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Equilibrium in a Discrete Exchange Economy with Money

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Abstract: We consider a market in which there are n traders each of whom owns an object and some amount of money. It is shown that under rather mild conditions on demand the market will have a price equilibrium. The proof makes use of a generalization of a well known result of combinatorial topology.

We consider an exchange economy in which there are n traders each of whom owns at most one indivisible object and some amount of a common divisible good which may be thought of as money. The traders have preferences over objects and money subject to the restriction that no trader desires more than one object. Typically, one may think of the objects as houses. An interesting special case is that in which some of the traders, called *sellers*, own houses but no money while the rest, the *buyers* own money but no houses. As another example, we may think of the sellers as workers who wish to sell their services and the buyers as employers who wish to fill certain jobs.

We shall be concerned with the question of existence of equilibrium prices for the general model described above. Although the problem seems a natural one, it was solved only recently by *Quinzii* [1984] who obtained an equilibrium existence theorem as a biproduct of some general results on the core of a certain class of games. In her model, agents are assumed to have utility functions on pairs consisting of an object and a quantity of money, these functions being continuous and increasing in money. It is also assumed that no trader is willing to part with all of his money to obtain an object.

The purpose of this note is to give a direct proof of existence of equilibrium under rather general assumptions on demand. Only boundedness and continuity are required. Further, our result applies to a model with "externalities." Thus, whether or not a trader prefers object i to j may depend on the price of a third object k. Also we do not require that demand is a decreasing function of price. Thus, a trader who lacks information may be more likely to choose an object when its price is higher on the assumption that if it costs more it must be better.

Our proof of the equilibrium theorem is based on a generalization of the well known lemma of Knaster, Kuratowski and Mazurkewicz (KKM) in combinatorial topology.

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To facilitate the analysis, we will assume that every trader owns an object. This involves no loss of generality since if some trader owns no object we may assume that he has a "dummy" object which is of no value to any of the traders. To be precise, a worthless object is one which no trader would accept in exchange for a positive amount of money. The set of all objects will be denoted by $N = \{1, 2, \ldots, n\}$ and the set of traders by T. Members of T will be represented by Greek letters. An *assignment* is a one to one mapping σ from T to N. A *price vector* p is a function from N to $\mathbf{R}^+$.

To specify demand, we assume that corresponding to each α in T, there is a Covering $C^\alpha = \{C_0^\alpha, C_1^\alpha, \ldots, C_n^\alpha\}$ of $\mathbf{R}_n$. The interpretation is that if $p \in C_j^\alpha$, $j > 0$, then trader α will demand object j at prices p while if $p \in C_0^\alpha$ then α will demand no object but will wish to exchange his object for money.

Definition: An *equilibrium* is a pair consisting of a price vector p and an assignment σ such that $p \in C_{\sigma(\alpha)}^\alpha$ for all α in T.

We will denote the boundary of $\mathbf{R}_n^+$ by B_n. Our assumptions on demand are now the following:

(1) The sets C_j^α are closed.

(2) The sets $C_1^\alpha, \ldots, C_n^\alpha$ cover B_n.

(3) There exists $M > 0$ such that if $p_i \geqq M$, then $p \notin C_i^\alpha$ for any α.

Assumption (1) is not quite as innocent as it appears. It implies, for example, that every trader has a positive amount of money, for suppose there was a trader α with no money, but positive utility for all of the objects. Then C_0^α would be the interior of $\mathbf{R}_n^+$ which is not closed. In fact, closedness of the C_i^α implies that no trader would give up all of his money to buy any object as assumed in *Quinzii* [1984].

Assumption (2) is a free disposal condition which says that if an object costs nothing a trader cannot hurt himself by buying it, since $p \in B_n$ implies at least one object is free.

Assumption (3) says that no trader is willing to spend an infinite amount of money on any object. By choice of units, we assume from now on that $M = 1$.

Equilibrium Theorem: Under Assumptions (1), (2) and (3), there is an equilibrium (p, σ) where $p \in B_n$.

Note that if we assume traders have positive utility for money, then at equilibrium all dummy goods will be free, since if a dummy good had positive price then no trader would be willing to hold it in preference to the free good because of Assumption (3).

We first prove a topological lemma. Let Δ_{n-1} denote the unit $(n-1)$-simplex in $\mathbf{R}_n$. For any $S \subset N$, we denote by F_S the face of Δ_{n-1} spanned by the unit vectors

e_i, $i \in S$. A closed covering $C = \{C_1, \ldots, C_n\}$ of Δ_{n-1} is called a *KKM covering* if

$$F_S \subset \bigcup_{i \in S} C_i. \tag{4}$$

The KKM Lemma asserts that $\bigcap_{i \in N} C_i$ is nonempty. Our generalization is the following.

Lemma: Let C^i, $i = 1, \ldots, n$ be n KKM coverings of Δ_{n-1}. Then there is a permutation σ on N and a point $p \in \Delta_{n-1}$ such that $p \in C^i_{\sigma(i)}$ for all i.

A colloquial statement of this result is the "red, white and blue lemma" which asserts that if each of three people paint a triangle red, white and blue according to the KKM rules, then there will be a point which is in the red set of one person, the white set of another, the blue of the third.

Proof: We first consider the case where the sets C^i_j are open. The proof for the closed case then follows by a routine limiting argument. Now if the C^i_j are open, we may define $\widetilde{C}^i_j = C^i_j - F_{N\text{-}\{j\}}$ and observe that the $\widetilde{C}^i_j$ still have the KKM property. Now for each covering $\widetilde{C}^i$ consider the corresponding *partition of unity*, thus, continuous non-negative functions f^i_j such that $f^i_j(p) = 0$ for $p \notin \widetilde{C}^i_j$ and $\Sigma f^i_j(p) = 1$. Define F^i from Δ_{n-1} to Δ_{n-1} by $F^i(p) = (f^i_1(p), \ldots, f^i_n(p))$ and let $F = 1/n\ \Sigma F^i$. Note that since $\widetilde{C}^i_j$ does not meet $F^i_{N\text{-}\{j\}}$, it follows that F^i maps each face of Δ_{n-1} into itself and the same will be true for F. By a standard topological result it follows that F is surjective so there is p such that $F(p) = 1/n\,(1, \ldots, 1)$ so $nF(p) = (1, \ldots, 1)$. Thus, the matrix $nF = (f^i_j(p))$ is doubly stochastic and, hence, by a standard result it is possible to find a permutation σ such that $f^i_{\sigma(i)}(p) > 0$ for all i, but, this means precisely that $p \in C^i_{\sigma(1)}$. □

In order to prove the Equilibrium Theorem, let Σ_n be the intersection of B_n and the unit n-cube. We will exhibit a homeomorphism ϕ from Σ_n to Δ_{n-1} such that the sets $\phi(C^\alpha_j)$ are a KKM cover. The result then follows from the Lemma. For each permutation $\tau = (j_1, j_2, \ldots, j_n)$ of N define $\Sigma_\tau \subset \Sigma_n$ by

$$\Sigma_\tau = \{p \in \Sigma_n \mid p_{j_1} \geqslant p_{j_2} \geqslant \ldots \geqslant p_{j_n} = 0\}$$

and define

$$\Delta_\tau = \{x \in \Delta_{n-1} \mid x_{j_1} \leqslant x_{j_2} \leqslant \ldots \leqslant x_{j_n}\}.$$

We then define ϕ from Σ_τ to Δ_τ by

$$(\phi(p))_{j_k} = x_{j_k} = \frac{1 - p_{j_1}}{n} + \frac{p_{j_1} - p_{j_2}}{n-1} + \ldots + \frac{p_{j_{k-1}} - p_{j_k}}{n-k+1}.$$

One verifies that $\phi(p) \in \Delta_{n-1}$, i.e., $\sum_{k=1}^{n} x_{j_k} = 1$ and also that ϕ maps Σ_τ onto Δ_τ, for, given x in Δ_τ, $p_{j_1} = 1 - nx_{j_1}$, $p_{j_2} = (n-1)(x_{j_2} - x_{j_1}) + p_{j_1}$ etc.

Note that $\phi^{-1}(e_j) = e - e_j$ and $\phi^{-1}(F_{N\text{-}\{j\}}) = \{p \mid p_j = 1\}$.

By Assumption (3) C_j^α does not meet $\phi^{-1}(F_{N\text{-}\{j\}})$ so it follows that $\phi(C_j^\alpha)$ does not meet $F_{N\text{-}\{j\}}$ giving the desired KKM covering of Δ_{n-1}. The equilibrium theorem now follows. □

Reference

Quinzii, M.: Core and Competitive Equilibria with Indivisibilities. International Journal of Game Theory, this issue.

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