproduct $\Delta\theta = \sum\theta_i'\otimes\theta_i''\in A_p\otimes A_p$ satisfies

$$\theta(xy) = \sum(-1)^{\deg\theta_i''\deg x}\theta_i'(x)\theta_i''(y).$$

In particular, $\Delta\beta_p = \beta_p \otimes 1 + 1 \otimes \beta_p$, $\Delta P^k = \sum_{l+j=k} P^l \otimes P^j$. The dual Hopf algebra $A_p^\vee = \mathrm{Hom}(A_p, \mathbb{F}_p)$ is called the *Milnor algebra*: a supercommutative graded algebra with a non-cocommutative coproduct.

3.3.1. *Theorem (Milnor, 1958).* $A_p^\vee \simeq \Lambda_{\mathbb{F}_p}(\tau_0, \tau_1, \ldots) \otimes \mathbb{F}_p[\xi_1, \xi_2, \ldots]$, where ξ_n is the linear dual of $P^{p^{n-1}} P^{p^{n-2}} \cdots P^p P^1$, and τ_n is the linear dual of $P^{p^{n-1}} P^{p^{n-2}} \cdots P^p P^1 \beta_p$. The coproduct is given by

$$\mu(\xi_n) = \sum_{i+j=n} \xi_i \otimes \xi_j^{p^i},\ \mu(\tau_n) = 1 \otimes \tau_n + \sum_{i+j=n} \tau_i \otimes \xi_j^{p^i},$$

(we set $\xi_0 = 1$).

4. April 17th

4.1. **Group automorphisms of the additive supergroup.** In this section p is odd. We set $\Lambda := \Lambda_{\mathbb{F}_p}(\varepsilon)$, $\deg \varepsilon = -1$. We view the algebra $H^\bullet(L_p^\infty, \Lambda)$ (see Exercise 3, April 10th) as the algebra $\Lambda[\mathbb{A}^{1|1}_{\mathbb{F}_p}]$ of Λ-valued functions on the superline $\mathbb{A}^{1|1}_{\mathbb{F}_p}$ with coordinates t, $\deg t = 2$, and τ, $\deg \tau = 1$. The Steenrod algebra A_p acts by cohomological operations on $H^\bullet(L_p^\infty, \Lambda)$ (more precisely, $a(x + \varepsilon y) := a(x) + (-1)^{\deg a} \varepsilon a(y)$). Composing with coproduct $\Delta\colon A_p \to A_p \otimes A_p$, one can take tensor product of Hopf algebra modules, so A_p acts on $H^\bullet((L_p^\infty)^{\times n}, \Lambda)$ for any n. This action is effective for $n \to \infty$, and thus A_p is realized in endomorphisms of something that we presently describe.

We will consider the Λ-valued functions $\Lambda[\![t, \tau]\!]$ on the *formal neighbourhood of the origin* $\widehat{\mathbb{A}}^{1|1}_{\mathbb{F}_p}$ in $\mathbb{A}^{1|1}_{\mathbb{F}_p}$, that is formal Taylor series with topological Λ-basis $t^n \tau^s$, $n \in \mathbb{N}$, $s = 0, 1$. We have

$$f(t + t_1, \tau + \tau_1) = \sum t_1^n \tau_1^s \Delta_{n,s} f(t, \tau),$$

where $\Delta_{n,s} = \frac{1}{n!} \frac{\partial^n}{\partial t^n} \frac{\partial^s}{\partial \tau^s}$ is a continuous Λ-linear endomorphism of $\Lambda[\![t, \tau]\!]$. The algebra $\mathrm{End}(\Lambda[\![t, \tau]\!])$ of all continuous Λ-linear endomorphisms of $\Lambda[\![t, \tau]\!]$ (*not* respecting the algebra structure!) is described in Exercise 5 (April 10th). Its multiplicative group contains the subgroup $\mathrm{Aut}(\Lambda[\![t, \tau]\!])$ of automorphisms *respecting* the algebra structure, i.e. automorphisms of the formal neighbourhood $\widehat{\mathbb{A}}^{1|1}_{\mathbb{F}_p}$ of the origin of our superline.

Moreover, there is an evident structure of the supergroup $\mathbb{G}_a^{1|1}$ on $\mathbb{A}^{1|1}_{\mathbb{F}_p}$, and we can consider the automorphisms of the formal supergroup $\widehat{\mathbb{G}}_a^{1|1}$ (i.e. the automorphisms of $\widehat{\mathbb{A}}^{1|1}_{\mathbb{F}_p}$ respecting the addition operation). Furthermore, among those we can consider the ones that act trivially on the associated graded of the filtration of its Lie algebra arising from $\widehat{\mathbb{G}}_a \subset \widehat{\mathbb{G}}_a^{1|1}$. A typical automorphism like this $\bar{\varphi} \in \mathrm{Aut}(\Lambda[\![t, \tau]\!])$ acts as

$$\bar{\varphi}(t) = t + \sum_{i \geq 1} \bar{\xi}_i t^{p^i},\ \bar{\varphi}(\tau) = \tau + \sum_{i \geq 0} \bar{\tau}_i t^{p^i} \tag{1}$$

for certain $\bar{\xi}_i \in \Lambda^0$, $\bar{\tau}_i \in \Lambda^{-1}$. Such automorphisms form a subgroup $\mathrm{Aut}_p(\Lambda[\![t, \tau]\!]) \subset \mathrm{Aut}(\Lambda[\![t, \tau]\!])$.

4.2. **Coaction of $A_p^\vee$ on $\Lambda[\![t,\tau]\!]$ (Buchstaber, 1978).** We consider the Λ-algebra $\mathcal{H} := \Lambda\otimes A_p^\vee$ with free polynomial generators $\xi_1,\xi_2,\dots,\ \deg\xi_i = -2(p^i-1)$ and free exterior generators $\tau_0,\tau_1,\dots,\ \deg\tau_i = -2p^i+1$. We have a coaction ring homomorphism

$$\varphi\colon \Lambda[\![t,\tau]\!]\to \mathcal{H}\otimes_\Lambda \Lambda[\![t,\tau]\!],\ t\mapsto t+\sum_{i\ge 1}\xi_i t^{p^i},\ \tau\mapsto \tau+\sum_{i\ge 0}\tau_i t^{p^i}$$

(the universal automorphism of $\widehat{\mathbb{G}}_a^{1|1}$ acting trivially on the associated graded of the filtration of its Lie algebra arising from $\widehat{\mathbb{G}}_a\subset\widehat{\mathbb{G}}_a^{1|1}$) and the composition (of universal automorphisms) ring homomorphisms

$$\Lambda[\![t,\tau]\!]\xrightarrow{\varphi_1}\mathcal{H}\otimes_\Lambda\Lambda[\![t,\tau]\!]\xrightarrow{\mathrm{Id}\otimes\varphi_2}\mathcal{H}\otimes_\Lambda(\mathcal{H}\otimes_\Lambda\Lambda[\![t,\tau]\!]),$$

$$t\mapsto\varphi_1(t)=t+\sum_{i\ge1}\xi_{1,i}t^{p^i}\mapsto\varphi_2(t)+\sum_{i\ge1}\xi_{1,i}\varphi_2(t)^{p^i}=t+\sum_{n\ge1}\Big(\sum_{i+j=n}\xi_{1,i}\otimes\xi_{2,j}^{p^i}\Big)t^{p^n},$$

$$\tau\mapsto\varphi_1(\tau)=\tau+\sum_{i\ge0}\tau_{1,i}t^{p^i}\mapsto\varphi_2(\tau)+\sum_{i\ge0}\tau_{1,i}\varphi_2(t)^{p^i}=\tau+\sum_{n\ge0}\Big(1\otimes\tau_{2,n}+\sum_{i+j=n}\tau_{1,i}\otimes\xi_{2,j}^{p^i}\Big)t^{p^n},$$

where $\xi_{1,0}=\xi_{2,0}=1\in\Lambda^0$.

It follows that the ring homomorphism $\mu\colon A_p^\vee\to A_p^\vee\otimes A_p^\vee$ of 3.3.1 equips $A_p^\vee$ with a structure of Hopf algebra, and φ equips $\Lambda[\![t,\tau]\!]$ with a structure of $\mathcal{H}=\Lambda\otimes A_p^\vee$-comodule. Note that $A_p=(A_p^\vee)^\vee=\mathrm{Hom}^{\mathrm{even}}(A_p^\vee,\Lambda)=\mathrm{Hom}_\Lambda^{\mathrm{even}}(\mathcal{H},\Lambda)$ (*restricted* dual). Hence we obtain an embedding $\gamma\colon\ \mathrm{Aut}_p(\Lambda[\![t,\tau]\!])\hookrightarrow\widehat{A}_p^\times$ into the multiplicative group $\widehat{A}_p^\times$ of the completed Steenrod algebra $\widehat{A}_p$. Its image consists of all the linear maps in $\widehat{\mathrm{Hom}}^{\mathrm{even}}(A_p^\vee,\Lambda)$ (*nonrestricted* dual) that are algebra homomorphisms.

4.3. **Action of A_p on $\Lambda[\![t,\tau]\!]$.** We write down explicit formula for the action of A_p on $\Lambda[\![t,\tau]\!]$ dual to the coaction of $A_p^\vee$ on $\Lambda[\![t,\tau]\!]$. For a sequence $I=(s_0,n_1,s_1,\dots,n_k,s_k,0,\dots)$ we set $(\xi,\tau)^I=\tau_0^{s_0}\xi_1^{n_1}\tau_1^{s_1}\cdots\xi_k^{n_k}\tau_k^{s_k}$. These monomials form a basis of $\mathbb{F}_p$-vector space $A_p^\vee$. We denote by $P_I\in A_p$ the dual basis element. In particular, we will have the Bockstein $\beta_p=P_{(1,0,\dots)}$, and the Steenrod powers $P^n=P_{(0,n,0,0,\dots)}$. Any element $a\in A_p$ can be written as $\sum a_{(\xi,\tau)^I}P_I$, where $a_{(\xi,\tau)^I}\in\Lambda$, and $\deg a_{(\xi,\tau)^I}\equiv\deg(\xi,\tau)^I \pmod 2$. We will also need the completed algebra $\widehat{A}_p$ with topological basis $\{P_I\}$. The embedding

$$\gamma\colon\ \mathrm{Aut}_p(\Lambda[\![t,\tau]\!])\hookrightarrow\widehat{A}_p^\times$$

takes $\bar\varphi\in\mathrm{Aut}_p(\Lambda[\![t,\tau]\!])$ with parameters $\bar\xi_i,\bar\tau_i$ (see (1)) to $\gamma(\bar\varphi)=\sum(\bar\xi,\bar\tau)^I P_I\in A_p$.

If we set $\bar\varphi_P(t)=t+t^p,\ \bar\varphi_P(\tau)=\tau$, and $\bar\varphi_\beta(t)=t,\ \bar\varphi_\beta(\tau)=\tau+\varepsilon t$, then

$$\gamma(\bar\varphi_P)=1+\sum_{n>0}P^n,\ \gamma(\bar\varphi_\beta)=1+\beta_p.$$

By Taylor expansion, for $f(t,\tau)\in\Lambda[\![t,\tau]\!]$ we have

$$\bar\varphi_P(f(t,\tau))=f(t+t^p,\tau)=\sum_{n\ge0}t^{pn}\Delta_{n,0}f(t,\tau),$$

$$\bar{\varphi}_\beta(f(t,\tau)) = f(t,\tau+\varepsilon t) = f(t,\tau) + \varepsilon t\Delta_{0,1} f(t,\tau),$$

so that $\bar{\varphi}_P = \sum_{n\geq 0} t^{pn}\Delta_{n,0},\ \bar{\varphi}_\beta = \mathrm{Id} + \varepsilon t\Delta_{0,1}$.

Looking at the composition

$$\mathrm{Aut}_p(\Lambda[\![t,\tau]\!]) \xrightarrow{\gamma} \widehat{A}_p^\times \hookrightarrow \widehat{A}_p \xrightarrow{\rho} \mathrm{End}(\Lambda[\![t,\tau]\!])$$

we conclude that

$$\rho(P^n) = t^{pn}\Delta_{n,0} = t^{pn}\frac{1}{n!}\frac{\partial^n}{\partial t^n},\ \rho(\beta_p) = \varepsilon t\frac{\partial}{\partial \tau}.$$

And this is nothing but the action of A_p on $H^\bullet(L_p^\infty, \Lambda)$.

4.4. **Thom isomorphism (1952).** Given a vector bundle $\mathcal{V} \to B$ of rank n we choose a fiberwise metric and consider the corresponding sphere bundle $S(\mathcal{V}) \to B$ bounding the disc bundle $D(\mathcal{V}) \to B$. The *Thom space* $Th(\mathcal{V})$ is the quotient $D(\mathcal{V})/S(\mathcal{V})$. If B is compact, this is the one point compactification of $\mathcal{V}$. We make one of the following two assumptions: either the coefficient ring of cohomology is $\mathbb{F}_2$, or $\mathcal{V}$ is oriented (i.e. $\Lambda^n\mathcal{V} \setminus B$ has 2 connected components, and we choose one of them) (and then the coefficient ring of cohomology is arbitrary). Then there is a unique *Thom class* $\mathsf{t}_\mathcal{V} \in H^n(Th(\mathcal{V}))$ such that
(a) the restriction of $\mathsf{t}_\mathcal{V}$ to any fiber $D^n/S^{n-1} \simeq S^n$ is the generator h of $H^n(S^n)$.
(b) For any $i \in \mathbb{N}$, the product with $\mathsf{t}_\mathcal{V}$ gives the *Thom isomorphism* Φ:

$$H^i(B) = H^i(D(\mathcal{V})) \xrightarrow[\mathsf{t}_\mathcal{V}\cdot ?]{\sim} H^{n+i}(D(\mathcal{V}), S(\mathcal{V})) = H^{n+i}(Th(\mathcal{V}))$$

(in particular, $\mathsf{t}_\mathcal{V} = \Phi(1)$).

Indeed, consider the fiberwise quotient $\mathcal{E} := D(\mathcal{V})/_B S(\mathcal{V})$ (a sphere bundle over B with a canonical section $s\colon B \to \mathcal{E}$ such that $Th(\mathcal{V}) = \mathcal{E}/s(B)$). The Gysin sequence of $\mathcal{E} \xrightarrow{p} B$:

$$\ldots \to H^{n+i}(B) \xrightarrow{p^*} H^{n+i}(\mathcal{E}) \to H^i(B) \to \ldots$$

is split by s^*, so that $H^{n+i}(\mathcal{E}) = H^{n+i}(B) \oplus H^i(B)$, and the relative cohomology $H^{n+i}(Th(\mathcal{V})) = H^{n+i}(\mathcal{E}/s(B)) = H^{n+i}(\mathcal{E}, s(B)) \cong H^i(B)$. By the construction of Gysin sequence, the latter $H^i(B)$ is actually the $E_2^{i,n} = H^i(B) \otimes H^n(S^n)$ term of the spectral sequence of $\mathcal{E} \xrightarrow{p} B$. We set $\mathsf{t}_\mathcal{V} := 1 \otimes h \in H^0(B) \otimes H^n(S^n)$ (or rather its image in E_∞), and we are done by the multiplicativity of spectral sequence.

4.5. **Stiefel-Whitney characteristic classes (1935-1936).** Now we consider the cohomology with coefficients in $\mathbb{F}_2$. We set $w_i(\mathcal{V}) := \Phi^{-1}Sq^i\Phi(1) \in H^i(B, \mathbb{F}_2)$. The *total Stiefel-Whitney class* is $w(\mathcal{V}) := \sum_i w_i(\mathcal{V})$. They enjoy the following basic properties.

4.5.1. *Dimension.* $w_0(\mathcal{V}) = 1$, and $w_{>n}(\mathcal{V}) = 0$ for $\mathrm{rk}\,\mathcal{V} = n$.

4.5.2. *Naturality.* Consider a cartesian diagram

$$\begin{array}{ccc} \mathcal{V}' & \xrightarrow{f} & \mathcal{V} \\ \downarrow & & \downarrow \\ B' & \xrightarrow{f} & B \end{array}$$

Then $w_i(\mathcal{V}') = f^* w_i(\mathcal{V})$. Indeed, by Exercise 2, the Thom isomorphism is natural: $\Phi' f^* = f^* \Phi$.

4.5.3. *Whitney product formula.* $w_k(\mathcal{V}_1 \oplus \mathcal{V}_2) = \sum_{l+j=k} w_l(\mathcal{V}_1) w_j(\mathcal{V}_2)$; in other words, the total class $w(\mathcal{V}_1 \oplus \mathcal{V}_2) = w(\mathcal{V}_1) w(\mathcal{V}_2)$. Indeed, for vector bundles $\mathcal{V}_1 \to B_1$, $\mathcal{V}_1 \to B_2$ over different bases (later on we will set $B_1 = B_2 = B$), we have

$$\Phi_{\mathcal{V}_1 \times \mathcal{V}_2}(w(\mathcal{V}_1 \times \mathcal{V}_2)) = Sq(\mathsf{t}_{\mathcal{V}_1 \times \mathcal{V}_2}) = Sq(\mathsf{t}_{\mathcal{V}_1} \otimes \mathsf{t}_{\mathcal{V}_2}) = Sq(\mathsf{t}_{\mathcal{V}_1}) \otimes Sq(\mathsf{t}_{\mathcal{V}_2})$$

by Cartan formula. The RHS is

$$\Phi_{\mathcal{V}_1}(w(\mathcal{V}_1)) \otimes \Phi_{\mathcal{V}_2}(w(\mathcal{V}_2)) = \Phi_{\mathcal{V}_1 \times \mathcal{V}_2}(w(\mathcal{V}_1) \otimes w(\mathcal{V}_2))$$

by Exercise 2. Applying $\Phi^{-1}_{\mathcal{V}_1 \times \mathcal{V}_2}$ we obtain $w(\mathcal{V}_1 \times \mathcal{V}_2) = w(\mathcal{V}_1) \otimes w(\mathcal{V}_2)$. Now set $B_1 = B_2 = B$ and restrict to the diagonal $\Delta_B \subset B \times B$.

4.5.4. *Normalization.* Let $\mathcal{O}(-1) = \mathcal{S}_1$ be the tautological line bundle over $\mathbb{RP}^1$. Then $w_1(\mathcal{S}_1)$ is the unique nonzero element $\tau \in H^1(\mathbb{RP}^1, \mathbb{F}_2)$. Indeed, the disc bundle $D(\mathcal{S}_1)$ is the Möbius band M bounded by the circle ∂M. On the other hand, M is homeomorphic to the closure of $\mathbb{RP}^2 \setminus D^2$. We get $H^\bullet(M, \partial M) = H^\bullet(\mathbb{RP}^2, D^2)$. Hence the natural isomorphisms

$$H^1(Th(\mathcal{S}_1)) = H^1(D(\mathcal{S}_1), S(\mathcal{S}_1)) = H^1(M, \partial M) \xleftarrow{\sim} H^1(\mathbb{RP}^2, D^2) \xrightarrow{\sim} H^1(\mathbb{RP}^2).$$

Hence the Thom class $\mathsf{t}_{\mathcal{S}_1} \in H^1(Th(\mathcal{S}_1), \mathbb{F}_2)$ corresponds to the generator $\tau \in H^1(\mathbb{RP}^2, \mathbb{F}_2)$, and $Sq^1(\mathsf{t}_{\mathcal{S}_1}) = \mathsf{t}^2_{\mathcal{S}_1}$ corresponds to $Sq^1(\tau) = \tau^2 \neq 0$. Hence $w_1(\mathcal{S}_1) = \Phi^{-1} Sq^1(\mathsf{t}_{\mathcal{S}_1}) \neq 0$.

5. April 24th

5.1. **Principal G-bundles.** A *principal G-bundle* or a *G-torsor* $\mathcal{E} \xrightarrow{p} B$ is a space $\mathcal{E}$ equipped with a fiberwise right G-action simply transitive on each fiber of p. For example, given a k-vector bundle $\mathcal{V} \to B$ of rank n, the space $\mathcal{E}$ of fiberwise bases is a $\mathrm{GL}(n, \mathsf{k})$-torsor over B. Conversely, $\mathcal{V} = \mathcal{E} \overset{\mathrm{GL}(n,\mathsf{k})}{\times} \mathsf{k}^n$ is the associated vector bundle (here $\mathsf{k} = \mathbb{R}, \mathbb{C}, \mathbb{H}$). If a vector bundle is equipped with a metric, and we consider the space of orthonormal fiberwise bases, we obtain a principal $\mathrm{O}(n)$-bundle (or a principal $\mathrm{U}(n)$-bundle).

We will only consider locally trivial bundles. They can be specified by a covering $B = \bigcup U_\alpha$ and transition functions $\varphi_{\alpha\beta} \colon U_\alpha \cap U_\beta \to G$.

5.1.1. *Lemma.* Let $\mathcal{E} \xrightarrow{p} B\times I$ be a G-torsor over the product of compact B with the interval $I=[0,1]$. Suppose G is a Lie group. Then $\mathcal{E}|_{B\times 0} \simeq \mathcal{E}|_{B\times 1}$.

Proof: We may assume $U_\alpha = V_\beta \times (a_k, a_{k+1})$ where $B = \bigcup V_\beta$, and $I = \bigcup (a_k, a_{k+1})$. This reduces the problem to the case $(a_k, a_{k+1}) = [0,1]$. So we have the transition functions $\varphi_{\gamma\beta}(x,t)\colon (V_\gamma \cap V_\beta)\times I \to G$. We may also assume that $\varphi_{\gamma\beta}$ extends continuously to the closure $\overline{V}_\gamma \cap \overline{V}_\beta$, and hence it is uniformly continuous. Hence there is an open neighbourhood $e \in U \subset G$ such that $U \simeq \mathbb{R}^r$ (where $r = \dim G$) and $\varphi_{\gamma\beta}(x,t_1)\varphi_{\gamma\beta}^{-1}(x,t_2) \in U\ \forall t_1, t_2 \in I$. Moreover, we may assume that $U \cdots U \subset G$ ($3N$-fold product, where N is the number of charts V_β) is also isomorphic to $\mathbb{R}^r$.

We will construct functions $h_\beta\colon V_\beta \to G,\ 1 \le \beta \le N$, by induction in β. We start with $h_1(x) \equiv e$. We set $h_2(x) := \varphi_{12}^{-1}(x,1)\varphi_{12}(x,0)$ on $\overline{V}_1 \cap \overline{V}_2$ and extend it to V_2 with values in $U\cdot U \simeq \mathbb{R}^r$. Now $h_\beta(x) := \varphi_{\gamma\beta}^{-1}(x,1)h_\gamma(x)\varphi_{\gamma\beta}(x,0)$ on $\overline{V}_\gamma \cap \overline{V}_\beta$ for $\gamma < \beta$. It matches on triple intersections by induction assumption, and takes values in $U\cdots U \simeq \mathbb{R}^r$ (3β-fold product). So it can be extended to the whole of V_β with values in $U \cdots U \simeq \mathbb{R}^r$.

Thus $\varphi_{\gamma\beta}(x,1) = h_\gamma(x)\varphi_{\gamma\beta}(x,0)h_\beta^{-1}(x)$, and hence $\mathcal{E}|_{B\times 0} \simeq \mathcal{E}|_{B\times 1}$. □

5.1.2. *Corollary.* Any G-torsor over a disc D^n is trivial.

Proof: Consider its pullback to $D^n \times I \to D^n,\ (x,t) \mapsto tx$. □

5.2. **Classifying spaces. Theorem.** Any G-torsor $\mathcal{E} \xrightarrow{p} B$ over a CW-complex B is isomorphic to a pullback of $EG \to BG$ for an appropriate $\phi\colon B \to BG$. The pullbacks ϕ_1^*EG and ϕ_0^*EG are isomorphic if and only if $\phi_1, \phi_0\colon B \to BG$ are homotopic.

Proof: We have to construct a G-equivariant map $F\colon \mathcal{E} \to EG$ by induction in skeleta of B. Suppose it is constructed for $\mathcal{E}|_{\mathrm{sk}_{n-1} B}$, and we want to extend it through a cell $\chi_\alpha\colon D^n \to e_\alpha^n \subset B$. The pullback $\chi_\alpha^*\mathcal{E}$ is trivial. Since $\chi_\alpha(S^{n-1}) \subset \mathrm{sk}_{n-1} B$, the existing F_{n-1} gives an equivariant map $F_{n-1}(x,g) = F_{n-1}(x,1)g \in EG,\ x \in S^{n-1},\ g \in G$.

Since $\pi_{n-1}(EG) = 0$, the map $F_{n-1}(x,1)$ extends to $F_n(x,1)\colon D^n \to EG$. We set $F_n(x,g) = F_n(x,1)g$, and thus add all the n-cells one by one.

The second claim follows if we replace B by $B\times I$, and $\mathrm{sk}_{n-1} B$ by $B\times\{0\} \cup B \times \{1\}$. □

Thus, the isomorphism classes of G-torsors over a CW-complex B are in a natural bijection with $\mathrm{Ho}(B, BG)$.

5.2.1. *Vector bundles.* Recall that $B\mathrm{GL}(n,\mathsf{k}) = \mathrm{Gr}(n,\infty,\mathsf{k})$. The classifying map $\phi\colon B \to \mathrm{Gr}(n,\infty,\mathsf{k})$ for the $\mathrm{GL}(n,\mathsf{k})$-torsor corresponding to a vector bundle $\mathcal{V} \xrightarrow{p} B$, is obtained from an embedding $\mathcal{V} \hookrightarrow B \times \mathsf{k}^N$ for $N \gg 0$. To construct this embedding for a compact B choose a trivializing covering $B = \bigcup U_\alpha$, inscribe a finer covering $V_\alpha \subset U_\alpha$ such that $\overline{V}_\alpha \subset U_\alpha$ and a still finer covering $W_\alpha \subset V_\alpha$ such that $\overline{W}_\alpha \subset V_\alpha$. Pick a continuous function $f_\alpha\colon B \to \mathsf{k}$ such that $f_\alpha|_{\overline{W}_\alpha} \equiv 1$ and $f_\alpha|_{B\setminus V_i} \equiv 0$.

Let $q_\alpha\colon \mathcal{V}|_{U_\alpha} \to \mathsf{k}^n$ be the trivializing projection. Set $q'_\alpha(v) = f_\alpha(p(v))q_\alpha(v)$ if $v \in p^{-1}(U_\alpha)$, and $q'_\alpha(v) = 0$ otherwise. Finally, $Q := (p, q'_\alpha)\colon \mathcal{V} \to B \times \bigoplus_\alpha \mathsf{k}^n$ is the desired embedding.

5.3. **Cohomology ring of Grassmannian.** **Theorem.** The cohomology ring of the Grassmannian $H^\bullet(\mathrm{Gr}(n,\infty,\mathbb{R}),\mathbb{F}_2) = \mathbb{F}_2[w_1(\mathcal{S}_n),\dots,w_n(\mathcal{S}_n)]$, where $\mathcal{S}_n$ is the tautological n-dimensional vector bundle over the Grassmannian.

Proof: First, there are no relations between $w_i(\mathcal{S}_n)$. Otherwise, since $\mathcal{S}_n$ is the universal bundle, such a relation would be universal (by naturality of Stiefel-Whitney classes). But $H^\bullet((\mathbb{RP}^\infty)^{\times n},\mathbb{F}_2) = \mathbb{F}_2[\tau_1,\dots,\tau_n]$, and $w(\mathcal{S}_1^{(1)}\oplus\ldots\oplus\mathcal{S}_1^{(n)}) = (1+\tau_1)\cdots(1+\tau_n)$, hence $w_i(\mathcal{S}_1^{(1)}\oplus\ldots\oplus\mathcal{S}_1^{(n)})$ is the elementary symmetric polynomial $e_i(\tau_1,\dots,\tau_n)$. And there are no relations between these guys.

By Exercise 2, $\dim H^k(\mathrm{Gr}(n,\infty,\mathbb{R}),\mathbb{F}_2)$ equals the dimension of k-th graded component of $\mathbb{F}_2[w_1(\mathcal{S}_n),\dots,w_n(\mathcal{S}_n)]$, and we are done. □

Thus for a vector bundle $\mathcal{V}\to B$ and the corresponding classifying map $\phi\colon B\to\mathrm{Gr}(n,\infty,\mathbb{R})$ the Stiefel-Whitney classes $w_i(\mathcal{V})$ are the pullbacks $\phi^*w_i(\mathcal{S}_n)$ of the canonical generators of the cohomology ring $H^\bullet(\mathrm{Gr}(n,\infty,\mathbb{R}),\mathbb{F}_2)$.

6. May 1st

6.1. **Obstructions.** Given a real vector bundle $\mathcal{V}\to B$ of rank n, we choose a metric and consider the corresponding $\mathrm{O}(n,\mathbb{R})$-torsor $\mathcal{E}\to B$. The group $\mathrm{O}(n,\mathbb{R})$ acts on the Stiefel variety $\mathrm{St}(k,n,\mathbb{R})$, and we consider the associated bundle

$$\mathcal{S}t_{\mathcal{V}}(k,n,\mathbb{R}) := \mathcal{E}\overset{\mathrm{O}(n,\mathbb{R})}{\times}\mathrm{St}(k,n,\mathbb{R})\to B.$$

Similarly to Exercises of 06.03.2020, there are obstruction classes in $H^{r+1}(B,\underline{\pi_r(\mathrm{St}(k,n,\mathbb{R}))})$ to constructing sections of $\mathcal{S}t_{\mathcal{V}}(k,n,\mathbb{R})$, i.e. k-tuples of orthonormal sections of $\mathcal{V}$. Here $\underline{\pi_r(\mathrm{St}(k,n,\mathbb{R}))}$ is a local system on B with stalks $\pi_r(\mathrm{St}(k,n,\mathbb{R}))$.

6.1.1. *Lemma.* a) $\pi_r(\mathrm{St}(k,n,\mathbb{R}))=0$ for $r<n-k$; b) $\pi_{n-k}(\mathrm{St}(k,n,\mathbb{R}))=\mathbb{Z}$ if $k=1$ or $n-k$ is even, and $\mathbb{Z}/2\mathbb{Z}$ otherwise.

Proof: We proceed by simultaneous induction in n,k starting with $\mathrm{St}(1,n,\mathbb{R})=S^{n-1}$. Consider the fibration projecting to the first vector

$$\mathrm{St}(k,n,\mathbb{R})\xrightarrow{\mathrm{St}(k-1,n-1,\mathbb{R})}S^{n-1}\tag{2}$$

and the long exact sequence

$$\ldots\to\pi_{r+1}(S^{n-1})\to\pi_r(\mathrm{St}(k-1,n-1,\mathbb{R}))\to\pi_r(\mathrm{St}(k,n,\mathbb{R}))\to\pi_r(S^{n-1})\to\ldots$$

If $r<n-2$, the middle arrow is an isomorphism, and we are done for $k>2$ by induction. If $k=1$, then $\mathrm{St}(k,n,\mathbb{R})=S^{n-1}$. If $k=2$, and $r=n-2$, then the above sequence takes form

$$\ldots\to\mathbb{Z}\to\mathbb{Z}\to\pi_{n-2}(\mathrm{St}(2,n,\mathbb{R}))\to 0,$$

and we need to prove that the left arrow is $1+(-1)^{n-1}$. The fibration (2) is the fibration of unit tangent vectors $T_1S^{n-1}\overset{S^{n-2}}{\longrightarrow}S^{n-1}$. We take the identity spheroid $D^{n-1}/S^{n-2}=S^{n-1}\xrightarrow{\mathrm{Id}}S^{n-1}$ and lift it to T_1S^{n-1} so that the boundary $S^{n-2}\subset D^{n-1}$ goes to a fiber S^{n-2}.

To this end we need a vector field on S^{n-1} with *one* singular point. There is a vector field with two singular points (the gradient flow from the north pole to the south pole). It follows that for a vector field with a unique singular point the index at this point is $1+(-1)^{n-1}$. $\square$

Thus the first nontrivial obstruction lies in $H^{n-k+1}(B,\underline{\pi_{n-k}(\mathrm{St}(k,n,\mathbb{R}))})$, and the corresponding local system has stalks $\mathbb{Z}$ or $\mathbb{Z}/2\mathbb{Z}$. For unification we reduce modulo 2 in both cases; then the local systems become trivial, and we obtain the reduced obstruction classes $o_i(\mathcal{V})\in H^i(B,\mathbb{F}_2)$.

6.2. **Theorem.** $o_i(\mathcal{V})=w_i(\mathcal{V})$.

Proof: Due to naturality of both theories, it suffices to consider the universal case $\mathcal{S}_n\to \mathrm{Gr}(n,\infty,\mathbb{R})$. Then $o_i(\mathcal{S}_n)=f_i(w_1(\mathcal{S}_n),\ldots,w_i(\mathcal{S}_n))=f_i'(w_1(\mathcal{S}_n),\ldots,w_{i-1}(\mathcal{S}_n))+\lambda w_i(\mathcal{S}_n)$, where $\lambda\in\mathbb{F}_2$. First, we claim $f_i'=0$. Indeed, consider an auxiliary vector bundle $\mathcal{V}=\mathcal{S}_{i-1}\oplus\mathbb{R}^{n-i+1}\to\mathrm{Gr}(i-1,\infty,\mathbb{R})$. This vector bundle has $n-i+1$ orthonormal sections for trivial reasons, hence the obstruction $o_i(\mathcal{V})\in H^i(\mathrm{Gr}(i-1,\infty,\mathbb{R}),\mathbb{F}_2)$ must vanish. But $o_i(\mathcal{V})=f'(w_1(\mathcal{S}_{i-1}),\ldots,w_{i-1}(\mathcal{S}_{i-1}))+0$, and since $w_1(\mathcal{S}_{i-1}),\ldots,w_{i-1}(\mathcal{S}_{i-1})$ are algebraically independent, $f'\equiv 0$.

It remains to prove $\lambda=1$, i.e. $o_i(\mathcal{V})$ does not vanish identically. We start with $i=n$. We take $\mathcal{V}=\mathcal{Q}_n\to\mathbb{RP}^n$ (the universal quotient bundle): its fiber over a point of $\mathbb{RP}^n$ represented by a point $u\in S^n\subset\mathbb{R}^{n+1}$ is the orthogonal hyperplane $u^\perp$. A section $u\mapsto u_0-(u_0,u)u$ vanishes at a unique point represented by u_0. If we take u_0 in the middle of the unique n-cell of $\mathbb{RP}^n$, then we obtain a section over $\mathrm{sk}_{n-1}\mathbb{RP}^n$, and the obstruction to its extension associates to the n-cell the generator of $\pi_{n-1}S^{n-1}=\mathbb{Z}$ (reduced modulo 2). So it does not vanish, and $\lambda=1$.

If $i<n$, we take $\mathcal{V}=\mathcal{Q}_i\oplus\mathbb{R}^{n-i}\to\mathbb{RP}^i$ and compare with the description of the generator of $\pi_{i-1}\mathrm{St}(n-i+1,n,\mathbb{R})$ in Lemma 6.1.1 to deduce $\lambda=1$.

In other words, the first obstruction to the existence of sections in the above cases happens to be the last one, and its vanishing would imply the existence of sections that would contradict the nontriviality of $w_i(\mathcal{V})$. $\square$

7. May 8th

7.1. **Stiefel-Whitney classes of the tangent bundle.** We will use the Poincaré duality for smooth compact manifolds with coefficients in $\mathbb{F}_2$. In particular, the isomorphism $D\colon H_\bullet(M,\mathbb{F}_2)\xrightarrow{\sim}H^{\dim M-\bullet}(M,\mathbb{F}_2)$. The fundamental class of a smooth submanifold $M'\subset M$ will be denoted $[M']\in H_{\dim M'}(M,\mathbb{F}_2)$. The tubular neighbourhood of M' in M is isomorphic to the disc bundle $D(\mathcal{N}_{M'/M})$ of the normal bundle of M' in M. Hence the Thom space $Th(\mathcal{N}_{M'/M})$ is isomorphic to the quotient of a tubular neighbourhood modulo its boundary, and $H^\bullet(Th(\mathcal{N}_{M'/M}))=H^\bullet(M,M\setminus M')$. The image of the Thom class $\mathsf{t}_{\mathcal{N}_{M'/M}}\in H^{\mathrm{codim}_M M'}(Th(\mathcal{N}_{M'/M}))=H^{\mathrm{codim}_M M'}(M,M\setminus M')$ in $H^{\mathrm{codim}_M M'}(M)$ is the fundamental class $D[M']$.

The fundamental class of the diagonal

$$D[\Delta_M] = \sum \xi_i \otimes \xi_i^* \in H^{\dim M}(M \times M, \mathbb{F}_2),$$

the sum is taken over a homogeneous basis of $H^\bullet(M)$, and $\{\xi_i^*\}$ is the dual basis (with respect to the Poincaré duality).

The *slant product* $H^{i+j}(M \times N) \otimes H_j(N) \to H^i(M)$, $\xi \otimes \eta \mapsto \xi/\eta$ arises from the Künneth formula and the usual pairing. In particular, $D[\Delta_M]/[M] = 1$.

The tangent bundle TM is isomorphic to the normal bundle $\mathcal{N}_{\Delta_M/M\times M}$, and so the disc bundle of the tangent bundle is isomorphic to a tubular neighbourhood of diagonal in $M \times M$. Hence the Thom space $Th(TM)$ is isomorphic to the quotient of a tubular neighbourhood modulo its boundary, and $H^\bullet(Th(TM)) = H^\bullet(M \times M, M \times M \setminus \Delta_M)$. The image of the Thom class $\mathsf{t}_{TM} \in H^{\dim M}(Th(TM)) = H^n(M \times M, M \times M \setminus \Delta_M)$ in $H^{\dim M}(M \times M)$ is the diagonal class $D[\Delta_M]$. By definition, $Sq^i(\mathsf{t}_{TM}) = (w_i(TM) \otimes 1) \cdot \mathsf{t}_{TM}$, hence $Sq^i(D[\Delta_M]) = (w_i(TM) \otimes 1) \cdot D[\Delta_M]$. Since the slant product commutes with the left multiplication by $H^\bullet(M)$, we get

$$(3)\quad Sq^i(D[\Delta_M])/[M] = \big((w_i(TM) \otimes 1) \cdot D[\Delta_M]\big)/[M] = w_i(TM) \cdot (D[\Delta_M]/[M]) = w_i(TM).$$

7.1.1. *Corollary.* If M_1 and M_2 are homotopy equivalent (in particular, the cohomology rings are isomorphic $H^\bullet(M_1, \mathbb{F}_2) \cong H^\bullet(M_2, \mathbb{F}_2)$), then $w_i(TM_1) = w_i(TM_2)$, i.e. the Stiefel-Whitney classes of the tangent bundle are independent of a choice of a smooth structure.

7.1.2. *Wu formula.* Note that the total Steenrod square Sq is an automorphism of the ring $H^\bullet(M, \mathbb{F}_2)$. We will denote Sq^{inv} the inverse automorphism. Then for any homology class $h \in H_\bullet(M, \mathbb{F}_2)$,

$$\langle Sq^{\mathrm{inv}} w(TM), h\rangle = \langle Sq(Dh), [M]\rangle.$$

Indeed, let $n = \dim M$, and define $v_i^M \in H^i(M, \mathbb{F}_2)$ by the identity $\langle v_i^M \cdot \xi, [M]\rangle = \langle Sq^i(\xi), [M]\rangle$ for any $\xi \in H^{n-i}(M, \mathbb{F}_2)$. Set $v^M = 1 + v_1^M + \ldots + v_n^M$. Then $\langle v^M \cdot \xi, [M]\rangle = \langle Sq(\xi), [M]\rangle$. In particular, $\langle Sq(Dh), [M]\rangle = \langle v^M \cdot Dh, [M]\rangle = \langle v^M, h\rangle$. We have to check $w(TM) = Sq(v^M)$. But

$$v^M = \sum \xi_i \langle v^M \cdot \xi_i^*, [M]\rangle = \sum \xi_i \langle Sq(\xi_i^*), [M]\rangle, \text{ hence}$$

$$Sq(v^M) = \sum Sq(\xi_i)\langle Sq(\xi_i^*), [M]\rangle = \sum (Sq(\xi_i) \otimes Sq(\xi_i^*))/[M] = Sq(D[\Delta_M])/[M] \overset{(3)}{=} w(TM).$$

7.2. **Stiefel theorem.** The tangent bundle of an orientable compact 3-dimensional manifold is trivial.

Proof: It suffices to construct two linearly independent vector fields on M, i.e. a section of $\mathcal{S}t_{TM}(2,3,\mathbb{R}) \to M$. Since $\mathrm{St}(2,3,\mathbb{R}) = \mathbb{RP}^3$, we have $\pi_1(\mathrm{St}(2,3,\mathbb{R})) = \mathbb{Z}/2\mathbb{Z}$, $\pi_2(\mathrm{St}(2,3,\mathbb{R})) = 0$. The first obstruction to the existence of our section is $w_2(TM) \in H^2(M, \mathbb{F}_2)$. If it vanishes, the next obstruction vanishes as well, and the desired section exists. So it remains to compute $w_2(TM)$.

The Bockstein homomorphism $Sq^1 = \beta_2 \colon H^2(M, \mathbb{F}_2) \to H^3(M, \mathbb{F}_2)$ is defined as the composition $H^2(M, \mathbb{Z}/2\mathbb{Z}) \to H^3(M, \mathbb{Z}) \to H^3(M, \mathbb{Z}/2\mathbb{Z})$. The middle group is $\mathbb{Z}$ since M is orientable, hence the left arrow vanishes, and $Sq^1 \equiv 0\colon H^2(M,\mathbb{F}_2) \to H^3(M,\mathbb{F}_2)$. Moreover, $Sq^2 \equiv 0\colon H^1(M,\mathbb{F}_2) \to H^3(M,\mathbb{F}_2)$ and $Sq^3 \equiv 0\colon H^0(M,\mathbb{F}_2) \to H^3(M,\mathbb{F}_2)$ for degree reasons. Hence $\langle Sq(Dh), [M]\rangle = 0$ for $\deg Dh < 3$, and hence $\langle Sq^{\mathrm{inv}} w(TM), h\rangle = 0$ for $\deg h > 0$. We conclude that $Sq^{\mathrm{inv}} w(TM) \in H^0(M,\mathbb{F}_2)$, and hence $w(TM) \in H^0(M,\mathbb{F}_2)$, and finally $w_2(TM) = 0$. □

7.3. **Chern classes.** We know from §1.5 that $B\mathrm{U}(k) = \mathrm{Gr}(k,\infty,\mathbb{C})$, and $H^\bullet(B\mathrm{U}(k),\mathbb{Q}) \subset H^\bullet(B(S^1)^k,\mathbb{Q}) = \mathbb{Q}[z_1,\ldots,z_k]$ is the subalgebra of symmetric polynomials generated by the elementary symmetric polynomials $e_1,\ldots,e_k$. In fact, $H^\bullet(B\mathrm{U}(k),\mathbb{Z}) = \mathbb{Z}[e_1,\ldots,e_k]$. For a *complex* rank k vector bundle $\mathcal{V} \to B$ with the classifying map $\phi\colon B \to B\mathrm{U}(k)$, the *Chern classes* are defined as $c_i(\mathcal{V}) := \phi^* e_i,\ c_0 = 1$. The *total Chern class* $c(\mathcal{V}) = \sum_{i=0}^k c_i(\mathcal{V})$. We have $c(\mathcal{V}\oplus\mathcal{V}') = c(\mathcal{V})\cdot c(\mathcal{V}')$ since

$$e_m(z_1,\ldots,z_k,z_{k+1},\ldots,z_{k+l}) = \sum_{i+j=m} e_i(z_1,\ldots,z_k)e_j(z_{k+1},\ldots,z_{k+l}).$$

7.4. **(Semi)-infinite Grassmannian.** The union $\lim\limits_{k\to\infty} \mathrm{Gr}(k,\infty,\mathbb{C}) =: \mathrm{Gr}(\infty,2\infty,\mathbb{C})$ is an H-space with respect to the direct sum. So its cohomology is a Hopf algebra $\mathbb{Q}[e_1,e_2,\ldots]$ with comultiplication $\Delta(e_i) = e_i\otimes 1 + 1\otimes e_i$.

More precisely, consider a category $\mathscr{C}$ of $\mathbb{C}$-vector spaces (of finite or countable dimension) equipped with positive definite hermitian form; the morphisms are isometric embeddings. For $V \in \mathscr{C},\ \dim V = k$, we set $B(V) := \mathrm{Gr}(k, V\otimes\mathbb{C}^\infty)$. For $f\colon V\to W$ and $P \in \mathrm{Gr}(k, V\otimes\mathbb{C}^\infty)$ we set $B(f)(P) := (f\otimes \mathrm{Id}_{\mathbb{C}^\infty})(P) \oplus f(V)^\perp \otimes y_1 \in B(W)$, where $y_1 = (1,0,0,\ldots) \in \mathbb{C}^\infty$. We get a functor from the category of finite dimensional hermitian spaces to topological spaces. If W is of countable dimension, then we set $B(W) := \bigcup\limits_{V\subset W}^{\dim V<\infty} B(V)$. We get a functor B from $\mathscr{C}$ to topological spaces. Clearly, $B(\mathbb{C}^k) = B\mathrm{U}(k),\ B(\mathbb{C}^\infty) = B\mathrm{U}(\infty)$. We define a monoidal structure $\oplus\colon B(V)\times B(W) \to B(V\oplus W)$ first for finite dimensional V,W, and then for arbitrary $V,W\in\mathscr{C}$ as the direct limit. We get $\oplus\colon B(\mathbb{C}^\infty)\times B(\mathbb{C}^\infty) \to B(\mathbb{C}^\infty\oplus\mathbb{C}^\infty)$. Now compose it with $B(f)$ for an isometry $f\colon \mathbb{C}^\infty\oplus\mathbb{C}^\infty \xrightarrow{\sim} \mathbb{C}^\infty$ to get the desired product $B\mathrm{U}(\infty)\times B\mathrm{U}(\infty) \to B\mathrm{U}(\infty)$. All the required properties follow from the fact that the space of isometries $\mathbb{C}^\infty \hookrightarrow \mathbb{C}^\infty$ is contractible. This in turn follows from the contractibility of $\mathrm{Isom}(\mathbb{C}^k,\mathbb{C}^\infty) = \mathrm{St}(k,\infty,\mathbb{C})$.

8. May 15th

8.1. **Rational cohomology of real Grassmannians. Theorem.** (a) There is an isomorphism $H^\bullet(\mathrm{Gr}_+(2n,\infty,\mathbb{R}),\mathbb{Q}) \simeq \mathbb{Q}[p_1,p_2,\ldots,p_{n-1},\mathsf{eu}_{2n}]$, where $\deg p_i = 4i,\ \deg \mathsf{eu}_{2n} = 2n$;

(b) $H^\bullet(\mathrm{Gr}_+(2n+1,\infty,\mathbb{R}),\mathbb{Q}) \simeq \mathbb{Q}[p_1,p_2,\ldots,p_n]$, where $\deg p_i = 4i$;

(c) The standard embedding $\mathrm{Gr}_+(2n,\infty,\mathbb{R}) \hookrightarrow \mathrm{Gr}_+(2n+1,\infty,\mathbb{R})$ induces the homomorphism $p_i \mapsto p_i,\ 1\le i\le n-1$, and $p_n \mapsto \mathsf{eu}_{2n}^2$. The standard embedding $\mathrm{Gr}_+(2n+1,\infty,\mathbb{R}) \hookrightarrow \mathrm{Gr}_+(2n+2,\infty,\mathbb{R})$ induces the homomorphism $p_i \mapsto p_i,\ 1\le i\le n$, and $\mathsf{eu}_{2n+2} \mapsto 0$;

(d) Let $\mathcal{V}$ be a real oriented vector bundle over B, and ϕ the corresponding classifying map. Setting $p_0 = 1$ and $p(\mathcal{V}) = \sum \phi^* p_i$ we get $p(\mathcal{V} \oplus \mathcal{V}') = p(\mathcal{V}) \cdot p(\mathcal{V}')$.

Proof: For a compact Lie group K with a maximal (Cartan) torus $T \subset K$, the Weyl group W is the quotient of the normalizer of T in K modulo T. It acts on the Lie algebra $\mathfrak{t}$, and we argued in §1.5 that $H^\bullet(BK, \mathbb{Q}) \cong \mathbb{Q}[\mathfrak{t}]^W$. For $K = \mathrm{SO}(2n+1)$, $\mathfrak{t}^*$ has a basis $y_1, \ldots, y_n$, where $W =: W(B_n) := S_n \ltimes \mathbb{F}_2^n$ acts by permutations and sign changes. For $K = \mathrm{SO}(2n)$, $\mathfrak{t}^*$ has a basis $y_1, \ldots, y_n$, where $W =: W(D_n)$ acts as the index 2 subgroup of $W(B_n)$: $S_n \ltimes \mathrm{Ker}(\sum \colon \mathbb{F}_2^n \to \mathbb{F}_2)$. In particular, $\mathsf{eu}_{2n} = y_1 \cdots y_n$. Everything follows.

By the way, by a Chevalley theorem, the restriction $\mathbb{Q}[\mathfrak{k}]^K \xrightarrow{\sim} \mathbb{Q}[\mathfrak{t}]^W$. The corresponding invariant functions on $\mathfrak{k}$ are $\mathrm{Tr}(\Lambda^{2i} X) \mapsto p_i$, and $\mathrm{Pfaff}(X) \mapsto \mathsf{eu}$.

Here is an alternative topological argument. Consider a fibration

$$\mathrm{St}(2, 2k, \mathbb{R}) \xrightarrow{S^{2k-2}} S^{2k-1}$$

identifying $\mathrm{St}(2, 2k, \mathbb{R})$ with the unit tangent bundle $T_1 S^{2k-1}$ as in the proof of Lemma 6.1.1. The action of S^1 on $S^{2k-1} \subset \mathbb{C}^k$ generates a global nonvanishing vector field, i.e. a section of the above fibration. It follows that $H^\bullet(\mathrm{St}(2, 2k, \mathbb{R}), \mathbb{Q}) \simeq \Lambda_{\mathbb{Q}}(a_{2k-2}, a_{2k-1})$. We prove (a) by induction in k considering the spectral sequence of the fibration

$$(4) \qquad B\mathrm{SO}(2k-2) \xrightarrow{\mathrm{St}(2,2k,\mathbb{R})} B\mathrm{SO}(2k).$$

We have $d_{2k-1}(a_{2k-2}) = 0$: otherwise $d_{2k-1}(a_{2k-2}) = v \neq 0$, and $d_{2k-1}(a_{2k-2} v) = v^2 = 0$, hence $E_\infty^{2k-1, 2k-2} \neq 0$; however, the odd cohomology $H^{4k-3}(B\mathrm{SO}(2k-2), \mathbb{Q}) = 0$ by the induction assumption. Thus $d_{2k-1} \equiv 0$.

At this stage we already see that $H^{\mathrm{odd}}(B\mathrm{SO}(2k), \mathbb{Q}) = 0$. Indeed, this is the first row $E_2^{\bullet,0}$ of our spectral sequence. Among the higher differentials only d_{2k} and d_{4k-2} can possibly land into the first row, but they can possibly kill only even degrees elements, so the odd degree elements in the first row will survive and give the odd degree elements in E_∞ — contradiction to the induction assumption.[1]

Since $\deg a_{2k-1}$ is odd, we must have $d_{2k} a_{2k-1} = w \neq 0$, and

$$d_{2k} \colon a_{2k-1} \cdot E_{2k}^{\bullet\bullet} \hookrightarrow w \cdot E_{2k}^{\bullet\bullet}.$$

Indeed, the kernel of d_{2k} in $a_{2k-1} \cdot E_{2k}^{\bullet\bullet}$ would contribute to the (nonexisting) odd degree terms of E_∞. In particular, d_{2k} embeds $a_{2k-1} \otimes E_2^{\bullet,0}$ into $w \cdot E_2^{\bullet,0}$, and thus w is not a zero divisor in $E_2^{\bullet,0}$. Also, $E_\infty^{\bullet\bullet} \simeq E_\infty^{\bullet,0} \oplus a_{2k-2} E_\infty^{\bullet,0}$ is the associated graded of $\mathbb{Q}[p_1, \ldots, p_{k-2}, \mathsf{eu}_{2k-2}]$. Hence

$$p_1, \ldots, p_{k-2} \in E_\infty^{\bullet,0}, \text{ and } \overline{\mathsf{eu}}_{2k-2} = a_{2k-2}, \ \overline{\mathsf{eu}}_{2k-2}^2 \in E_\infty^{\bullet,0} \implies$$

$$E_\infty^{\bullet,0} = \mathbb{Q}[p_1, p_2, \ldots, p_{k-2}, \mathsf{eu}_{2k-2}^2], \text{ and } E_2^{\bullet,0} = \mathbb{Q}[p_1, p_2, \ldots, p_{k-2}, \mathsf{eu}_{2k-2}^2, w].$$

Indeed, if w is not algebraically independent, the relation it satisfies is divisible by w: otherwise after d_{2k} kills w we are left with a relation on $p_1, \ldots, p_{k-2}, \mathsf{eu}_{2k-2}^2$. But a minimal degree relation divisible by w implies that w is a zero divisor, while we know already that w is not a zero divisor in $E_2^{\bullet,0}$.

[1] I emphasize the claims omitted in the argument that I screwed up during the lecture.

Set $p_{k-1} := \mathsf{eu}_{2k-2}^2$. Then $E_2^{\bullet,0} = H^\bullet(B\mathrm{SO}(2k),\mathbb{Q}) = \mathbb{Q}[p_1,\dots,p_{k-1},w]$. We define the Euler class $\mathsf{eu}_{2k} \in H^\bullet(B\mathrm{SO}(2k),\mathbb{Q})$ as w (the transgression of a_{2k-1}), and (a) is proved.

Consider the spectral sequence of the fibration

$$B\mathrm{SO}(2k-2) \xrightarrow{S^{2k-2}} B\mathrm{SO}(2k-1).$$

It has just two nonzero rows: $E_2^{\bullet,0}$ and $E_2^{\bullet,2k-2} = a_{2k-2}E_2^{\bullet,0}$. If $d_{2k-1}a_{2k-2} = v \neq 0$, then $d_{2k-1}(a_{2k-2}v) = v^2 = 0$, and hence $E_\infty^{2k-1,2k-2} \neq 0$. It contradicts the odd cohomology vanishing of $B\mathrm{SO}(2k-2)$. Hence $d_{2k-1}a_{2k-2} = 0$, and $d_{2k-1} \equiv 0$, and $E_2^{\bullet\bullet} = E_\infty^{\bullet\bullet}$. Hence the odd cohomology of $B\mathrm{SO}(2k-1)$ vanish. Now consider the spectral sequence of the fibration

$$B\mathrm{SO}(2k-1) \xrightarrow{S^{2k-1}} B\mathrm{SO}(2k)$$

with $E_2^{\bullet\bullet} = H^\bullet(B\mathrm{SO}(2k),\mathbb{Q}) \otimes \Lambda_\mathbb{Q}(a_{2k-1})$. The odd cohomology vanishing of $B\mathrm{SO}(2k-1)$ implies $d_{2k}a_{2k-1} \neq 0$. More precisely, comparing with the above computation for the fibration (4) we see $d_{2k}a_{2k-1} = w$, hence $d_{2k}(a_{2k-1}x) = wx$. Hence $E_{2k+1}^{\bullet\bullet} = E_\infty^{\bullet\bullet} = \mathbb{Q}[p_1,\dots,p_{k-1}] = H^\bullet(B\mathrm{SO}(2k-1),\mathbb{Q})$.

To prove (d) we trace back the above construction of p_i and check that its pullback under $B\mathrm{SO}(2)^{\times n} \to B\mathrm{SO}(2n+1)$ is the i-th elementary symmetric polynomial of $y_1^2,\dots,y_n^2$. ☐

8.1.1. *Nonoriented case.* We have a two-fold covering $B\mathrm{SO}(k) \to B\mathrm{O}(k) = \mathrm{Gr}(k,\infty,\mathbb{R})$. Hence $H^\bullet(\mathrm{Gr}(k,\infty,\mathbb{R}),\mathbb{Q}) = H^\bullet(B\mathrm{SO}(k),\mathbb{Q})^{\mathbb{Z}/2\mathbb{Z}}$. The classes p_i are $\mathbb{Z}/2\mathbb{Z}$-invariant and descend to the same named classes in $H^\bullet(\mathrm{Gr}(k,\infty,\mathbb{R}),\mathbb{Q})$. The class eu_{2n} is multiplied by $(-1)^n$.

8.2. **Pontriagin classes.** The natural embedding $\mathrm{SO}(n) \hookrightarrow \mathrm{U}(n)$ gives rise to $\Xi\colon B\mathrm{SO}(n) \to B\mathrm{U}(n)$ corresponding to the operation of *complexification* of real vector bundles. At the level of Cartan algebras (i.e. for the induced map $B\mathrm{SO}(2)^{\times\lfloor\frac{n}{2}\rfloor} \to B\mathrm{U}(1)^{\times n}$), say for $n = 2k$ even, the corresponding map (on cohomology) is $z_i \mapsto y_i,\ z_{k+i} \mapsto -y_i,\ 1 \leq i \leq k$. The odd elementary symmetric polynomials in z go to zero, and $\Xi^* e_{2m}(z_1,\dots,z_n) = (-1)^m e_m(y_1^2,\dots,y_k^2)$. In other words, $\Xi^* c_{2m+1} = 0,\ \Xi^* c_{2m} = (-1)^m p_m$.

Conversely, the natural embedding $\mathrm{U}(n) \hookrightarrow \mathrm{SO}(2n)$ gives rise to $\Theta\colon B\mathrm{U}(n) \to B\mathrm{SO}(2n)$ corresponding to the operation of *restriction of scalars* from $\mathbb{C}$ to $\mathbb{R}$ (taking a complex vector bundle $\mathcal{V}$ of rank n to a rank $2n$ real vector bundle $\mathcal{V}_\mathbb{R}$). At the level of Cartan algebras (i.e. for the induced map $B\mathrm{U}(1)^{\times n} \to B\mathrm{SO}(2)^{\times n}$), the corresponding map (on cohomology) is $y_i \mapsto z_i,\ 1 \leq i \leq n$. Hence $\Theta^*\mathsf{eu}_{2n} = c_n$.

8.2.1. *Definition.* For a real rank k vector bundle $\mathcal{V} \to B$ the *Pontriagin classes* are defined as $p_i(\mathcal{V}) := (-1)^i c_{2i}(\mathcal{V} \otimes_\mathbb{R} \mathbb{C}),\ p_0 = 1$. The *total Pontriagin class* $p(\mathcal{V}) = \sum_{i=0}^k p_i(\mathcal{V})$. We have $p(\mathcal{V} \oplus \mathcal{V}') = p(\mathcal{V}) \cdot p(\mathcal{V}')$.

Equivalently, $p_i(\mathcal{V}) = \phi^* p_i$ for the classifying map $\phi\colon B \to \mathrm{Gr}(k,\infty,\mathbb{R})$.

9. May 22nd

9.1. **The Euler class.** Recall the setup of §4.4. Let $\mathcal{V} \to B$ be an *oriented* real vector bundle of rank m. The Thom class

$$\mathsf{t}_{\mathcal{V}} \in H^m(Th(\mathcal{V}), \mathbb{Z}) = H^m(D(\mathcal{V}), S(\mathcal{V}); \mathbb{Z}) \to H^m(D(\mathcal{V}), \mathbb{Z}) = H^m(B, \mathbb{Z}) \ni \mathsf{eu}_m(\mathcal{V})$$

goes to the *Euler class*. Equivalently, $\mathsf{eu}_m(\mathcal{V}) = \Phi^{-1}(\mathsf{t}_{\mathcal{V}} \cdot \mathsf{t}_{\mathcal{V}})$; in particular $w_m(\mathcal{V}) \equiv \mathsf{eu}_m(\mathcal{V})$ (mod 2). The Euler class is natural for the pullbacks of oriented vector bundles, and changes sign if we change the orientation. In particular, if the rank m is odd, then $2\mathsf{eu}_m(\mathcal{V}) = 0$ (since the opposition changes orientation), so that $\mathsf{eu}_m(\mathcal{V}) = w_m(\mathcal{V})$. The same argument as in §4.5.3 proves $\mathsf{eu}_{m+m'}(\mathcal{V} \oplus \mathcal{V}') = \mathsf{eu}_m(\mathcal{V}) \cdot \mathsf{eu}_{m'}(\mathcal{V}')$.

The same argument as in §6.1 defines the obstruction $o_m(\mathcal{V}) \in H^m(B, \mathbb{Z})$ (where $\mathbb{Z} = \pi_{m-1}(\mathrm{St}(1, m, \mathbb{R}))$) to constructing a nowhere vanishing section of $\mathcal{V}$. The same argument as in §6.2 proves $o_m(\mathcal{V}) = \mathsf{eu}_m(\mathcal{V})$. In case m is even, and $\phi\colon B \to B\mathrm{SO}(m)$ is the classifying map for $\mathcal{V}$, the same argument proves $\mathsf{eu}_m(\mathcal{V}) = \phi^*\mathsf{eu}_m$. In particular, the top Chern class of a complex vector bundle is the Euler class of its restriction of scalars from $\mathbb{C}$ to $\mathbb{R}$.

9.2. **Chern and Pontriagin numbers.** A partition $\lambda = (\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_\ell \geq 0)$ can be written in the form (i^{m_i}) where m_i is the number of occurences of i among $\{\lambda_r\}_{r=1}^{\ell}$. For a smooth compact complex manifold M of complex dimension n and a partition $\mathfrak{P}(n) \ni \lambda = (i^{m_i})$ we set $C_\lambda[M] = \langle c_1^{m_1}(T_{\mathbb{C}}M) \cdots c_n^{m_n}(T_{\mathbb{C}}M), [M]\rangle$ (Chern classes of complex tangent bundle). Similarly, for a smooth compact oriented manifold N of real dimension $4n$ we set $P_\lambda[N] = \langle p_1^{m_1}(TN) \cdots p_n^{m_n}(TN), [N]\rangle$. For example,

$$C_\lambda[\mathbb{CP}^n] = \prod_{r=1}^{\ell} \binom{n+1}{\lambda_r},\ P_\lambda[\mathbb{CP}^{2n}] = \prod_{r=1}^{\ell} \binom{2n+1}{\lambda_r},$$

since $c_i(T_{\mathbb{C}}\mathbb{CP}^n) = \binom{n+1}{i} z^i$ where z is the oriented generator of $H^2(\mathbb{CP}^n, \mathbb{Z})$.

9.3. **Monomial symmetric functions.** A basis in the ring Λ of symmetric polynomials in $x_1, x_2, \ldots$ with integral coefficients is formed by $\{e_\lambda := e_{\lambda_1} \cdots e_{\lambda_\ell}\}_{\lambda \in \mathfrak{P}}$ (elementary symmetric functions). Another basis is formed by the monomial functions $m_\lambda := \sum x_{\sigma(1)}^{\lambda_1} \cdots x_{\sigma(\ell)}^{\lambda_\ell}$ (sum over all permutations, but every monomial enters with coefficient 1). For example, $m_{(k)} = \sum_j x_j^k$. Indeed, for the transposed partition λ^t we have $e_\lambda = m_{\lambda^t} +$ combination of m_μ for μ lexicographically smaller than λ^t. Hence e_λ is an integral linear combination of $\{m_\nu\}$, and m_λ is an integral linear combination of $\{e_\nu\}$.

Substituting c_i (resp. p_i) in place of e_i we get $m_\lambda^c \in H^{2|\lambda|}(\mathrm{Gr}(\infty, 2\infty, \mathbb{C}), \mathbb{Z})$ (resp. $m_\lambda^p \in H^{4|\lambda|}(\mathrm{Gr}_+(\infty, 2\infty, \mathbb{R}), \mathbb{Z})$). For a complex vector bundle $\mathcal{V} \to B$, the pullbacks of these classes with respect to the classifying map $B \to B\mathrm{U}(k)$ are denoted $m_\lambda^c(\mathcal{V}) \in H^{2|\lambda|}(B, \mathbb{Z})$. Similarly, for an oriented real vector bundle $\mathcal{V} \to B$, the pullbacks of these classes with respect to the classifying map $B \to B\mathrm{SO}(k)$ are denoted $m_\lambda^p(\mathcal{V}) \in H^{4|\lambda|}(B, \mathbb{Z})$. Pairing these classes for the tangent bundle with the fundamental class of a complex manifold M

(resp. a real oriented manifold N) we get the integers $m^c_\lambda[M]$ (resp. $m^p_\lambda[N]$). In particular, $m^c_{(n)}[\mathbb{CP}^n] = n+1$, $m^p_{(n)}[\mathbb{CP}^{2n}] = 2n+1$.

9.3.1. *Lemma.* For complex vector bundles $\mathcal{V}, \mathcal{V}' \to B$, we have

$$m^c_\lambda(\mathcal{V} \oplus \mathcal{V}') = \sum_{\mu\nu=\lambda} m^c_\mu(\mathcal{V}) m^c_\nu(\mathcal{V}'),$$

where for two partitions $\mu = (i^{m_i})$ and $\nu = (i^{n_i})$ their shuffle $\mu\nu$ is defined as $(i^{m_i+n_i})$.
Proof: $m_\lambda(\underline{x}, \underline{y}) = \sum_{\mu\nu=\lambda} m_\mu(\underline{x}) m_\nu(\underline{y})$. □

9.3.2. *Corollary.* $m^c_\lambda[M \times M'] = \sum\limits_{\substack{\mu\nu=\lambda \\ |\mu|=\dim M,\ |\nu|=\dim M'}} m^c_\mu[M] m^c_\nu[M']$.

9.3.3. *Proposition.* Let $M^4, M^8, \ldots, M^{4n}$ be smooth oriented compact manifolds such that $m^p_{(k)}[M^{4k}] \neq 0$ (e.g. $M^{4k} = \mathbb{CP}^{2k}$). Then the $p(n) \times p(n)$-matrix $\mathfrak{M}$ with matrix elements $P_\mu[M^{4\lambda_1} \times \ldots \times M^{4\lambda_\ell}]$ numbered by pairs of partitions $|\lambda| = |\mu| = n$, is nondegenerate.
Proof: It suffices to prove the nondegeneracy of another matrix $\mathfrak{M}'$ with matrix elements $m^p_\mu[M^{4\lambda_1} \times \ldots \times M^{4\lambda_\ell}] = \sum_{\nu^{(1)}\nu^{(2)}\ldots\nu^{(\ell)}=\mu} m^p_{\nu^{(1)}}[M^{4\lambda_1}] \cdots m^p_{\nu^{(\ell)}}[M^{4\lambda_\ell}]$, where the sum runs over ℓ-tuples of partitions such that $|\nu^{(i)}| = \lambda_i$. This matrix element vanishes unless μ is a refinement of λ, in particular, if $\ell(\mu) < \ell = \ell(\lambda)$. Hence for a total order on $\mathfrak{P}(n)$ compatible with the partial order of refinement, the matrix $\mathfrak{M}'$ is lower-triangular. The diagonal entries are $m^p_{(\lambda_1)}[M^{4\lambda_1}] \cdots m^p_{(\lambda_\ell)}[M^{4\lambda_\ell}] \neq 0$, hence $\mathfrak{M}'$ and $\mathfrak{M}$ are nondegenerate. □

10. May 29th

10.1. **Oriented cobordisms.** The oriented real manifolds M, M' of dimension k are *oriented cobordant* if there is an oriented real manifold N with boundary $M \sqcup M'$ with induced orientation $M \sqcup -M'$. This is an equivalence relation, and the quotient is an abelian group Ω_k with operation $M + M' := M \sqcup M'$ (and with inverse $-M := M$ with the opposite orientation). The direct product induces a homomorphism $\Omega_k \times \Omega_m \to \Omega_{k+m}$, hence the supercommutative graded ring $\Omega_\bullet$. Similarly to Exercise 3 of 08.05, if M^{4n} is a boundary of N^{4n+1}, then any $P_\lambda[M^{4n}] = 0$. Hence any partition λ of n defines a homomorphism $\Omega_{4n} \to \mathbb{Z}$, $M^{4n} \mapsto P_\lambda[M^{4n}]$. It follows from Proposition 9.3.3 that $\{\mathbb{CP}^{2\lambda_1} \times \ldots \times \mathbb{CP}^{2\lambda_\ell}\}_{\lambda \in \mathfrak{P}(n)}$ are linearly independent in Ω_{4n}.
According to R. Thom,

$$\Omega_n \cong \pi_{k+n}(Th(\mathcal{S}_k), t_0) \tag{5}$$

for the Thom space of the universal tautological bundle $\mathcal{S}_k \to \mathrm{Gr}_+(k, \infty, \mathbb{R})$ and $k > n+1$. The homomorphism from the RHS to the LHS is constructed by the following

10.1.1. *Proposition.* Let $\mathcal{V} \xrightarrow{p} B$ be a rank k smooth real oriented vector bundle over a smooth manifold B. Then any continuous map $f\colon S^m \to Th(\mathcal{V})$ is homotopic to a continuous map g smooth away from the base point $t_0 \in Th(\mathcal{V})$ and transversal to the zero section $B \subset Th(\mathcal{V})$. The oriented cobordism class of the resulting smooth oriented $m-k$-dimensional manifold $g^{-1}(B)$ depends only on the homotopy class of f. Hence we obtain a homomorphism $\pi_m(Th(\mathcal{V}), t_0) \to \Omega_{m-k}$.

Proof: First we approximate f by f_0 smooth away from t_0. Choose an open covering $W_1 \cup \ldots \cup W_r$ of the compact $f_0^{-1}(B)$ such that each open $W_i \subset S^m$ lands into a local chart $U_i \times D_1(0)$ (where $D_1(0)$ is the open unit ball in $\mathbb{R}^k$) in the disc bundle $D(\mathcal{V})|_{U_i}$. Choose compacts $K_i \subset W_i$ such that $f_0^{-1}(B)$ is contained in the interior of $K_1 \cup \ldots \cup K_r$. We will successively modify f_0 on the open sets $W_1, \ldots, W_r$ to obtain $f_1, \ldots, f_r$ satisfying the following conditions:

(a) f_i is smooth away from t_0 and coincides with f_{i-1} away from a compact in W_i;

(b) $f_i|_{K_1 \cup \ldots \cup K_i}$ is transversal to B: if $f_i(x) \in B$, then $df_i(T_x S^m) + T_{f_i(x)} B = T_{f_i(x)} Th(\mathcal{V})$;

(c) If $f_0(x) \neq t_0$, then $p(f_i(x)) = p(f_0(x))$.

At the i-th step we know that $f_{i-1}(W_i) \subset U_i \times D_1(0) \subset p^{-1}(U_i)$. We denote by q_i the projection $U_i \times D_1(0) \to D_1(0)$. We know $p \circ f_i$ by (c) and have to define $q_i \circ f_i$. By (b), $0 \in D_1(0)$ is a regular value of $q_i \circ f_{i-1}|_{(K_1 \cup \ldots \cup K_{i-1}) \cap W_i}$. Hence $q_i \circ f_{i-1}$ can be approximated by a map $\varphi_i\colon W_i \to D_1(0)$ that coincides with $q_i \circ f_{i-1}$ away from a compact subset of W_i and such that $0 \in D_1(0)$ is a regular value of $\varphi_i|_{(K_1 \cup \ldots \cup K_i) \cap W_i}$. We set $q_i \circ f_i := \varphi_i$.

Now the desired $g := f_r$. Then $g|_{K_1 \cup \ldots \cup K_r}$ is transversal to B. It remains to make sure that $g^{-1}(B) \subset K_1 \cup \ldots \cup K_r$. Since $K_1 \cup \ldots \cup K_r$ is a neighbourhood of $f_0^{-1}(B) \subset S^m$, there exists $0 < c < 1$ such that $|f_0(y)| \geq c$ for any $y \notin K_1 \cup \ldots \cup K_r$ (here $|t|$ is the euclidean norm of a point $t \in Th(\mathcal{V})$; in particular, $|t_0| = 1$). Additionally to (a–c) above, we will choose f_i close enough to f_{i-1}, so that $|f_i(x) - f_{i-1}(x)| < c/r$ for any x. Then $|g(x) - f_0(x)| < c$, and hence $|g(y)| \neq 0$ for $y \notin K_1 \cup \ldots \cup K_r$. So $g^{-1}(B) \subset K_1 \cup \ldots \cup K_r$, and g is everywhere transversal to B, and $g^{-1}(B)$ is a smooth compact oriented $m-k$-dimensional manifold.

If g and g' are homotopic maps $S^m \to Th(\mathcal{V})$ smooth away from t_0 and transversal to B, then we can construct a homotopy $h_0\colon S^m \times [0,3] \to Th(\mathcal{V})$ smooth away from t_0 and such that $h_0(x,s) = g(x)$ for $s \in [0,1]$, while $h_0(x,s) = g'(x)$ for $s \in [2,3]$. Similarly to above, we can modify it to $h\colon S^m \times [0,3] \to Th(\mathcal{V})$ coinciding with h_0 away from a compact subset of $S^m \times (0,3)$ and transversal to B. The preimage $h^{-1}(B)$ realizes an oriented cobordism between $g^{-1}(B)$ and $g'^{-1}(B)$. Thus, the oriented cobordism class of $g^{-1}(B)$ is well defined.

Finally, the addition in $\pi_m(Th(\mathcal{V}), t_0)$ corresponds to the disjoint union of preimages $g^{-1}(B)$, and hence our map $\pi_m(Th(\mathcal{V}), t_0) \to \Omega_{m-k}$ is a homomorphism. □

10.1.2. *Proposition.* The homomorphism $\pi_{k+n}(Th(\mathcal{S}_k), t_0) \to \Omega_n$ for the Thom space of the universal tautological bundle $\mathcal{S}_k \to \mathrm{Gr}_+(k, k+p, \mathbb{R})$ is surjective for $k \geq n$ and $p \geq n$.

Proof: Given a smooth compact oriented n-dimensional manifold M we can embed it into $\mathbb{R}^{k+n}$ (Whitney). Consider the Gauß map from the total space of the normal bundle $\mathcal{N}_{M/\mathbb{R}^{k+n}}$ to the total space of the tautological bundle over $\mathrm{Gr}_+(k, k+n)$. Compose it with the embedding into the total space of the tautological bundle $\mathcal{S}_k$ over $\mathrm{Gr}_+(k, k+p)$. Restricting it to a tubular neighbourhood $\mathbb{R}^{k+n} \supset U \supset M$ and projecting to the Thom space we obtain $g\colon U \to Th(\mathcal{S}_k)$ transversal to the zero section $\mathrm{Gr}_+(k, k+p) \hookrightarrow \mathcal{S}_k$ such that $g^{-1}(\mathrm{Gr}_+(k, k+p)) = M$. We extend g to the one-point compactification $S^{k+n} = (\mathbb{R}^{k+n})_+$ sending the complement of U to t_0. The resulting map $g\colon S^{k+n} \to Th(\mathcal{S}_k)$ gives rise to the cobordism class of M by the rule of Proposition 10.1.1. □

10.1.3. *Proof of (5) modulo torsion.* According to Proposition 10.1.2, the homomorphism $\pi_{k+n}(Th(\mathcal{S}_k), t_0) \to \Omega_n$ is surjective for $k \geq n$. The Thom space $Th(\mathcal{S}_k)$ is $k-1$-connected. It follows that the Hurewicz homomorphism $\pi_r(Th(\mathcal{S}_k), t_0) \to H_r(Th(\mathcal{S}_k), \mathbb{Z})$ is an isomorphism modulo torsion for $r < 2k-1$ (use Theorem 1.3). We know the dimension of $H_{n+k}(Th(\mathcal{S}_k), \mathbb{Q}) = H_n(\mathrm{Gr}_+(k, \infty, \mathbb{R}), \mathbb{Q})$ by Theorem 8.1. By Proposition 9.3.3, the rank of $\Omega_n \otimes_{\mathbb{Z}} \mathbb{Q}$ is at least the rank of $H_n(\mathrm{Gr}_+(k, \infty, \mathbb{R}), \mathbb{Q})$. □

10.1.4. *Corollary.* $\Omega_\bullet \otimes_{\mathbb{Z}} \mathbb{Q} = \mathbb{Q}[\mathbb{CP}^2, \mathbb{CP}^4, \mathbb{CP}^6, \ldots]$.

10.1.5. *Notation.* The Thom space of the universal tautological bundle $\mathcal{S}_k$ over $B\mathrm{SO}(k)$ is denoted $M\mathrm{SO}(k)$: the classifying space of oriented cobordisms. Similarly, the Thom space of the universal tautological bundle over $B\mathrm{U}(k)$ is denoted $M\mathrm{U}(k)$: the classifying space of *stable complex* cobordisms. These are equivalence classes of manifolds whose tangent bundle is equipped with a stable complex structure, that is for some r, $TM \oplus \mathbb{R}^r$ is equipped with a complex structure.

10.2. **Pontriagin numbers and homology of $B\mathrm{SO}$.** The tangent bundle of a smooth compact oriented real $4n$-dimensional manifold M defines the classifying map $\phi\colon M \to B\mathrm{SO}$. Given an element $h \in H^\bullet(B\mathrm{SO}, \mathbb{Q})$ we can integrate $\langle \phi^* h, [M] \rangle$ to obtain a number. This way M gives rise to a linear functional $H^\bullet(B\mathrm{SO}, \mathbb{Q}) \xrightarrow{[M]} \mathbb{Q}$.

More precisely, M defines a degree $4n$ homogeneous linear functional on the *completed* cohomology ring $\widehat{H}^\bullet(B\mathrm{SO}, \mathbb{Q}) = \mathbb{Q}[\![p_1, p_2, \ldots]\!]$. This functional is a degree $4n$ element of the *homology ring* $H_\bullet(B\mathrm{SO}, \mathbb{Q})$ (recall that $B\mathrm{SO}$ is an H-space).

If M is the boundary of N, then the above functional vanishes. Thus we obtain a linear map $\mathcal{T}\colon \Omega_\bullet \otimes_{\mathbb{Z}} \mathbb{Q} \to H_\bullet(B\mathrm{SO}, \mathbb{Q})$. Furthermore, since $T(M_1 \times M_2) = TM_1 \oplus TM_2$, and the monoidal structure on $B\mathrm{SO}$ is given by the direct sum, $\mathcal{T}$ is a ring homomorphism. Moreover, by Thom theorem 10.1.4, $\mathcal{T}$ is an isomorphism.

So any linear functional $\varphi\colon \Omega_\bullet \to \mathbb{Q}$ can be viewed as an element $\sum_{i=0}^\infty K_i \in \widehat{H}^\bullet(B\mathrm{SO}, \mathbb{Q}) = \mathbb{Q}[\![p_1, p_2, \ldots]\!]$, for a sequence of homogeneous polynomials $K_i(p_1, \ldots, p_i)$, $\deg K_i = i$. The condition that φ is an algebra homomorphism (a character) is equivalent to the condition that $\mathbb{Q} \to \widehat{H}^\bullet(B\mathrm{SO}, \mathbb{Q})$, $1 \mapsto \sum_{i=0}^\infty K_i$, is a homomorphism of *coalgebras*. It can be formulated in terms of the sequence (K_i) as follows. Suppose the formal variables p_i, p'_j, p''_k satisfy an equality

$$1 + p_1 + p_2 + \ldots = (1 + p'_1 + p'_2 + \ldots) \cdot (1 + p''_1 + p''_2 + \ldots). \text{ Then}$$
$$\sum_{i\geq 0} K_i(p_1, \ldots, p_i) = \sum_{j\geq 0} K_j(p'_1, \ldots, p'_j) \cdot \sum_{k\geq 0} K_k(p''_1, \ldots, p''_k).$$

Indeed, recall that the symmetric functions ring Λ is a Hopf ring with respect to the coproduct $\Delta(e_i) = e_i \otimes 1 + 1 \otimes e_i$.[2]

Such a sequence (K_i) is called a *multiplicative Hirzebruch sequence.*

10.2.1. *Proposition.* A multiplicative Hirzebruch sequence (K_i) is completely determined by the *characteristic power series* $Q(x) = 1 + q_1 x + q_2 x^2 + \ldots \in 1 + x\mathbb{Q}[\![x]\!]$, where $x = p_1$, and $q_i = K_i(1, 0, \ldots, 0)$. Moreover, any formal series $Q(x)$ gives rise to a multiplicative Hirzebruch sequence.

Proof: Recall that the Pontriagin classes correspond to the elementary symmetric polynomials (in "Pontriagin roots" x_i), so that $1 + p_1 + \ldots + p_n = (1 + x_1) \cdots (1 + x_n)$. Hence

$$Q(x_1) \cdots Q(x_n) = 1 + K_1(p_1) + K_2(p_1, p_2) + \ldots + K_n(p_1, \ldots, p_n) + K_{n+1}(p_1, \ldots, p_n, 0, \ldots) + \ldots$$

$\square$

11. June 5th

11.1. **Hirzebruch genera.** A ring homomorphism $\Omega_\bullet \to \mathbb{Q}$ (a character) is called a *genus*. Thus any characteristic power series $Q(x) = 1 + q_1 x + q_2 x^2 + \ldots \in 1 + x\mathbb{Q}[\![x]\!]$ determines a *Hirzebruch genus* $\varphi_Q[M^{4n}] = \langle \prod_{i=1}^n Q(x_i), [M^{4n}] \rangle$, where $(1 + x_1) \cdots (1 + x_n) = p(TM^{4n})$. Conversely, any genus arises from an appropriate characteristic power series.

The most famous genus is the signature (of the Poincaré pairing on the middle cohomology).

11.1.1. *Theorem.* (F. Hirzebruch, 1954) The signature $\operatorname{sign}(M)$ is given by the L-genus

$$L(x) = \frac{\sqrt{x}}{\tanh(\sqrt{x})} = \sum_{k \geq 0} \frac{2^{2k} B_{2k} x^k}{(2k)!} = 1 + \frac{x}{3} - \frac{x^2}{45} + \ldots$$

(Bernoulli numbers). The corresponding multiplicative Hirzebruch sequence is

$$L_0 = 1,\ L_1 = \frac{p_1}{3},\ L_2 = \frac{7p_2 - p_1^2}{45},\ L_3 = \frac{62p_3 - 13p_1p_2 + 2p_1^3}{945},$$

$$L_4 = \frac{381p_4 - 71p_1p_3 - 19p_2^2 + 22p_1^2p_2 - 3p_1^4}{14175}, \ldots$$

Proof: It suffices to compare the signature and L-genus on generators of $\Omega_\bullet \otimes_{\mathbb{Z}} \mathbb{Q}$. Evidently, $\operatorname{sign}(\mathbb{CP}^{2k}) = 1$. On the other hand, $p(T\mathbb{CP}^{2k}) = (1+z^2)^{2k+1}$, and $L(z^2) = z/\tanh(z)$; hence $L(p(T\mathbb{CP}^{2k})) = (z/\tanh(z))^{2k+1}$. So we have to find the degree $2k$ term of this series, that is $\dfrac{1}{2\pi i} \oint \dfrac{dz}{z^{2k+1}} \left(\dfrac{z}{\tanh(z)} \right)^{2k+1}$. The variable change $u = \tanh(z)$, so that

[2] By the way, Λ is graded *selfdual*, and an orthonormal base of Λ is formed by the *Schur functions* that correspond to the fundamental classes of *Schubert cells*. This is the unique integral orthonormal base up to permutations and sign changes.

$dz = \dfrac{du}{1-u^2} = (1+u^2+u^4+\dots)du$, proves

$$\frac{1}{2\pi i}\oint \frac{dz}{\tanh^{2k+1}(z)} = \frac{1}{2\pi i}\oint \frac{(1+u^2+u^4+\dots)du}{u^{2k+1}} = 1. \qquad \square$$

11.1.2. *Corollary.* $L[M]$ is an integer. For example, $p_1[M^4]$ is divisible by 3, while $7p_2[M^8] - p_1^2[M^8]$ is divisible by 45.

11.2. **Complex version.** There is a parallel story for the (stable) complex cobordism ring $\Omega_\bullet^{\mathbb{C}}$ in place of $\Omega_\bullet$, and Chern classes in place of Pontriagin classes. In particular, the *Todd genus*

$$Td(x) = \frac{x}{1-\exp(-x)} = 1 + \frac{x}{2} + \sum_{k\geq 1}(-1)^{i+1}\frac{B_{2k}x^{2k}}{(2k)!} = 1 + \frac{x}{2} + \frac{x^2}{12} + \dots$$

The corresponding multiplicative Hirzebruch sequence is

$$Td_0 = 1,\ Td_1 = \frac{c_1}{2},\ Td_2 = \frac{c_2+c_1^2}{12},\ Td_3 = \frac{c_1c_2}{24},\ Td_4 = \frac{-c_4+c_1c_3+3c_2^2+4c_1^2c_2-c_1^4}{720}, \dots$$

We have $Td(\mathbb{CP}^n) = 1$ for any n.

11.3. **PL Pontriagin classes.** We will study the simplicial complexes that have a structure of a topological (*not* necessarily smooth) manifold, and their piecewise linear (PL) morphisms (that is, linear on simplices after an appropriate subdivision). For example, S^k: the boundary of the standard $k+1$-simplex. More generally, any *smooth* compact manifold has a PL structure (Whitehead); it is unique up to a PL isomorphism.

11.3.1. *Lemma.* Let M^n be a compact PL manifold of dimension n, and $f\colon \mathsf{M}^n \to \mathsf{S}^k$ a PL morphism, $n-k=4i$. Then for almost all $y \in \mathsf{S}^k$, the preimage $f^{-1}(y)$ is a compact PL manifold of dimension $4i$. A choice of orientations of M^n and S^k defines the induced orientation of $f^{-1}(y)$, and the signature $\operatorname{sign}(f^{-1}(y))$ is independent of y for almost all y. $\square$

This common value of $\operatorname{sign}(f^{-1}(y))$ is denoted $\operatorname{sign}(f)$. Similarly to Exercises 4,5, we have

11.3.2. *Lemma.* a) $\operatorname{sign}(f)$ depends only on the homotopy class of f in $\pi^k(\mathsf{M}^n)$;
b) If $8i < n-1$, so that $\pi^k(\mathsf{M}^n)$ is a group (as in Exercise 4a), then $f \mapsto \operatorname{sign}(f)$ is a homomorphism $\pi^k(\mathsf{M}^n) \to \mathbb{Z}$. $\square$

11.3.3. *Theorem.* a) If $8i < n-1$, there is a unique cohomology class $L_i(\mathsf{M}^n) \in H^{4i}(\mathsf{M}^n, \mathbb{Q})$ such that $\langle L_i(\mathsf{M}^n)\cdot f^*s, [\mathsf{M}^n]\rangle = \operatorname{sign}(f)$;
b) If M^n is a PL structure on a smooth manifold M^n, then $L_i(\mathsf{M}^n) = L_i(TM^n)$. $\square$

If the condition $8i < n-1$ is not satisfied, we can consider $\mathsf{M}^n \times \mathsf{S}^m$ for $m \gg 0$, and define $L_i(\mathsf{M}^n) \in H^{4i}(\mathsf{M}^n, \mathbb{Q})$ as the pullback of $L_i(\mathsf{M}^n \times \mathsf{S}^m) \in H^{4i}(\mathsf{M}^n \times \mathsf{S}^m, \mathbb{Q})$ for a natural embedding $\mathsf{M}^n \hookrightarrow \mathsf{M}^n \times \mathsf{S}^m$. In particular, $\langle L_i(\mathsf{M}^{4i}), [\mathsf{M}^{4i}]\rangle = \operatorname{sign}(\mathsf{M}^{4i})$.

11.4. **Lemma.** a) There is a rank 4 oriented vector bundle $\mathcal{V}$ over S^4 such that $p_1(\mathcal{V}) = -2s$, and $\mathsf{eu}_4(\mathcal{V}) = s$, where s is the generator of $H^4(S^4, \mathbb{Z})$;

b) For any $k, m \in \mathbb{Z}$ such that $k \equiv 2m \pmod 4$, there is a rank 4 oriented vector bundle $\mathcal{W}$ over S^4 such that $p_1(\mathcal{W}) = ks$, and $\mathsf{eu}_4(\mathcal{W}) = ms$.

Proof: a) $\mathcal{V}$ is the tautological vector bundle over $\mathbb{HP}^1 = S^4$. Then $c(\mathcal{V}) = 1 + \mathsf{eu}_4(\mathcal{V}) = 1 + s \implies p(\mathcal{V}) = (1-s)^2 = 1 - 2s$.

b) $\pi_4(B\mathrm{SO}(4)) = \pi_3(\mathrm{SO}(4)) = \mathbb{Z} \oplus \mathbb{Z}$. Given $f \colon S^4 \to B\mathrm{SO}(4)$, $p_1(f^*\mathcal{S}_4)$ and $\mathsf{eu}_4(f^*\mathcal{S}_4)$ are linear functionals $\mathbb{Z} \oplus \mathbb{Z} = \pi_4(B\mathrm{SO}(4)) \to H^4(S^4, \mathbb{Z}) = \mathbb{Z}$. The values of these functionals on the classifying map of $\mathcal{V}$ are $-2, 1$. The values of these functionals on the classifying map of the tangent bundle TS^4 are $0, 2$. □

11.5. **Example.** (Milnor, 1956) For any $k \equiv 2 \pmod 4$ let $\mathcal{W}_k$ be a rank 4 oriented vector bundle over S^4 with $p_1(\mathcal{W}_k) = ks$, $\mathsf{eu}_4(\mathcal{W}_k) = s$. From the Gysin sequence for the sphere bundle $S(\mathcal{W}_k) \xrightarrow{S^3} S^4$ we conclude that $S(\mathcal{W}_k)$ is homotopic to S^7. Actually it is homeomorphic to S^7, and even PL equivalent to S^7. It follows that $Th(\mathcal{W}_k)$ (the one-point compactification of the open disc bundle) is a PL manifold. But $H^\bullet(Th(\mathcal{W}_k), \mathbb{Z}) = \mathbb{Z}[0] \oplus \mathbb{Z}[-4] \oplus \mathbb{Z}[-8]$, hence $\mathrm{sign}(Th(\mathcal{W}_k)) = 1$ (with an appropriate choice of orientation).

The total Pontriagin class of the tangent bundle of the total space of $\mathcal{W}_k$ is

$$p(T\mathcal{W}_k) = \pi^*(p(TS^4) \cdot p(\mathcal{W}_k)) = \pi^*(1 + ks)$$

up to 2-torsion, where π is the projection from the total space of $\mathcal{W}_k$ to S^4. Hence $p_1(TTh(\mathcal{W}_k)) = ku$, where u is a generator of $H^4(Th(\mathcal{W}_k), \mathbb{Z})$. Indeed, passing to the one-point compactification of $\mathcal{W}_k$ does not affect H^4. Hence $P_{(1,1)}[Th(\mathcal{W}_k)] = k^2$ (by Poincaré duality with integral coefficients, u^2 is a generator of $H^8(Th(\mathcal{W}_k), \mathbb{Z})$). By Hirzebruch formula, $1 = \mathrm{sign}(Th(\mathcal{W}_k)) = \frac{7}{45} P_{(2)}[Th(\mathcal{W}_k)] - \frac{1}{45} P_{(1,1)}[Th(\mathcal{W}_k)]$, hence $P_{(2)}[Th(\mathcal{W}_k)] = (45 + k^2)/7$ is *not* integral if $k \not\equiv \pm 2 \pmod 7$. Hence the PL manifold $Th(\mathcal{W}_k)$ admits *no smooth structure* if $k \not\equiv \pm 2 \pmod 7$ (but $k \equiv 2 \pmod 4$). In particular, the smooth manifold $S(\mathcal{W}_k)$ is not diffeomorphic to S^7 (though it is PL equivalent to S^7): otherwise $Th(\mathcal{W}_k)$ would be smooth.