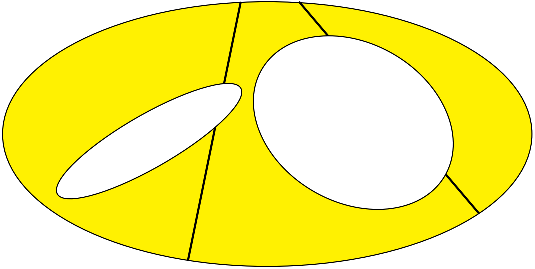
Michael Lvovich Blank, HSE + IITP RAS, 02.11.20

A cake cutting problem



$$\mu_i(\Delta_i, \Delta) \geq \mu_i(\Delta_j, \Delta) \quad \forall i, j$$

$$\mu_i(\Delta_i, \Delta) \geq \frac{1}{n} \sum_j \mu_i(\Delta_j, \Delta) \quad \forall i$$

$$\mu_i(\Delta_i, \Delta) = \mu_j(\Delta_j, \Delta) \quad \forall i, j$$

Optimal, democratic and fair division from a math. viewpoint

Экономика в большой степени является гуманитарной дисциплиной и мы (математики) можем всерьез обсуждать только ее малую часть, в которой есть надежда на формализацию. Нобель по экономике (2018): за исследования интеграции в долгосрочный макроэкономический анализ: изменения климата, технологических инноваций. (2017): за иррациональность поведения эк. агентов. Базовый пример "пирога": division of the faculty teaching load + rewards (salary, bonuses, etc.). Каждый преподаватель получает кусок "по справедливости".

Экономист изучает "рынок" (пирог), создает его модель. В результате (в идеале) появляется оптимизационная постановка типа $\min_x F(x),\ x \in \Omega$. Решая ее получаем "оптимальное" распределение ресурса. Особенно грамотный экономист может догадаться, что оптимума здесь наверное вовсе нет и перейдет к поиску Парето-оптимального решения (нет решений строго лучших по одним переменным и хотя бы не худших по другим - седловая точка).

У нас вместо этого создается комиссия, которая делает в целом то же самое, но неформально. Это образец централизации (диктатуры).

Демократическое деление

Предлагается вариант => голосование: за только если выгодно, иначе против.

Пример: n участников с капиталами x_i^t в момент $t \geq 0$. Начальный капитал $x_i^0 = 1 \ \forall i = 1$.

"Наивная версия": на $(i < n)$-м шагу предлагается обанкротить (i+1)-го участника и раздать его деньги поровну остальным. Дальше продолжаем периодически. В пределе все деньги скопятся у 1-го участника (первичное накопление капитала). Квалифицированное большинство $n - 1$ из n каждый раз голосует "за".

Демократическое деление - 2

Приведенный вариант далеко не худший - здесь суммарный капитал $S^t := \sum_i x_i^t$ хотя бы сохраняется. Приведем чуть более сложную, но весьма реалистическую ситуацию:

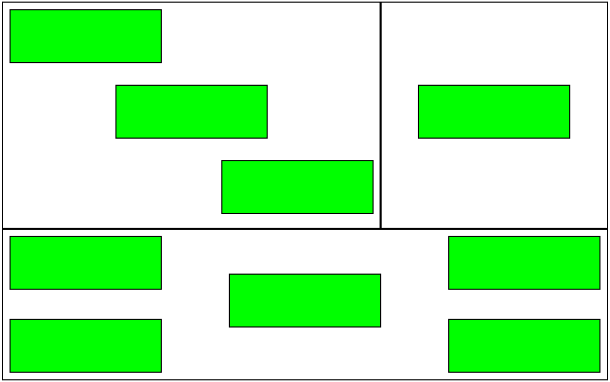
При $t = i (\mathrm{mod}\ \mathrm{n})$ "разоряем" i-го участника и раздаем всем часть (но не все) его капитала ($n\varepsilon \sum_j x_j^t < x_i^t, \quad \varepsilon < 1/n^2$):

$$x_i^{t+1} := (1 - 1_{t=np+i}) x_i^t + \varepsilon \sum_j x_j^t$$

$$S^{t+1} < \frac{n-1}{n} S^t + n^2 \varepsilon < (1 - \gamma_{\varepsilon,n})^t S^0 \xrightarrow{t\to\infty} 0.$$

Квалифицированное большинство $n-1$ голос из n каждый раз голосует "за".

Rent division



n rooms + n agents. The total rent is fixed by $= 1$. The i-th agent gets his own room and a part of the rent x_i ($\sum_i x_i = 1$). The problem is to distribute the rooms and the rent such that according to each of the agents own estimate he/she is not offended.

Formal definitions

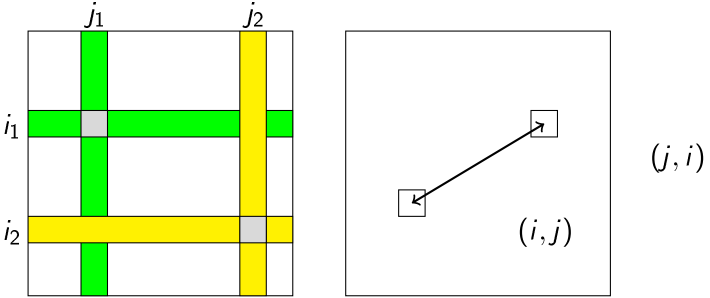
Solution: (π, x), where $\pi : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$ is a permutation, describing the room distribution (i.e. the i-th agent gets the room number π_i), and $x = (x_1, \ldots, x_n)$ is the rent distribution (i.e. $\sum x_i = 1$). Let $A := (a_{ij})$ be the matrix of a priory estimates of room prices by the agents. This means that the i-th agent is ready to ge the j-th room if his part of the rent $x_i \leq a_{ij}$. Valuations $\mu_i(\cup_{j \in \pi_i} \Delta_j, x_i) := \mathrm{sign}(\sum_{j \in \pi_i} a_{ij} - x_i)$.

We say that the rent partition (π, x) is *admissible* if $x_i \leq a_{i\pi_i} \quad \forall i$. Here π is the discrete part of the solution, while x is the continuous part.

Theorem 1.

A solution (π, x) is admissible iff $a(\pi) := \sum_i a_{i\pi_i} \geq 1$. An admissible solution exists if one of the inequalities bellow holds true: (i) $\sum_{i,j} a_{ij} \geq n, \quad (ii) \sum_j a_{ij} \geq 1 \ \forall i, \quad (iii) \sum_i a_{ij} \geq 1 \ \forall j.$

Solution for the rent division problem



At the k-th step we choose the locally maximal elements $a_{i_k j_k}$ of the matrix A, which define the map π (room distribution) and reset the corresponding row and column. Then $a(\pi) := \sum_i a_{i\pi_i} \geq 1$ and the rent of the i-th agent is calculated as $x_i := \frac{a_{i\pi_i}}{a(\pi)}$.

Comparison with the combinatorial-topological approach

A now classical combinatorial-topological approach to solving this problem [F.E.Su-1999] is based on the assumption that for each rent partition each participant is able to find an admissible room. This condition is satisfied only in the case (ii).

The matrix $A := \begin{pmatrix} 1/3 & 1/3 & 1/3 \\ 1/2 & 0 & 0 \\ 1/2 & 2/3 & 2/3 \end{pmatrix}$ satisfies the condition (iii) (but not (ii)), so our solution exists, however under the division of the rooms with the rent distribution $\{3/4, 1/8, 1/8\}$ there is no admissible room for the 2-nd agent.

General Fair division

For a given measurable partition Δ of a space X into n elements consider their (subjective) estimates $\mu_i(\Delta_j, \Delta)$, $i, j \in \{1, \dots, n\}$ by each of the agents. Then each solution of the division problem is a pair (π, Δ), where π – is a permutation of n elements, which we call the *discrete* component of the solution. The map $i \to \pi_i$ describes the division of the partition Δ between the agents.
The pair (π, Δ) is said to be

- the *weak* solution, if $\mu_i(\Delta_{\pi_i}, \Delta) \geq \frac{1}{n} \sum_j \mu_i(\Delta_j, \Delta) \;\; \forall i$;
- the *strong* solution, if $\mu_i(\Delta_{\pi_i}, \Delta) \geq \mu_i(\Delta_j, \Delta) \;\; \forall i, j$;
- the *homogeneous* solution, if $\mu_i(\Delta_{\pi_i}, \Delta) = \mu_j(\Delta_{\pi_j}, \Delta) \;\; \forall i, j$.

Proportional, envy-free, equitable – economic terminology.

Example, when the “standard” partitioning into two groups doesn't work: $7 + 3 + 2 * (2k + 1)$. Then there is a partition with equal sums, but it inevitably has 7 and 3 in one group.

Massive non-divisible components

Let the resource under division be purely discrete and consists of of a number of non-divisible parts, e.g. *gems*, having very different subjective estimates by the agents. Then there is no hope for a fair division. The situation changes if we add a continuous part which we refer as *gold*: there are m gems and a certain amount $|x| > 0$ of gold to be divided between n agents. United m gems into n groups we reduce the problem to the case $m = n$. (Multiple Knapsack Problem is NP-complete task!)

Massive non-divisible components-2

Let a_{ij} be a subjective estimate by the i-th agent of price of the j-th gem and by the x_i the amount of gold he gets after division $(\sum_i x_i = |x|)$.

For a given permutation π we get $\mu_i(\Delta_{\pi_i}, \Delta) := a_{i\pi_i} + x_i$. Thus a solution (π, Δ) is

weak, if $x_i + a_{i\pi_i} \geq \frac{1}{n}(|x| + \sum_j a_{ij}) \quad \forall i$;

strong, if $x_i + a_{i\pi_i} \geq x_j + a_{i\pi_j} \quad \forall i, j$;

homogeneous, if $x_i + a_{i\pi_i} = x_j + a_{j,\pi_j} \quad \forall i, j$.

Set $a(\pi) := \sum_i a_{i\pi_i}$.

Theorem 2.

A homogeneous solutions exists iff

$$\max_i a_{i\pi_i} - \min_j a_{j\pi_j} \leq |x|/(n-1).$$

Massive non-divisible components-3

On the other hand a weak solution may exist even with zero gold reserve (i.e. $|x| = 0$), but there are neither strong nor homogeneous solutions: $A := \begin{pmatrix} 2 & 1 & 0 \\ 2 & 1 & 0 \\ 2 & 0 & 1 \end{pmatrix}$ – weak solution $(\pi_i := i, x_i := 0)$, but the minimal value of $|x|$, admitting strong or homogeneous solutions is 2.

A naive idea that having a lot of gold one always can get a strong solution if wrong: $A := \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ for $(\pi_i := i)$ there is no strong solution for arbitrary large $|x|$. Of interest that for $(\pi_1 := 2, \pi_2 := 1)$ there is a strong solution with $|x| = 0$.

Main result 2

Theorem 3.

A necessary condition for the existence of a strong solution is
(a) $a(\pi) \geq \frac{1}{n} \sum_{i,j} a_{ij}$, (b) $a_{i\pi_i} + a_{j\pi_j} \geq a_{i\pi_j} + a_{j\pi_i} \quad \forall i, j$.
A sufficient condition for the strong solution is
$\mathrm{Osc}(A) := \max_{i,j} a_{ij} - \min_{i,j} a_{ij} \leq |x|/(n-1)$.

The last assumption is almost necessary: $\forall \varepsilon > 0$ there exists a matrix A with $\mathrm{Osc}(A) = \varepsilon + |x|/(n-1)$ which does not admit even weak solutions.

Construction of the strong solution

Consider a map T from the set of all permutations π of n elements Π_n into itself, defined as follows. Consider the set of pairs $(i,j) \in \{1,\dots,n\}^2$ with the lexicographic order, and denote by $\pi(i,j)$ a new permutation obtained from π by exchanging of the corresponding entries: $\pi_i \longleftrightarrow \pi_j$.
Then for a given permutation π we check the pairs (i,j) successively with respect to their order until the inequality $a_{i\pi_i} + a_{j\pi_j} \ge a_{i\pi_j} + a_{j\pi_i}$ fails. In the latter case we set $T\pi := \pi(i,j)$.
Applying the map T iteratively in a finite number of steps we obtain the "maximal" permutation, satisfying the above condition. Indeed, if $T\pi \neq \pi$ then $a(T\pi) > a(\pi)$ by the construction. Since the total number of permutations is finite this proves the claim.

Connection to the rent division problem

Let $n = N$, i.e. each agent gets a single room. Then we set $\mu_i(\Delta_j, \Delta) := a_{ij} - x_i$ as an estimate of the fairness of the division. Using this set of preferences one can construct all types of the solutions that we have discussed in the 2nd part of the talk. On the other hand, neither of these solutions coincide with the admissible solution for the rent division problem.

– *Steinhaus, H.* // Econometrica, 1948, V.16(1), P.101-104.
– *Brams, S.J., Jones M.A., C. Klamler.* // Notices of AMS. 2006. V. 53 (11), 1314–1321.
– *Moulin, H.* // Fair division and collective welfare. Cambridge, 2003. – *Barbanel J., Brams S.J., Stromquist W.*// American Mathematical Monthly. 2009, V.116, P.496-514.
– *Dubins L. E., Spanier E. H.* // Amer. Math. Monthly. 1961, V.68:1, P.1-17. – *Stromquist W.* // Electronic Journal of Combinatorics. 2008, V.15(1), R11. – [Su] *Su F.E.* // American Mathematical Monthly. 1999, 106:10, 930–942.
– *Musin O.* // arXiv:1512.04612 [math.AT] (2015)
– *Blank M.* // Doklady Mathematics, 94:3(2016), 688–691.
– *Blank M.* // Problems of Information Transmission, 52:3(2016), 201-209.

1. Пусть $(X, \mathcal{B})$ – измеримое пространство с борелевской σ-алгеброй измеримых множеств и пусть φ – непрерывный функционал на нем. Можно ли разделить X "пополам", т.е. найти $A \in \mathcal{B}: \ \varphi(A) = \varphi(X \setminus A)$?
2. Пусть A – многогранник в $\mathbb{R}^d$ (не обязательно выпуклый). Можно ли его разделить гиперплоскостью на многогранники "слева" и "справа" с суммарно одинаковыми площадями поверхности?
3. Как разделить набор $a_1, a_2, \ldots, a_n \in \mathbb{N}$ на m групп с максимально близкими суммами?
4. Как ограничить демократическое деление, чтобы не могло быть "ухудшения в среднем"?
5. Как справедливо разделить квартиру с фиксированнтой рентой из m комнат для $n \neq m$ жильцов?