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Abstract. We consider generalizations of Gale's colored KKM lemma and Shapley's KKMS theorem. It is shown that spaces and covers can be much more general and the boundary KKM rules can be substituted by more weaker boundary assumptions.

Mathematics Subject Classification. 55M20, 55M25, 55P05.

Keywords. Sperner lemma, KKM theorem, KKMS theorem, Gale lemma, rental harmony, degree of mapping, homotopy classes of mappings, partition of unity.

1. Introduction

Sperner's lemma [22] is a combinatorial analog of the Brouwer fixed point theorem, which is equivalent to it. This lemma and its extension for covers, the Knaster–Kuratowski– Mazurkiewicz (KKM) theorem [9], have many applications in combinatorics, algorithms, game theory and mathematical economics.

There are many extensions of the KKM theorem. In this paper we consider two of them: Gale's lemma [6] and Shapley's KKMS theorem [17].

David Gale in [6] proved an existence theorem for an exchange equilibrium in an economy with indivisible goods and only one perfectly divisible good, which can be thought of as money. The main lemma for this theorem is [6, Lemma, p. 63].

Gale's lemma can be considered as a *colored KKM* theorem. In [6, p. 63], Gale wrote about his lemma: "A colloquial statement of this result is the *red, white and blue lemma* which asserts that *if each of three people paint a triangle red, white and blue according to the KKM rules, then there will be a point which is in the red set of one person, the white set of another, the blue of the third.*" Note that Bapat [3] found an analog of Gale's lemma for Sperner's lemma.

In fact, Gale's lemma (or its discrete analogs) can be applied for fair division problems, namely for the envy-free cake-cutting and rental harmony

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problems . In the envy-free *cake-cutting problem,* a "cake" (a heterogeneous divisible resource) has to be divided among n partners with different preferences over parts of the cake [5,19–21]. The cake has to be divided into n pieces such that: (a) each partner receives a single connected piece, and (b) each partner believes that his/her piece is (weakly) better than all other pieces. An algorithm for solving this problem was developed by Forest Simmons in 1980, in a correspondence with Michael Starbird. It was first publicized by Francis Su in 1999 [21].

Suppose a group of friends consider renting a house but they shall first agree on how to allocate its rooms and share the rent. They will rent the house only if they can find a room assignment-rent division which appeals to each of them. Following Su [21], we call such a situation *rental harmony.* In [1] consideration is given to different aspects of this model.

In 1967 Scarf [16] proved that any non-transferable utility game whose characteristic function is balanced, has a non-empty core. His proof is based on an algorithm which approximates fixed points. Lloyd Shapley [17] replaced the Scarf algorithm by a covering theorem (the KKMS theorem) being a generalization of the KKM theorem. Now Shapley's KKMS theorem [7,8, 17,18] is an important tool in the general equilibrium theory of economic analysis.

The main goal of this paper to considere generalizations of Gale's and Shapley's KKMS theorems with general boundary conditions. In our paper [15] with any cover of a space T we associate certain homotopy classes of maps from T to n-spheres. These homotopy invariants can then be considered as obstructions for extending covers of a subspace $A \subset X$ to a cover of all of X. We are using these obstructions to obtain generalizations of the KKM and Sperner lemmas. In particular, we show that in the case when A is a k-sphere and X is a $(k+1)$-disk there exist KKM type lemmas for covers by $n+2$ sets if and only if the homotopy group $\pi_k(\mathbb{S}^n) \neq 0$. In Sect. 2 is given a review of main results of [15].

In Sect. 3 we generalize Gale's lemma. In particular, see Corollary 3.1, we pove that *if each of n people paint a k-simplex with n colors such that the union of these covers is not null-homotopic on the boundary, then there will be a point which is in the first color set of one person, the second color set of another, and so on.*

In Sect. 4 we consider KKMS type theorems. Actually, these theorems are analogs for covers of a polytopal type Sperner's lemmas, see [4,13,15]. Let V be a set of m points in $\mathbb{R}^n$. Then, see Corollary 4.1, *if $\mathcal{F} = \{F_1, \ldots, F_m\}$ is a cover of a k-simplex that is not null-homotopic on the boundary, then there is a balanced with respect to V subset $\mathcal{B}$ in $\{1, \ldots, m\}$ such that all the F_i, for $i \in \mathcal{B}$, have a common point.*

If V is the set of vertices of a k-simplex Δ^k, then this corollary implies the KKM theorem and if V is the set of all centers of Δ^k it yields the KKMS theorem. As an example, we consider a generalization of Tucker's lemma (Corollary 4.3). (Note that David Gale, Lloyd Shapley as well as John F. Nash were Ph.D. students of Albert W. Tucker in Princeton.)

Notations. Throughout this paper we consider only normal topological spaces, all simplicial complexes be finite, all manifolds be compact and piecewise linear, I_n denotes the set $\{1, \dots, n\}$, Δ^n denotes the n-dimensional simplex, $\mathbb{S}^n$ denotes the n-dimensional unit sphere, $\mathbb{B}^n$ denotes the n-dimensional unit disk and $|K|$ denotes the underlying space of a simplicial complex K. We shall denote the set of homotopy classes of continuous maps from X to Y as $[X, Y]$.

2. Sperner–KKM lemma with boundary conditions

The $(n-1)$-dimensional unit simplex Δ^{n-1} is defined by

$$\Delta^{n-1} := \{x \in \mathbb{R}^n \mid x_i \geq 0, x_1 + \cdots + x_n = 1\}.$$

Let $v_i := (x_1, \dots, x_n)$ with $x_i = 1$ and $x_j = 0$ for $j \neq i$. Then $v_1, \dots, v_n$ is the set of vertices of Δ^{n-1} in $\mathbb{R}^n$.

Let K be a simplicial complex. Denote by $\mathrm{Vert}(K)$ the vertex set of K. An n-labeling L is a map $L\colon \mathrm{Vert}(K) \to \{1, 2, \dots, n\}$. Setting

$$f_L(u) := v_k, \quad \text{where} \quad u \in \mathrm{Vert}(K) \text{ and } k = L(u),$$

we have a map $f_L\colon \mathrm{Vert}(K) \to \mathbb{R}^n$. Every point $p \in |K|$ belongs to the interior of exactly one simplex in K. Letting $\sigma = \mathrm{conv}\{u_0, u_1, \dots, u_k\}$ be the simplex, we have $p = \sum_0^k \lambda_i(p) u_i$ with $\sum_0^k \lambda_i(p) = 1$ and all $\lambda_i > 0$. (Actually, $\lambda_i(p)$ are the barycentric coordinates of p). Then f_L can be extended to a continuous (piecewise linear) map $f_L\colon |K| \to \Delta^{n-1} \subset \mathbb{R}^n$ defined by

$$f_L(p) = \sum_{i=0}^{k} \lambda_i(p) f_L(u_i).$$

We say that a simplex s in K is *fully labeled* if s is labeled with a complete set of labels $\{1, 2, \dots, n\}$. Suppose there are no fully labeled simplices in K. Then $f_L(p)$ lies in the boundary of Δ^{n-1}. Since the boundary $\partial\Delta^{n-1}$ is homeomorphic to the sphere $\mathbb{S}^{n-2}$, we have a continuous map $f_L\colon |K| \to \mathbb{S}^{n-2}$. Denote the homotopy class $[f_L] \in [|K|, \mathbb{S}^{n-2}]$ by $\mu(L) = \mu(L, K)$.

Example 2.1. Let $L\colon \mathrm{Vert}(K) \to \{1, 2, 3\}$ be a labelling of a closed planar polygonal line K with vertices $p_1 p_2 \cdots p_k$. Then f_L is map from $|K| = \mathbb{S}^1$ to $\mathbb{S}^1$. Not that $\mu(L) \in [\mathbb{S}^1, \mathbb{S}^1] = \mathbb{Z}$ is the degree of the map f_L. Moreover,

$$\mu(L) = \deg(f_L) := p_* - n_*,$$

where p_* (respectively, n_*) is the number of (ordering) pairs (p_i, p_{i+1}) such that $L(p_i) = 1$ and $L(p_{i+1}) = 2$ (respectively, $L(p_i) = 2$ and $L(p_{i+1}) = 1$). It is clear, that instead of $[1, 2]$ we can take $[2, 3]$ or $[3, 1]$.

For instance, let $L = (1221231232112231231)$. Then $p_* = 5$ and $n_* = 2$. Thus,

$$\mu(L) = 5 - 2 = 3.$$

Example 2.2. Let K be a triangulation of the boundary of a simplex Δ^{k+1}. In other words, K is a triangulation of of $\mathbb{S}^k$). Let $L : \mathrm{Vert}(K) \to \{1, \dots, n\}$ be a labeling such that K has no simplices with n distinct labels. Then $f_L \in \pi_k(\mathbb{S}^{n-2})$.

In the case $k = n-2$ we have $\pi_k(\mathbb{S}^k) = \mathbb{Z}$ and

$$[f_L] = \deg(f_L) \in \mathbb{Z}.$$

(Here by $\deg(f)$ is denoted the degree of a continuous map f from $\mathbb{S}^k$ to itself.)

For instance, let L be a *Sperner labeling* of a triangulation K of $\partial\Delta^{n-1} = u_1 \cdots u_n$. The rules of this labeling are:

(i) The vertices of Δ^{n-1} are colored with different colors, i.e. $L(u_i) = i$ for $1 \le i \le n$.

(ii) Vertices of K located on any m-dimensional subface of the large simplex $u_{i_0} u_{i_1} \cdots u_{i_m}$ are colored only with the colors $i_0, i_1, \dots, i_m$.

Then $\mu(L) = \deg(f_L) = 1$ in $[\mathbb{S}^{n-2}, \mathbb{S}^{n-2}] = \mathbb{Z}$.

In [15] we proved the following theorem, see [15, Corollary 3.1].

Theorem 2.1. *Let T be a triangulation of a simplex Δ^{k+1}. Let $L\colon \mathrm{Vert}(T) \to \{1, \dots, n\}$ be a labeling such that T has no simplices on the boundary with n distinct labels. If $\mu(L, \partial T) \neq 0$, then T must contain a fully labeled simplex.*

(Here by $\mu(L, \partial T)$ we denote the invariant μ on the boundary of T.)

Since for a Sperner labeling $\mu(L, \partial T) = 1 \neq 0$, Theorem 2.1 implies:
(Sperner's lemma [22]) *Every Sperner labeling of a triangulation of Δ^{n-1} contains a cell labeled with a complete set of labels: $\{1, 2, \dots, n\}$.*

Consider an oriented manifold M of dimension $(n-1)$ with boundary. Then $[\partial M, \mathbb{S}^{n-2}] = \mathbb{Z}$ and for any continuous $f\colon \partial M \to \mathbb{S}^{n-2}$ we have $[f] = \deg f$. If T is a triangulation of M and $L\colon \mathrm{Vert}(T) \to \{1, \dots, n\}$ is a labeling then we denote by $\deg(L, \partial T)$ the class $\mu(L, \partial T)$.

Theorem 2.2 [15, Theorem 3.4]. *Let T be a triangulation of an oriented manifold M of dimension $(n-1)$ with boundary. Then for a labeling $L\colon \mathrm{Vert}(T) \to \{1, \dots, n\}$ the triangulation must contain at least $|\deg(L, \partial T)|$ fully labelled simplices.*

In Fig. 1 is shown an illustration of Theorem 2.2. We have a labeling with $\deg(L, \partial T) = 3$. Therefore, the theorem garantee that there are at least three fully labeled triangles.

Actually, a labeling can be considered as a particular case of a covering. For any labeling L there is a natural open cover of $|K|$. The open star of a vertex $u \in \mathrm{Vert}(K)$ (denoted $\mathrm{St}(u)$) is $|S| \backslash |B|$, where S is the set of all simplices in K that contain u, and B is the set of all simplices in S that contain no u. Let

$$\mathcal{U}_L(K) = \{U_1(K), \dots, U_n(K)\},$$

where

$$U_\ell(K) := \bigcup_{u \in W_\ell} \mathrm{St}(u), \quad W_\ell := \{u \in \mathrm{Vert}(K)\colon L(u) = \ell\}.$$

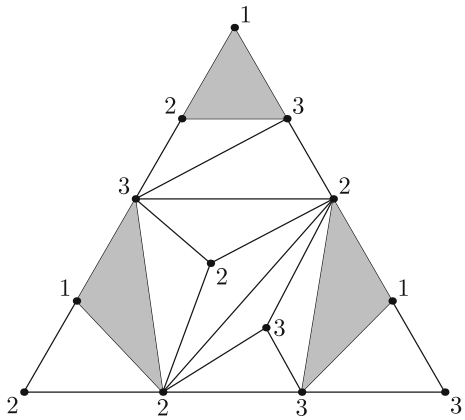

FIGURE 1. $\deg(L, \partial T) = 3$. There are three fully labeled triangles.

It is clear, that $K = U_1(K) \bigcup \cdots \bigcup U_n(K)$, i.e. $\mathcal{U}_L(K)$ is a cover of K.

Now we extend the definition of $\mu(L)$ for covers.

Let $\mathcal{U} = \{U_1, \ldots, U_n\}$ be a collection of open sets whose union contains a space T. In other words, $\mathcal{U}$ is a cover of T. Let $\Phi = \{\varphi_1, \ldots, \varphi_n\}$ be a partition of unity subordinate to $\mathcal{U}$, i.e. Φ is a collection of non-negative functions on T such that $\operatorname{supp}(\varphi_i) \subset U_i$, $i = 1, \ldots, n$, and for all $x \in T$, $\sum_1^n \varphi_i(x) = 1$. Let

$$f_{\mathcal{U},\Phi}(x) := \sum_{i=1}^{n} \varphi_i(x) v_i,$$

where $v_1, \ldots, v_n$, as above, are vertices of Δ^{n-1}.

Suppose the intersection of all U_i is empty. Then $f_{\mathcal{U},\Phi}$ is a continuous map from T to $\mathbb{S}^{n-2}$.

In [15, Lemmas 2.1 and 2.2] we proved that a homotopy class $[f_{\mathcal{U},\Phi}]$ in $[T, \mathbb{S}^{n-2}]$ does not depend on Φ. We denote it by $\mu(\mathcal{U})$.

Note that for a labeling $L\colon \operatorname{Vert}(K) \to \{1, 2, \ldots, n\}$ we have

$$\mu(L) = \mu(\mathcal{U}_L(K)) \in [|K|, \mathbb{S}^{n-2}].$$

Example 2.3. (a) Let $T = \mathbb{S}^k$ and $\mathcal{U} = \{U_1, \ldots, U_n\}$ be an open cover of $T = \mathbb{S}^k$ such that the intersection of all U_i is empty. Then $\mu(\mathcal{U}) \in \pi_k(\mathbb{S}^{n-2})$. In the case $k = n - 2$ we have

$$\mu(\mathcal{U}) = \deg(f_{\mathcal{U}}) \in \mathbb{Z}.$$

(b) Let $h\colon \mathbb{S}^k \to \mathbb{S}^{n-2}$ be any continuous map. Actually, $\mathbb{S}^{n-2}$ can be considered as the boundary of the simplex Δ^{n-1}. Let

$$U_i := h^{-1}(U_i(\Delta^{n-1}),\ i = 1, \ldots, n\ \mathcal{U} = \{U_1, \ldots, U_n\}.$$

Then $\mu(\mathcal{U}) = [h] \in \pi_k(\mathbb{S}^{m-1})$.

For instance, if $h\colon \mathbb{S}^3 \to \mathbb{S}^2$ is the Hopf fibration, then $\mu(\mathcal{U}) = 1 \in \pi_3(\mathbb{S}^2) = \mathbb{Z}$.

In fact, see [15, Lemma 2.4], the homotopy classes of covers are also well defined for closed sets.

Definition 2.1. We call a family of sets $\mathcal{S} = \{S_1, \dots, S_n\}$ as a cover of a space T if $\mathcal{S}$ is either an open or closed cover of T.

Definition 2.2. Let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a cover of a space T. We say that $\mathcal{S}$ is not null-homotopic if the intersection of all S_i is empty and $\mu(\mathcal{S}) \neq 0$ in $[T, \mathbb{S}^{n-2}]$.

Note that covers in Examples 2.1, 2.2 and 2.3 are not null-homotopic. Actually, these examples and [15, Theorems 2.1–2.3] imply the following theorem.

Theorem 2.3. *Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a cover of a space T. Suppose the intersection of all S_i is empty.*

1. *Let T be $\mathbb{S}^k$. Then $\mathcal{S}$ is not null-homotopic if and only if $\pi_k(\mathbb{S}^{n-1})$ is not trivial and $\mu(\mathcal{S}) \neq 0$ in this group.*
2. *Let T be an oriented $(n-2)$-dimensional manifold. Then $\mathcal{S}$ is not null-homotopic if and only if $\deg(f_{\mathcal{S}}) \neq 0$.*

Now we consider a generalization of the KKM theorem for covers of spaces.

Definition 2.3. Consider a pair (X, A), where A is a subspace of a space X. Let $\mathcal{S} = \{S_1, \dots, S_m\}$ be a cover of X and $\mathcal{C} = \{C_1, \dots, C_m\}$ be a cover of A. We say that $\mathcal{S}$ is an extension of $\mathcal{C}$ and write $\mathcal{C} = \mathcal{S}|_A$ if $C_i = S_i \cap A$ for all i.

Definition 2.4. We say that a pair of spaces (X, A), where $A \subset X$, belongs to EP_n and write $(X, A) \in \mathrm{EP}_n$ if there is a continuous map $f\colon A \to \mathbb{S}^n$ with $[f] \neq 0$ in $[A, \mathbb{S}^n]$ that cannot be extended to a continuous map $F\colon X \to \mathbb{S}^n$ with $F|_A = f$.

We denoted this class of pairs by EP after S. Eilenberg and L. S. Pontryagin who initiated obstruction theory in the late 1930s. Note that [15, Theorem 2.3] yield that

(i) if $\pi_k(\mathbb{S}^n) \neq 0$, then $(\mathbb{B}^{k+1}, \mathbb{S}^k) \in \mathrm{EP}_n$,
(ii) if X is an oriented $(n+1)$-dimensional manifold and $A = \partial X$, then $(X, A) \in \mathrm{EP}_n$.

Theorem 2.4. *Let A be a subspace of a space X. Let $(X, A) \in \mathrm{EP}_{n-2}$. Let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a cover of (X, A). Suppose the cover $\mathcal{S}$ is not null-homotopic on A. Then all the S_i have a common intersection point.*

Corollary 2.1. *Let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a cover of $\mathbb{B}^k$. Suppose $\mathcal{S}$ is not null-homotopic on the boundary of $\mathbb{B}^k$. Then all the S_i have a common intersection point.*

Corollary 2.2. *Let $\mathcal{S} = \{S_1, \dots, S_n\}$ be a cover of a manifold M of dimension $(n-1)$ with boundary. Let $\mathcal{C} := \mathcal{S}|_{\partial M}$. Suppose $\deg f_{\mathcal{C}} \neq 0$. Then all the S_i have a common intersection point.*

We say that a cover $\mathcal{S} := \{S_1, \ldots, S_m\}$ of a simplex Δ^{m-1} is a *KKM cover* if for all $J \subset I_m$ the face of Δ^{m-1} that is spanned by vertices v_i for $i \in J$ is covered by S_i for $i \in J$.

Let $\mathcal{C} := \mathcal{S}|_{\partial\Delta^{n-1}}$. Note that if all the C_i have no a common intersection point, then the KKM assumption implies that $\mu(\mathcal{C}) = \deg(f_{\mathcal{C}}) = 1$. Thus, Corollary 2.2 yields

Corollary 2.3 (KKM theorem [9]). *If $\mathcal{S} := \{S_1, \ldots, S_n\}$ is a KKM cover of Δ^{n-1}, then all the S_i have a common intersection point.*

3. Gale's lemma with boundary conditions

David Gale [6, Lemma, p. 63] proved the following lemma:

For $i, j = 1, 2, \ldots, n$ let S^i_j be closed sets such that for each i, $\{S^i_1, \ldots, S^i_n\}$ is a KKM covering of Δ^{n-1}. Then there exists a permutation π of $1, 2, \ldots, n$ such that

$$\bigcap_{i=1}^{n} S^i_{\pi(i)} \neq \emptyset.$$

Now we generalize this lemma for pairs of spaces.

Theorem 3.1. *Let A be a subspace of a space X. Let $(X, A) \in \mathrm{EP}_{n-2}$. Let $\mathcal{S}^i = \{S^i_1, \ldots, S^i_n\}$, $i = 1, \ldots, n$, be covers of (X, A). Let*

$$F_j := \bigcup_{i=1}^{n} S^i_j, \quad \mathcal{F} := \{F_1, \ldots, F_n\}, \quad \mathcal{C} := \mathcal{F}|_A.$$

Suppose $\mathcal{C}$ is not null-homotopic. Then there exists a permutation π of $1, 2, \ldots, n$ such that

$$\bigcap_{i=1}^{n} S^i_{\pi(i)} \neq \emptyset.$$

Proof. Here we use Gale's proof of his lemma. We consider the case where the sets S^i_j are open. As above the proof of the closed case then follows by a routine limiting argument. Now for each cover $\mathcal{S}^i$ consider the corresponding partition of unity $\{\varphi^i_1, \ldots, \varphi^i_n\}$.

Define $\Phi^i \colon X \to \Delta^{n-1}$ and $\Phi \colon X \to \Delta^{n-1}$ by

$$\Phi^i(p) := (\varphi^i_1(p), \ldots, \varphi^i_n(p)), \quad \Phi(p) := \frac{\Phi^1(p) + \ldots + \Phi^n(p)}{n},$$

where $\Delta^{n-1} = \{(x_1, \ldots, x_n) \in \mathbb{R}^n \mid x_i \geq 0,\ i = 1, \ldots, n,\ x_1 + \cdots + x_n = 1\}$. Since $\mathcal{C}$ is not null-homotopic, Φ is a map from A to $\partial\Delta^{n-1} \equiv \mathbb{S}^{n-2}$ and $\Phi(X) = \Delta^{n-1}$. Therefore, there is $p \in X$ such that $\Phi(p) = (1/n, \ldots, 1/n)$, so $n\Phi(p) = (1, \ldots, 1)$. Thus, the matrix $M := n\Phi(p) = \big(\varphi^i_j(p)\big)$ is a *doubly stochastic* matrix, is a square matrix of nonnegative real numbers, each of whose rows and columns sums to 1. By Mirsky's lemma [11] for any doubly stochastic matrix M it is a possible to find a permutation π such that $\varphi^i_{\pi(i)}(p) > 0$ for all i, but this means precisely that $p \in S^i_{\pi(i)}$. $\square$

Remark. This proof is constructive. If $p \in \Phi^{-1}(1/n, \dots, 1/n)$, then there is a permutation π with $\varphi^i_{\pi(i)}(p) > 0$, i.e. the intersection of all the $S^i_{\pi(i)}$, $i = 1, \dots, n$, is not empty.

Theorem 3.1 and [15, Theorem 2.3] imply the following corollaries.

Corollary 3.1. *Let $\mathcal{S}^i = \{S^i_1, \dots, S^i_n\}$, $i = 1, \dots, n$, be covers of $\mathbb{S}^k$. Let F_j is the union of S^i_j, $i = 1, \dots, n$, and $\mathcal{F} := \{F_1, \dots, F_n\}$. Suppose $\mathcal{F}$ is not null-homotopic on the boundary of $\mathbb{S}^k$. Then there exists a permutation π of $1, 2, \dots, n$ such that*

$$\bigcap_{i=1}^{n} S^i_{\pi(i)} \neq \emptyset.$$

Corollary 3.2. *Let $\mathcal{S}^i = \{S^i_1, \dots, S^i_n\}$, $i = 1, \dots, n$, be covers of a manifold M of dimension $(n-1)$ with boundary. Let F_j is the union of S^i_j, $i = 1, \dots, n$, $\mathcal{F} := \{F_1, \dots, F_n\}$, and $\mathcal{C} := \mathcal{F}|_{\partial M}$. Suppose $\deg \mathcal{C} \neq 0$. Then there exists a permutation π of $1, 2, \dots, n$ such that*

$$\bigcap_{i=1}^{n} S^i_{\pi(i)} \neq \emptyset.$$

Now consider the rental harmony problem. Following Su [21], suppose there are n housemates, and n rooms to assign, numbered $1, \dots, n$. Let x_i denote the price of the i-th room, and suppose that the total rent is 1. Then $x_1 + \ldots + x_n = 1$ and $x_i \geq 0$. From this we see that the set of all pricing schemes forms a simplex Δ^{n-1}.

Denote by S^i_j a set of price vectors p in Δ^{n-1} such that housemate i likes room j at these prices. Consider the following conditions:

(C1) In any partition of the rent, each person finds some room acceptable. In other words, $\mathcal{S}^i = \{S^i_1, \dots, S^i_n\}$, $i = 1, \dots, n$, is a (closed or open) cover of Δ^{n-1}.

(C2) Each person always prefers a free room (one that costs no rent) to a non-free room. In other words, for all i and j, S^i_j contains $\Delta^{n-1}_j := \{(x_1, \dots, x_n) \in \Delta^{n-1} \mid x_j = 0\}$.

In fact, (C2) is the "dual" boundary KKM condition. Su [21, Sect. 7] using the "dual" simplex Δ^* and "dual" Sperner lemma proves that there exists a permutation π of $1, 2, \dots, n$, such that the intersection of the $S^i_{\pi(i)}$, $i = 1, \dots, n$, is not empty. It proves the rental harmony theorem. Also, this theorem can be derived from Corollary 3.2.

Rental Harmony Theorem [21]. *Suppose n housemates in an n-bedroom house seek to decide who gets which room and for what part of the total rent. Also, suppose that the conditions (C1) and (C2) hold. Then there exists a partition of the rent so that each person prefers a different room.*

Let us consider an extension of this theorem. Suppose there are some constraints $f_i(p) \leq 0$, $i = 1, \dots, n$, for price vectors. Let $M := \{p \in \Delta^{n-1} \mid f_1(p) \leq 0, \dots, f_k(p) \leq 0\}$ be a manifold of dimension $n-1$. Let S^i_j

be sets of price vectors p in M such that housemate i likes room j at these prices. Consider the following conditions:

(A1) $\mathcal{S}^i = \{S_1^i, \dots, S_n^i\}$, $i = 1, \dots, n$, is a cover of M.
(A2) $\deg \mathcal{C} \neq 0$, where $\mathcal{C} := \mathcal{F}|_{\partial M}$, $\mathcal{F} := \{F_1, \dots, F_n\}$, and $F_j := \bigcup_{i=1}^n S_j^i$.

The following theorem is equivalent to Corollary 3.2.

Theorem 3.2. *Suppose n housemates in an n-bedroom house seek to decide who gets which room and for what part of the total rent. Also, suppose that the conditions (A1) and (A2) hold. Then there exists a partition of the rent so that each person prefers a different room.*

4. KKMS type theorems with boundary conditions

Let us extend definitions from Sect. 2 for any set of points (vectors) $V := \{v_1, \dots, v_m\}$ in $\mathbb{R}^n$. Denote by c_V the center of mass of V, $c_V := (v_1 + \cdots + v_m)/m$.

Let $\mathcal{U} = \{U_1, \dots, U_m\}$ be an open cover of a space T and $\Phi = \{\varphi_1, \dots, \varphi_m\}$ be a partition of unity subordinate to $\mathcal{U}$. Let

$$\rho_{\mathcal{U},\Phi,V}(x) := \sum_{i=1}^m \varphi_i(x) v_i.$$

Suppose c_V lies outside of the image $\rho_{\mathcal{U},\Phi,V}(T)$ in $\mathbb{R}^n$. Let for all $x \in T$

$$f_{\mathcal{U},\Phi,V}(x) := \frac{\rho_{\mathcal{U},\Phi,V}(x) - c_V}{||\rho_{\mathcal{U},\Phi,V}(x) - c_V||}.$$

Then $f_{\mathcal{U},\Phi,V}$ is a continuous map from T to $\mathbb{S}^{n-1}$.

In [15, Lemmas 2.1 and 2.2] we proved that the homotopy class $[f_{\mathcal{U},\Phi,V}] \in [T, \mathbb{S}^{n-1}]$ does not depend on Φ and then the homotopy class $[f_{\mathcal{U},V}]$ in $[T, \mathbb{S}^{n-1}]$ is well define.

Notations 4.1. Denote the homotopy class $[f_{\mathcal{U},V}]$ in $[T, \mathbb{S}^{n-1}]$ by $\mu(\mathcal{U}, V)$.

Note that for the case $V = \mathrm{Vert}(\Delta^n)$ we have $\mu(\mathcal{U}, V) = \mu(\mathcal{U})$.

In [15, Lemma 2.4] we show that this invariant is well defined also for closed covers. As above we call a family of sets $\mathcal{S} = \{S_1, \dots, S_m\}$ as a cover of a space T if $\mathcal{S}$ is either an open or closed cover of T.

Definition 4.1. Let I be a set of labels of cardinality m. Let $V := \{v_i,\, i \in I\}$, be a set of points in $\mathbb{R}^n$. Then a nonempty subset $\mathcal{B} \subset I$ is said to be balanced with respect to V if for all $i \in \mathcal{B}$ there exist non-negative λ_i such that

$$\sum_{i \in \mathcal{B}} \lambda_i v_i = c_V, \quad \text{where} \quad \sum_{i \in \mathcal{B}} \lambda_i = 1 \ \text{ and } \ c_V := \frac{1}{m} \sum_{i \in I} v_i.$$

In other words, $c_V \in \mathrm{conv}\{v_i,\, i \in \mathcal{B}\}$, where $\mathrm{conv}(Y)$ denote the convex hull of Y in $\mathbb{R}^n$.

First we consider an extension of Theorems 2.4.

Theorem 4.1. *Let $V := \{v_1, \dots, v_m\}$ be a set of points in $\mathbb{R}^n$. Let A be a subspace of a space X. Let $(X, A) \in \mathrm{EP}_{n-1}$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a cover of (X, A). Suppose $\mathcal{S} := \mathcal{F}|_A$ is not null-homotopic. Then there is a balanced subset $\mathcal{B}$ in I_m with respect to V such that*

$$\bigcap_{i \in \mathcal{B}} F_i \neq \emptyset.$$

Proof. Assume the converse. Then there are no balanced subsets $\mathcal{B}$ in I_m such that $\{F_i,\, i \in \mathcal{B}\}$ have a common point. It implies that $c_V \notin \rho_{\mathcal{F},V}(X)$ and therefore, $f_{\mathcal{F},V} \colon X \to \mathbb{S}^{n-1}$ is well defined. On the other side, it is an extension of the map $f_{\mathcal{S},V} \colon A \to \mathbb{S}^{n-1}$ with $[f_{\mathcal{S},V}] \neq 0$, a contradiction. □

Remark. The assumption: "$c_V \in \rho_{\mathcal{S},V}(A)$ in $\mathbb{R}^n$" is equivalent to the assumption: "there is a balanced $\mathcal{B}$ in I_m with respect to V such that the intersection of all S_i, $i \in \mathcal{B}$, is not empty." Thus, if $c_V \in \rho_{\mathcal{S},V}(A)$ or $c_V \notin \rho_{\mathcal{S},V}(A)$ and $\mu(\mathcal{S}, V) \neq 0$, then the intersection of the F_i, $i \in \mathcal{B}$, is not empty.

Theorem 4.1 and [15, Theorem 2.3] imply the following extension of Corollary 2.1.

Corollary 4.1. *Let $V := \{v_1, \dots, v_m\}$ be a set of points in $\mathbb{R}^n$. Let $\mathcal{F} = \{F_1, \dots, F_m\}$ be a cover of $\mathbb{B}^k$ that is not null-homotopic on the boundary. Then there is a balanced with respect to V subset $\mathcal{B}$ in $I_m := \{1, \dots, m\}$ such that the intersection of all F_i, $i \in \mathcal{B}$, is not empty.*

If $V = \mathrm{Vert}(\Delta^n)$, then this corollary implies the KKM theorem. It is also implies the KKMS theorem.

Corollary 4.2 (KKMS theorem [17]). *Let $\mathcal{K}$ be the collection of all non-empty subsets of I_{k+1}. Consider a simplex S in $\mathbb{R}^k$ with vertices $x_1, \dots, x_{k+1}$. Let $V := \{v_\sigma,\, \sigma \in \mathcal{K}\} \subset \mathbb{R}^k$, where v_σ denotes the center of mass of $S_\sigma := \{x_i,\, i \in \sigma\}$.*

Let $\mathcal{C} := \{C_\sigma,\, \sigma \in \mathcal{K}\}$ be a cover of $|\Delta^k|$ such that for every $J \subset I_{k+1}$ the simplex Δ_J that is spanned by vertices from J is covered by $\{C_\sigma,\, \sigma \in J\}$. Then there exists a balanced collection $\mathcal{B}$ in $\mathcal{K}$ with respect to V such that

$$\bigcap_{\sigma \in \mathcal{B}} C_\sigma \neq \emptyset.$$

Proof. The assumptions of the corollary imply that $\mu(\mathcal{C}, V) = \deg(f_{\mathcal{C},V}) = 1$. Thus, Corollary 4.1 yields the corollary. □

There are many extensions of the Sperner and KKM lemmas, in particular, that are Tucker's and Ky Fan's lemmas [2,4,12–15]. Actually, these lemmas can be derived from Theorem 4.1 using certain sets V. Let us consider as an example a generalization of Tucker's lemma.

Corollary 4.3. *Let $V := \{\pm e_1, \dots, \pm e_n\}$, where $e_1, \dots, e_n$ is a basis in $\mathbb{R}^n$. Let $\mathcal{F} = \{F_1, F_{-1} \dots, F_n, F_{-n}\}$ be a cover of $\mathbb{B}^k$ that is not null-homotopic on the boundary. Then there is i such that the intersection of F_i and F_{-i} is not empty.*

In particular, if $\mathcal{F}$ is antipodally symmetric on the boundary of $\mathbb{B}^k$, i.e. for all i we have $S_{-i} = -S_i$, where $S_j := F_j|_{\mathbb{S}^{k-1}}$, then there is i such that $F_i \cap F_{-i} \neq \emptyset$.

Proof. Note that any balanced subset with respect to V consists of pairs $(i, -i)$, $i = 1, \ldots, n$. It yields the first part of the theorem.

By assumption, $\mathcal{S}$ is antipodally symmetric on $\mathbb{S}^{k-1}$. Then $\rho_{\mathcal{S},V} \colon \mathbb{S}^{k-1} \to \mathbb{R}^n$ is an odd (antipodal) map. If $k > n$, then the Borsuk-Ulam theorem implies there $x \in \mathbb{S}^{k-1}$ such that $\rho_{\mathcal{S},V}(x) = 0$. Therefore, there is i such that $S_i \cap S_{-i} \neq \emptyset$.

If for all i we have $S_i \cap S_{-i} = \emptyset$, then $k \leq n$. Then the odd mapping theorem (see [14]) implies that $k = n$ and $\deg(f_{\mathcal{S},V})$ is odd. Thus, $\mu(\mathcal{S}, V) \neq 0$ and from Theorem 4.1 follows the second part of the corollary. □

Definition 4.2. Let $V := \{v_1, \ldots, v_m\}$ be a set of points in $\mathbb{R}^n$. Let T be a triangulation of an n-dimensional manifold M. Let $L \colon \mathrm{Vert}(T) \to I_m$ be a labeling. We say that a simplex $s \in T$ is BL (Balanced Labelled) if the set of labels $L(s) := \{L(v), v \in \mathrm{Vert}(s)\}$ is balanced with respect to V

[15, Theorems 3.3 and 3.5] yield the following theorem.

Theorem 4.2. *Let $V := \{v_1, \ldots, v_m\}$ be a set of points in $\mathbb{R}^n$. Let T be a triangulation of an oriented manifold M of dimension n with boundary. Let $L \colon \mathrm{Vert}(T) \to \{1, 2, \ldots, m\}$ be a labeling such that T has no BL-simplices on the boundary. Then T must contain at least $|\deg(L, \partial T)|$ distinct (internal) BL-simplices.*

Corollary 4.4. *Let T be a triangulation of a compact oriented PL-manifold M of dimension $(m-1)$ with boundary. Then for any labeling $L \colon \mathrm{Vert}(T) \to \{1, 2, \ldots, m\}$ the triangulation T must contain at least $|\deg(L, \partial T)|$ fully colored $(m-1)$-simplices.*

Proof. Let $V := \{v_1, \ldots, v_m\}$ be the set of vertices of an $(m-1)$-simplex in $\mathbb{R}^{m-1}$. Then I_m is the only balanced subset in I_m with respect to V. Thus, Theorem 4.2 yields the corollary. □

Corollary 4.5. *Let T be a triangulation of $\mathbb{B}^n$ that antipodally symmetric on the boundary. Let $L \colon \mathrm{Vert}(T) \to \{+1, -1, \ldots, +n, -n\}$ be a labelling that is antipodal on the boundary. Suppose there are no complementary edges on the boundary. Then $\deg(L, \partial T)$ is an odd integer and there are at least $|\deg(L, \partial T)|$ internal complementary edges.*

(An edge in T is called complementary if its two vertices are labelled by opposite numbers.)

Proof. Let $V := \{\pm e_1, \ldots, \pm e_n\}$, where $e_1, \ldots, e_n$ is an orthonormal basis in $\mathbb{R}^n$. Then any balanced subset with respect to V consists of pairs $(i, -i)$, $i = 1, \ldots, n$. The fact that $\deg(L, \partial T)$ is odd follows from the odd mapping theorem, see [12]. Thus, Theorem 4.2 completes the proof. □

In Fig.2 is shown a labeling with $\deg(L, \partial T) = 3$. Then Corollary 4.5 yield that there are at least three complimentary edges.

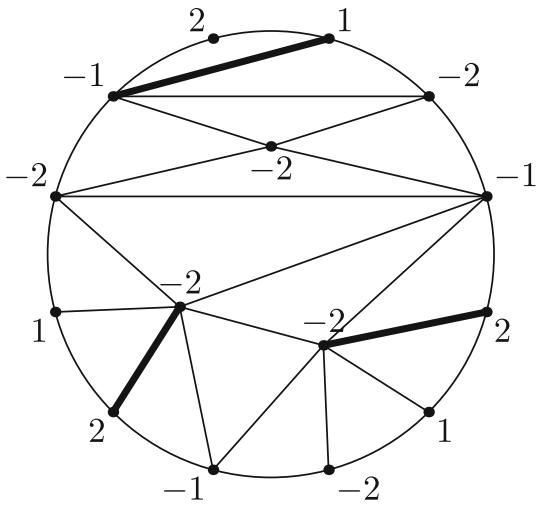

FIGURE 2. Since $\deg(L, \partial T) = 3$, there are three complementary edges

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