CHAPTER 23

NORMAL DISTRIBUTION

23.1 Introduction

In Chapter 5 [see Eq. (5.39)] we considered a sequence of random variables

$$W_n = \frac{X_n - np}{\sqrt{np(1-p)}}, \qquad n = 1, 2, \ldots,$$

where X_n has binomial $B(n,p)$ distribution and showed that the distribution of W_n converges, as $n \to \infty$, to a limiting distribution with characteristic function

$$f(t) = e^{-t^2/2}. \tag{23.1}$$

Similar result was also obtained for some sequences of negative binomial, Poisson, and gamma distributions [see Eqs. (7.30), (9.33), and (20.27)].

Since

$$\int_{-\infty}^{\infty} |f(t)|\, dt < \infty,$$

the corresponding limiting distribution has a density function $\varphi(x)$, which is given by the inverse Fourier transform

$$\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-itx} f(t)\, dt. \tag{23.2}$$

Since $f(t)$ in (23.1) is an even function, we can express $\varphi(x)$ in (23.2) as

$$\varphi(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{itx} e^{-t^2/2}\, dt. \tag{23.3}$$

Note that $\varphi(x)$ is differentiable and its first derivative is given by

$$\begin{aligned}
\varphi'(x) &= \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{itx} t e^{-t^2/2}\, dt \\
&= \frac{i}{2\pi} \int_{-\infty}^{\infty} e^{itx}\, d\left(-\, e^{-t^2/2}\right) \\
&= -\,\frac{x}{2\pi} \int_{-\infty}^{\infty} e^{itx} e^{-t^2/2}\, dt \\
&= -x\, \varphi(x).
\end{aligned} \tag{23.4}$$

$$
\begin{aligned}
\varphi(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2/2}\, dt \\
&= \frac{1}{\pi}\int_{0}^{\infty} e^{-t^2/2}\, dt \\
&= \frac{1}{\pi\sqrt{2}}\int_{0}^{\infty} e^{-u}u^{-1/2}\, du \\
&= \frac{1}{\pi\sqrt{2}}\Gamma\left(\frac{1}{2}\right) \\
&= \frac{1}{\sqrt{2\pi}}, \qquad (23.5)
\end{aligned}
$$

since $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$. Upon solving the differential equation in (23.4) using (23.5), we readily obtain

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}, \quad -\infty < x < \infty, \tag{23.6}$$

as the pdf corresponding to the characteristic function $f(t) = e^{-t^2/2}$, i.e.,

$$f(t) = e^{-t^2/2} = \int_{-\infty}^{\infty} e^{itx}\varphi(x)\, dx.$$

A book-length account of normal distributions, discussing in great detail their various properties and applications, is available [Patel and Read (1997)].

23.2 Notations

We say that a random variable X has the *standard normal distribution* if its pdf is as given in (23.6), and its cdf is given by

$$\Phi(x) = \int_{-\infty}^{x} \varphi(t)\, dt = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{x} e^{-t^2/2}\, dt. \tag{23.7}$$

The linear transformation $Y = a + \sigma X$ $(-\infty < a < \infty,\ \sigma > 0)$ generates a normal random variable with pdf

$$
\begin{aligned}
p_Y(x) &= p(a,\sigma,x) = \frac{1}{\sigma}\,\varphi\left(\frac{x-a}{\sigma}\right) \\
&= \frac{1}{\sigma\sqrt{2\pi}}\exp\left(-\frac{(x-a)^2}{2\sigma^2}\right), \quad -\infty < x < \infty, \qquad (23.8)
\end{aligned}
$$

and cdf

$$\Phi(a,\sigma,x) = \Phi\left(\frac{x-a}{\sigma}\right) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{(x-a)/\sigma} e^{-t^2/2}\, dt. \tag{23.9}$$

We also have from (23.3) that

$$
\begin{aligned}
\varphi(0) &= \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-t^2/2}\,dt \\
&= \frac{1}{\pi}\int_{0}^{\infty} e^{-t^2/2}\,dt \\
&= \frac{1}{\pi\sqrt{2}}\int_{0}^{\infty} e^{-u}u^{-1/2}\,du \\
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$$

parameter $\sigma > 0$. Shortly, we will show that a and σ^2 are, in fact, the mean and variance of Y, respectively. Then, $X \sim N(0, 1)$ will denote that X has the standard normal distribution with pdf and cdf as in (23.6) and (23.7), respectively.

The normal density function first appeared in the papers of de Moivre at the beginning of the eighteenth century as an auxiliary function that approximated binomial probabilities. Some decades later, the normal distribution was given by Gauss and Laplace in the theory of errors and the least squares method, respectively. For this reason, the normal distribution is also sometimes referred to as *Gaussian law*, *Gauss–Laplace distribution*, *Gaussian distribution*, and the *second law of Laplace*.

23.3 Mode

It is easy to see that normal $N(a, \sigma^2)$ distribution is unimodal. From (23.8), we see that

$$p'_Y(x) = -\frac{1}{\sigma^3}\sqrt{\frac{2}{\pi}}\,(x-a)\exp\left(-\frac{(x-a)^2}{2\sigma^2}\right),$$

which when equated to 0, yields the mode to be the location parameter a, and the maximal value of the pdf $p(a, \sigma, x)$ is then readily obtained from (23.8) to be $1/(\sigma\sqrt{2\pi})$.

23.4 Entropy

The entropy of a normal distribution possesses an interesting property.

Exercise 23.1 Let $Y \sim N(a, \sigma^2)$. Show that its entropy $H(Y)$ is given by

$$H(Y) = \frac{1}{2} + \log(\sigma\sqrt{2\pi}). \tag{23.10}$$

It is of interest to mention here that among all distributions with fixed mean a and variance σ^2, the maximal value of the entropy is attained for the normal $N(a, \sigma^2)$ distribution.

auxiliary function – вспомогательная

[illegible] [illegible]

In the sequel, we will use the notation $Y \sim N(a, \sigma^2)$ to denote a random variable Y having the normal distribution with location parameter a and scale parameter $\sigma > 0$. Shortly, we will show that a and σ^2 are, in fact, the mean and variance of Y, respectively. Then, $X \sim N(0,1)$ will denote that X has the standard normal distribution with pdf and cdf as in (23.6) and (23.7), respectively.

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23.5 Tail Behavior

The normal distribution function has light tails. Let $X \sim N(0,1)$. From Table 23.1, which presents values of the standard normal distribution function $\Phi(x)$, we have

$$\begin{aligned}
P\{|\xi| > 1\} &= \Phi(-1) + 1 - \Phi(1) = 2\{1 - \Phi(1)\} = 0.3173\ldots, \\
P\{|\xi| > 2\} &= 2\{1 - \Phi(2)\} = 0.0455\ldots, \\
P\{|\xi| > 3\} &= 2\{1 - \Phi(3)\} = 0.0027\ldots, \\
P\{|\xi| > 4\} &= 2\{1 - \Phi(4)\} = 0.000063\ldots.
\end{aligned}$$

It is easy to obtain that for any $x > 0$,

$$\begin{aligned}
1 - \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt \\
&\leq \frac{1}{x\sqrt{2\pi}} \int_x^\infty t e^{-t^2/2}\, dt \\
&= \frac{1}{x\sqrt{2\pi}} \int_x^\infty d\left(-\, e^{-t^2/2}\right) \\
&= \frac{1}{x\sqrt{2\pi}}\, e^{-x^2/2}. \qquad (23.11)
\end{aligned}$$

Similarly, for any $x > 0$,

$$\begin{aligned}
\frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{1}{t^2}\, e^{-t^2/2}\, dt &\leq \frac{1}{x^2\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt \\
&\leq \frac{1}{x^3\sqrt{2\pi}}\, e^{-x^2/2}. \qquad (23.12)
\end{aligned}$$

Using (23.12), we get the following lower bound for the tail of the normal distribution function:

$$\begin{aligned}
1 - \Phi(x) &= \frac{1}{\sqrt{2\pi}} \int_x^\infty e^{-t^2/2}\, dt \\
&= \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{1}{t}\, d\left(-\, e^{-t^2/2}\right) \\
&= \frac{1}{x\sqrt{2\pi}}\, e^{-x^2/2} - \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{1}{t^2}\, e^{-t^2/2}\, dt \\
&\geq \frac{1}{x\sqrt{2\pi}}\, e^{-x^2/2} - \frac{1}{x^3\sqrt{2\pi}}\, e^{-x^2/2} \\
&= \left(\frac{1}{x} - \frac{1}{x^3}\right) \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}. \qquad (23.13)
\end{aligned}$$

0.00	0.5000	0.88	0.8106	1.76	0.9608	2.64	0.9959
0.02	0.5080	0.90	0.8159	1.78	0.9625	2.66	0.9961
0.04	0.5160	0.92	0.8212	1.80	0.9641	2.68	0.9963
0.06	0.5239	0.94	0.8264	1.82	0.9656	2.70	0.9965
0.08	0.5319	0.96	0.8315	1.84	0.9671	2.72	0.9967
0.10	0.5398	0.98	0.8365	1.86	0.9686	2.74	0.9969
0.12	0.5478	1.00	0.8413	1.88	0.9699	2.76	0.9971
0.14	0.5557	1.02	0.8461	1.90	0.9713	2.78	0.9973
0.16	0.5636	1.04	0.8508	1.92	0.9726	2.80	0.9974
0.18	0.5714	1.06	0.8554	1.94	0.9738	2.82	0.9976
0.20	0.5793	1.08	0.8599	1.96	0.9750	2.84	0.9977
0.22	0.5871	1.10	0.8643	1.98	0.9761	2.86	0.9979
0.24	0.5948	1.12	0.8686	2.00	0.9772	2.88	0.9980
0.26	0.6026	1.14	0.8729	2.02	0.9783	2.90	0.9981
0.28	0.6103	1.16	0.8770	2.04	0.9793	2.92	0.9982
0.30	0.6179	1.18	0.8810	2.06	0.9803	2.94	0.9984
0.32	0.6255	1.20	0.8849	2.08	0.9812	2.96	0.9985
0.34	0.6331	1.22	0.8888	2.10	0.9821	2.98	0.9986
0.36	0.6406	1.24	0.8925	2.12	0.9830	3.00	0.9987
0.38	0.6480	1.26	0.8962	2.14	0.9838	3.02	0.9987
0.40	0.6554	1.28	0.8997	2.16	0.9846	3.04	0.9988
0.42	0.6628	1.30	0.9032	2.18	0.9854	3.06	0.9989
0.44	0.6700	1.32	0.9066	2.20	0.9861	3.08	0.9990
0.46	0.6772	1.34	0.9099	2.22	0.9868	3.10	0.9990
0.48	0.6844	1.36	0.9131	2.24	0.9875	3.12	0.9991
0.50	0.6915	1.38	0.9162	2.26	0.9881	3.14	0.9992
0.52	0.6985	1.40	0.9192	2.28	0.9887	3.16	0.9992
0.54	0.7054	1.42	0.9222	2.30	0.9893	3.18	0.9993
0.56	0.7123	1.44	0.9251	2.32	0.9898	3.20	0.9993
0.58	0.7190	1.46	0.9278	2.34	0.9904	3.22	0.9994
0.60	0.7257	1.48	0.9306	2.36	0.9909	3.24	0.9994
0.62	0.7324	1.50	0.9332	2.38	0.9913	3.26	0.9994
0.64	0.7389	1.52	0.9357	2.40	0.9918	3.28	0.9995
0.66	0.7454	1.54	0.9382	2.42	0.9922	3.30	0.9995
0.68	0.7517	1.56	0.9406	2.44	0.9927	3.32	0.9995
0.70	0.7580	1.58	0.9429	2.46	0.9931	3.34	0.9996
0.72	0.7642	1.60	0.9452	2.48	0.9934	3.36	0.9996
0.74	0.7704	1.62	0.9474	2.50	0.9938	3.38	0.9996
0.76	0.7764	1.64	0.9495	2.52	0.9941	3.40	0.9997
0.78	0.7823	1.66	0.9515	2.54	0.9945	3.42	0.9997
0.80	0.7881	1.68	0.9535	2.56	0.9948	3.44	0.9997
0.82	0.7939	1.70	0.9554	2.58	0.9951	3.46	0.9997
0.84	0.7995	1.72	0.9573	2.60	0.9953	3.48	0.9997
0.86	0.8051	1.74	0.9591	2.62	0.9956	3.50	0.9998

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Table 23.1: Standard Normal Cumulative Probabilities

which are valid for any positive x. It readily follows from (23.14) that

$$1 - \Phi(x) \sim \frac{1}{x}\,\varphi(x) \tag{23.15}$$

as $x \to \infty$, where

$$\varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2}$$

is the pdf of the standard normal distribution [see (23.6)].

23.6 Characteristic Function

From (23.1), we have the characteristic function of the standard normal $N(0,1)$ distribution to be

$$f(t) = e^{-t^2/2}.$$

If Y has the general $N(a, \sigma^2)$ distribution, we can use the relation $Y = a + \sigma X$, where $X \sim N(0,1)$, in order to obtain the characteristic function of Y as

$$f_Y(t) = Ee^{itY} = e^{iat} f(\sigma t) = \exp\left\{ iat - \frac{\sigma^2 t^2}{2} \right\}. \tag{23.16}$$

We see that $f_Y(t)$ in (23.16) has the form $\exp\{P_2(t)\}$, where $P_2(t)$ is a polynomial of degree two. Marcinkiewicz (1939) has shown that if a characteristic function $g(t)$ is expressed as $\exp\{P_n(t)\}$, where $P_n(t)$ is a polynomial of degree n, then there are only the following two possibilities:

(a) $n = 1$, in which case $g(t) = e^{iat}$ (degenerate distribution);

(b) $n = 2$, in which case $g(t) = \exp\{iat - \sigma^2 t^2/2\}$ (normal distribution).

In Chapter 12 we presented the definition of stable characteristic functions and stable distributions [see Eq. (12.10)].

Exercise 23.2 Prove that the characteristic function in (23.16) is stable, and hence all normal distributions are stable distributions.

Exercise 23.3 Find the characteristic function of XY when X and Y are independent standard normal random variables.

Exercise 23.4 Let X_1, X_2, X_3, and X_4 be independent standard normal random variables. Then, show that the random variable $Z = X_1X_2 + X_3X_4$ has

The exponentially decreasing nature of the pdf in (23.8) entails the existence of all the moments of the normal distribution.

Moments about zero:

Let $X \sim N(0,1)$. Then, it is easy to see that the pdf $\varphi(x)$ in (23.6) is symmetric about 0, and hence

$$\alpha_{2n+1} = EX^{2n+1} = 0 \tag{23.17}$$

for $n = 1, 2, \ldots$. In particular, we have from (23.17) that

$$\alpha_1 = EX = 0 \tag{23.18}$$

and

$$\alpha_3 = EX^3 = 0. \tag{23.19}$$

Next, the moments of even order are obtained as follows:

$$\begin{aligned}
\alpha_{2n} = EX^{2n} &= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} x^{2n} e^{-x^2/2}\, dx \\
&= \sqrt{\frac{2}{\pi}} \int_0^{\infty} x^{2n} e^{-x^2/2}\, dx \\
&= \sqrt{\frac{2}{\pi}}\, 2^{n-\frac{1}{2}} \int_0^{\infty} e^{-u} u^{n-\frac{1}{2}}\, du \\
&= \frac{2^n}{\sqrt{\pi}} \Gamma\left(n + \frac{1}{2}\right) \\
&= \frac{2^n}{\sqrt{\pi}} \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) \cdots \frac{1}{2} \Gamma\left(\frac{1}{2}\right) \\
&= 2^n \left(n - \frac{1}{2}\right)\left(n - \frac{3}{2}\right) \cdots \frac{1}{2} \\
&= \frac{(2n)!}{2^n\, n!}, \qquad n = 1, 2, \ldots.
\end{aligned} \tag{23.20}$$

In particular, we obtain from (23.20) that

$$\alpha_2 = EX^2 = 1 \tag{23.21}$$

and

$$\alpha_4 = EX^4 = 3. \tag{23.22}$$

In general, if $Y = a + \sigma X \sim N(a, \sigma^2)$, we can obtain its moments using the formula

$$EY^n = E(a + \sigma X)^n = \sum_{r=0}^{n} \binom{n}{r} a^r \sigma^{n-r} EX^{n-r}, \qquad n = 1, 2, \ldots, \tag{23.23}$$

which immediately yields

$$EY = a \tag{23.24}$$

and

$$EY^2 = a^2 + \sigma^2. \tag{23.25}$$

Central moments:

Let $X \sim N(0,1)$, $V \sim N(0,\sigma^2)$, and $Y \sim N(a,\sigma^2)$. Then, we have the following relations between central moments of these random variables:

$$\begin{aligned} E(Y-EY)^n &= E(V-EV)^n = \sigma^n E(X-EX)^n \\ &= \sigma^n EX^n = \sigma^n \alpha_n, \qquad n = 1,2,\ldots, \end{aligned} \tag{23.26}$$

where α_n are as given in (23.17) and (23.20). In particular, we have

$$\text{Var } Y = \sigma^2 \text{ Var } X = \sigma^2 \alpha_2 = \sigma^2. \tag{23.27}$$

We have thus shown that the location parameter a and the scale parameter σ^2 of normal $N(a,\sigma^2)$ distribution are simply the mean and variance of the distribution, respectively.

Cumulants:

In situations when the logarithm of a characteristic function is simpler to deal with than the characteristic function itself, it is convenient to use the *cumulants*. If $f(t)$ is the characteristic function of the random variable X, then the cumulant γ_k of degree k is defined as follows:

$$\gamma_k = \frac{1}{i^k}\left[\frac{d^k}{dt^k}\log f(t)\right]_{t=0}, \qquad k = 1,2,\ldots. \tag{23.28}$$

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In particular, we have

$$\gamma_1 = EX, \tag{23.29}$$
$$\gamma_2 = \text{Var } X, \tag{23.30}$$
$$\gamma_3 = E(X-EX)^3. \tag{23.31}$$

If the moment EX^k exists, then all cumulants $\gamma_1, \gamma_2, \ldots, \gamma_k$ also exist.

Let us now consider $Y \sim N(a,\sigma^2)$. Since its characteristic function $f_Y(t)$ has the form

$$f_Y(t) = \exp\left\{iat - \frac{\sigma^2 t^2}{2}\right\},$$

we have

$$\log f_Y(t) = iat - \frac{\sigma^2 t^2}{2}.$$

Hence, we find $\gamma_1 = a$, $\gamma_2 = \sigma^2$, and $\gamma_k = 0$ for $k = 3, 4$

$$E(Y - EY)^3 = \sigma^3 \alpha_3 = 0, \tag{23.32}$$

from which we immediately obtain the Pearson coefficient of skewness to be 0.

Next, we have from (23.26) and (23.22) that

$$E(Y - EY)^4 = \sigma^4 \alpha_4 = 3\sigma^4, \tag{23.33}$$

from which we immediately obtain the Pearson coefficient of kurtosis to be 3. Thus, we have the normal distributions to be symmetric, unimodal, bell-shaped, and mesokurtic distributions. Plots of normal density function presented in Figure 23.1 (for $a = 0$ and different values of σ) reveal these properties.

Remark 23.1 Recalling now that (see Section 1.4) distributions with coefficient of kurtosis smaller than 3 are classified as platykurtic (light-tailed) and those with larger than 3 are classified as leptokurtic (heavy-tailed), we simply realize that a distribution is considered to be light-tailed or heavy-tailed relative to the normal distribution.

23.9 Convolutions and Decompositions

In order to find the distribution of the sum $Y = Y_1 + Y_2$ of two independent random variables $Y_k \sim N(a_k, \sigma_k^2)$, $k = 1, 2$, we must recall from (23.16) that characteristic functions $f_k(t)$ of these random variables have the form

$$f_k(t) = \exp\left\{ia_k t - \frac{\sigma_k^2 t^2}{2}\right\}, \qquad k = 1, 2,$$

and hence

$$f_Y(t) = Ee^{itY} = f_1(t) f_2(t) = \exp\left\{iat - \frac{\sigma^2 t^2}{2}\right\},$$

where $a = a_1 + a_2$ and $\sigma^2 = \sigma_1^2 + \sigma_2^2$. This immediately implies that Y has normal $N(a_1 + a_2, \sigma_1^2 + \sigma_2^2)$ distribution. Of course, a more general result is also valid.

Exercise 23.5 For any $n = 1, 2, \ldots$, the sum $Y_1 + Y_2 + \cdots + Y_n$ of independent random variables $Y_k \sim N(a_k, \sigma_k^2)$, $k = 1, 2, \ldots, n$, has normal $N(a, \sigma^2)$ distribution with mean $a = a_1 + \cdots + a_n$ and variance $\sigma^2 = \sigma_1^2 + \cdots + \sigma_n^2$.

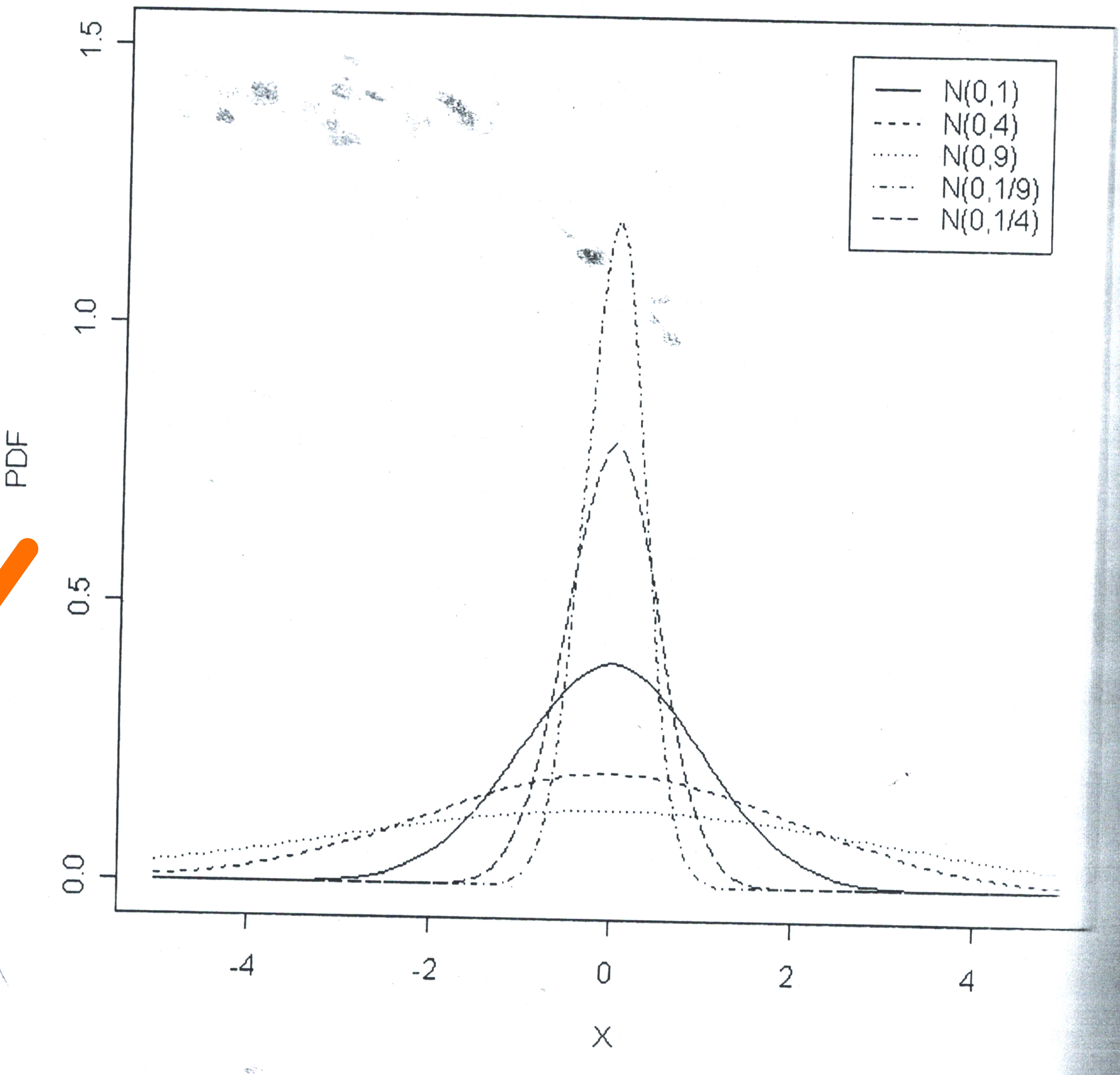

Figure 23.1. Plots of normal density function when $\sigma = 0$

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value. In fact, Cramér (1936) has proved that only "normal" decompositions are possible for normal distributions, viz., if $Y \sim N(a, \sigma^2)$, $-\infty < a < \infty$, $\sigma^2 > 0$, and $Y = Y_1 + Y_2$, where Y_1 and Y_2 are independent nondegenerate random variables, then there exist $-\infty < a_1 < \infty$ and $0 < \sigma_1^2 < \sigma^2$ such that $Y_1 \sim N(a_1, \sigma_1^2)$ and $Y_2 \sim N(a - a_1, \sigma^2 - \sigma_1^2)$. Thus, the family of normal distributions is closed with respect to the operations of convolution and decomposition. Recall that families of binomial and Poisson distributions also possess the same property.

Furthermore, the characteristic function of normal $N(a, \sigma^2)$ distribution which is

$$f(t) = \exp\left\{iat - \frac{\sigma^2 t^2}{2}\right\}$$

can be presented in the form

$$f(t) = \{f_n(t)\}^n,$$

where

$$f_n(t) = \exp\left\{i\left(\frac{a}{n}\right)t - \frac{\left(\sigma/\sqrt{n}\right)^2 t^2}{2}\right\}$$

is also the characteristic function of a normal distribution. Thus, we have established that any normal distribution is infinitely divisible.

23.10 Conditional Distributions

Consider independent random variables $X \sim N(0,1)$ and $Y \sim N(0,1)$. Then, $V = X + Y$ has normal $N(0,2)$ distribution. Probability density functions of X, Y, and V are given by

$$p_X(x) = p_Y(x) = \varphi(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2} \tag{23.34}$$

and

$$p_V(x) = \frac{1}{\sqrt{2}}\,\varphi\left(\frac{x}{\sqrt{2}}\right) = \frac{1}{2\sqrt{\pi}}\, e^{-x^2/4}, \tag{23.35}$$

from which we can find the conditional distribution of X given $V = v$. The conditional pdf $p_{X|V}(x|v)$ is given by

$$\begin{aligned} p_{X|V}(x|v) &= \frac{p_X(x)\, p_V(v-x)}{p_V(v)} \\ &= \sqrt{2}\,\varphi(x)\,\frac{\varphi(v-x)}{\varphi(v/\sqrt{2})} \\ &= \frac{1}{\sqrt{\pi}} \exp\left\{-\left(x - \frac{v}{2}\right)^2\right\} \\ &= \sqrt{2}\varphi\left(\sqrt{2}\left(x - \frac{v}{2}\right)\right). \end{aligned} \tag{23.36}$$

Observing that the RHS of (23.36) is the pdf of normal $N(v/2, 1/2)$ dis bution, we conclude that the conditional distribution of X, given $V =$ is normal with mean $v/2$ and variance $1/2$. A similar result is valid in general case too.

Exercise 23.6 Find the conditional pdf $p_{X|V}(x|v)$ for the case when X $N(a_1, \sigma_1^2)$, $Y \sim N(a_2, \sigma_2^2)$, with X and Y being independent, and $V = X +$

23.11 Independence of Linear Combinations

As noted earlier (see Exercise 23.3), the sum $\sum_{k=1}^n X_k$ of independent n mally distributed random variables $X_k \sim N(a_k, \sigma_k^2)$, $k = 1, 2, \ldots, n$, also h normal

$$N\left(\sum_{k=1}^n a_k, \sum_{k=1}^n \sigma_k^2\right)$$

distribution. The following general result can also be proved similarly.

Exercise 23.7 For any coefficients $b_1, \ldots, b_n$, prove that the linear combin tion

$$L = \sum_{k=1}^n b_k X_k$$

of independent normally distributed random variables $X_k \sim N(a_k, \sigma_k^2)$, k $1, 2, \ldots, n$, also has normal

$$N\left(\sum_{k=1}^n b_k a_k, \sum_{k=1}^n b_k^2 \sigma_k^2\right)$$

distribution.

Let us now consider two different linear combinations

$$L_1 = \sum_{k=1}^n b_k X_k \quad \text{and} \quad L_2 = \sum_{k=1}^n c_k X_k$$

Theorem 23.1 *Let X and Y be i.i.d. random variables, and let $L_1 = X +$ and $L_2 = X - Y$ also be independent. Then, X and Y have either degener or normal distribution.*

We will present here briefly the main arguments used to prove this theorer

(1) Indeed, the statement of the theorem is valid if X and Y have deger erate distribution. Hence, we will focus only on the nontrivial situatio wherein X and Y have a nondegenerate distribution.

(2) Without loss of generality, we can suppose that X and Y are symmetric random variables with a common nonnegative real characteristic function $f(t)$. This is so because if X and Y have any characteristic function $g(t)$, we can produce symmetric random variables $V = X - X_1$ and $U = Y - Y_1$, where X_1 and Y_1 have the same distribution as X and Y, respectively, and the random variables X, X_1, Y, and Y_1 are all independent. This symmetrization procedure gives us new independent random variables with the common characteristic function $f(t) = g(t)g(-t) = |g(t)|^2$, which is real and nonnegative. Due to the conditions of the theorem, $X + Y$ and $X - Y$ as well as $X_1 + Y_1$ and $X_1 - Y_1$ are independent, and so $X+U$ and $X-U$ are also independent. Thus, the random variables V and U with a common real nonnegative characteristic function satisfy the conditions of the theorem. Suppose now that we have proved that V and U are normally distributed random variables. Since V is the sum of the initial independent random variables X and Y, we can apply Cramér's result on decompositions of normal distributions stated earlier and obtain immediately that X and Y are also normally distributed random variables.

(3) Since X and Y have a common real characteristic function $f(t)$, we have the characteristic functions of L_1 and L_2 as

$$f_1(u) = Ee^{iuL_1} = Ee^{iuX}Ee^{iuY} = f^2(u) \qquad (23.4)$$

and

$$f_2(v) = Ee^{ivL_2} = Ee^{ivX}Ee^{-ivY} = f(v)f(-v) = f^2(v). \qquad (23.5)$$

We also have the joint characteristic function of L_1 and L_2 as

$$\begin{aligned} f(u,v) &= Ee^{iuL_1+ivL_2} \\ &= Ee^{iu(X+Y)+iv(X-Y)} \\ &= Ee^{i(u+v)X+i(u-v)Y} \\ &= Ee^{i(u+v)X}Ee^{i(u-v)Y} \end{aligned}$$

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In particular, we have 2.23

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$$f(2t) = f^4(t). \tag{23.47}$$

It follows immediately from (23.47) that $f(t) \neq 0$ for any t. Also, since $f(0) = 1$ and $f(t)$ is a continuous function, there is a value $a > 0$ such that $f(t) \neq 0$ if $|t| < a$. Then, $f(2t) \neq 0$ if $|t| < a$, which means that $f(t) \neq 0$ for all t in the interval $(-2a, 2a)$. Proceeding this way, for any $n = 1, 2, \ldots$, we obtain that $f(t) \neq 0$ if $|t| < 2^n a$, and hence the nonnegative function $f(t)$ must be strictly positive for any t. Now, from the equality [see (23.46) and (23.47)],

$$f(3t)f(t) = f^2(2t)f^2(t) = \{f^4(t)\}^2 f^2(t) = f^{10}(t),$$

we get the relation

$$f(3t) = f^9(t). \tag{23.48}$$

Following this procedure, we obtain

$$f(mt) = f^{m^2}(t) \tag{23.49}$$

which is true for any $m = 1, 2, \ldots$ and any t.

(5) The standard technique now allows us to get from (23.49) that

$$f\left(\frac{t}{n}\right) = \{f(t)\}^{1/n^2} \tag{23.50}$$

and

$$f\left(\frac{m}{n}\right) = c^{(m/n)^2} \tag{23.51}$$

for any integers m and n, where $c = f(1)$, $0 < c \leq 1$. Since $f(t)$ is a continuous function, (23.51) holds true for any positive t:

$$f(t) = c^{t^2}. \tag{23.52}$$

Since any real characteristic function is even, (23.52) holds for any t. If $c = 1$, we obtain $f(t) \equiv 1$ (the characteristic function of the degenerate distribution). If $0 < c < 1$, then

$$f(t) = e^{-\sigma^2 t^2/2},$$

where $\sigma^2 = -2 \log c > 0$, and the theorem is thus proved.

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$$
\begin{aligned}
f(t, t_1, t_2, \ldots, t_n) &= E \exp\left\{ it\bar{X} + i\sum_{k=1}^{n} t_k(X_k - \bar{X}) \right\} \\
&= E \exp\left\{ i\sum_{k=1}^{n} X_k \left(t_k + \frac{t - (t_1 + t_2 + \cdots + t_n)}{n} \right) \right\}.
\end{aligned}
\tag{23.55}
$$

Since X's in (23.55) are independent random variables having a common characteristic function $f(t) = e^{-t^2/2}$, we obtain

$$
\begin{aligned}
f(t, t_1, t_2, \ldots, t_n) &= \prod_{k=1}^{n} f\left(t_k + \frac{t - (t_1 + t_2 + \cdots + t_n)}{n} \right) \\
&= \prod_{k=1}^{n} \exp\left\{ -\frac{1}{2}\left(t_k + \frac{t - (t_1 + t_2 + \cdots + t_n)}{n} \right)^2 \right\} \\
&= \exp\left\{ -\frac{1}{2}\sum_{k=1}^{n}\left(t_k + \frac{t - (t_1 + t_2 + \cdots + t_n)}{n} \right)^2 \right\}.
\end{aligned}
\tag{23.56}
$$

We can rewrite (23.56) as

$$
f(t, t_1, t_2, \ldots, t_n) = g(t)h(t_1, t_2, \ldots, t_n), \tag{23.57}
$$

where

$$
g(t) = \exp\left\{ -\frac{t^2}{2n} \right\} \tag{23.58}
$$

and

$$
h(t_1, t_2, \ldots, t_n) = \exp\left\{ -\frac{1}{2}\sum_{k=1}^{n}\left(t_k - \frac{t_1 + t_2 + \cdots + t_n}{n} \right)^2 \right\}. \tag{23.59}
$$

Equation (23.57) then readily implies the independence of $\bar{X}$ and the random vector $(X_1 - \bar{X}, X_2 - \bar{X}, \ldots, X_n - \bar{X})$.

Furthermore, $\bar{X}$ and any random variable $T(X_1 - \bar{X}, \ldots, X_n - \bar{X})$ are also independent. For example, if $X_1, X_2, \ldots, X_n$ is a random sample from normal $N(a, \sigma^2)$ distribution, then we can conclude that the sample mean $\bar{X}$ and the sample variance

$$
S^2 = \frac{1}{n-1}\sum_{k=1}^{n}(X_k - \bar{X})^2 \tag{23.60}
$$

are independent. Note that the converse is also true: If $X_1, X_2, \ldots, X_n$ is a random sample from a distribution and that $\bar{X}$ and S^2 are independent, then X's are all normally distributed.

Since this is the joint density function of $n-1$ independent normal $N(0,\sigma^2)$ random variables, and that

$$\frac{1}{\sigma^2}\sum_{k=1}^{n-1} w_k^2 = \frac{1}{\sigma^2}\sum_{k=1}^{n} y_k^2 = \frac{1}{\sigma^2}\sum_{k=1}^{n}(x_k-\bar{x})^2 = \frac{(n-1)S^2}{\sigma^2},$$

Helmert concluded that the variable $(n-1)S^2/\sigma^2$ has a chi-square distribution with $n-1$ degrees of freedom. The elegant transformation above given is referred to in the literature as *Helmert's transformation.*

23.15 Identity of Distributions of Linear Combinations

Once again, let $X_k \sim N(a_k, \sigma_k^2)$, $k = 1, 2, \ldots, n$, be independent random variables, and

$$L_1 = \sum_{k=1}^{n} b_k X_k \quad \text{and} \quad L_2 = \sum_{k=1}^{n} c_k X_k.$$

It then follows from (23.39) and (23.40) that these linear combinations have the same distribution if

$$\sum_{k=1}^{n} b_k a_k = \sum_{k=1}^{n} c_k a_k \tag{23.61}$$

and

$$\sum_{k=1}^{n} b_k^2 \sigma_k^2 = \sum_{k=1}^{n} c_k^2 \sigma_k^2. \tag{23.62}$$

If X's have a common standard normal distribution, then the condition in (23.62) is equivalent to

$$\sum_{k=1}^{n} b_k^2 = \sum_{k=1}^{n} c_k^2. \tag{23.63}$$

For example, in this situation,

$$\frac{X_1 + \cdots + X_m}{\sqrt{m}} \stackrel{d}{=} \frac{X_1 + \cdots + X_n}{\sqrt{n}} \tag{23.64}$$

for any integers m and n and, in particular,

$$\frac{X_1 + \cdots + X_n}{\sqrt{n}} \stackrel{d}{=} X_1. \tag{23.65}$$

Pólya (1932) showed that if X_1 and X_2 are independent and identically distributed nondegenerate random variables having finite variances, then the equality in distribution of the random variables X_1 and $(X_1+X_2)/\sqrt{2}$ characterizes

$$\frac{1}{\sigma^2}\sum_{k=1}^{n-1} w_k^2 = \frac{1}{\sigma^2}\sum_{k=1}^{n} y_k^2 = \frac{1}{\sigma^2}\sum_{k=1}^{n}(x_k - \bar{x})^2 = \frac{(n-1)S^2}{\sigma^2},$$

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Pólya (1932) showed that if X_1 and X_2 are independent and identically distributed nondegenerate random variables having finite variances, then the equality in distribution of the random variables X_1 and $(X_1 + X_2)/\sqrt{2}$ characterizes the normal distribution. Marcinkiewicz (1939) later proved, under some restrictions on the coefficients b_k and c_k ($k = 1, \ldots, n$), that if $X_1, \ldots, X_n$ are independent random variables having a common distribution and finite moments of all order, then

$$\sum_{k=1}^{n} b_k X_k \stackrel{d}{=} \sum_{k=1}^{n} c_k X_k$$

implies that X's all have normal distribution.

23.16 Asymptotic Relations

As already seen in Chapters 5, 7, 9, and 20, the normal distribution arises naturally as a limiting distribution of some sequences of binomial, negative binomial, Poisson, and gamma distributed random variables. A careful look at these situations reveals that the normal distribution has appeared there as a limiting distribution for suitably normalized sums of independent random variables. There are several modifications to the *central limit theorem*, which provide (under different restrictions on the random variables $X_1, X_2, \ldots$) the convergence of sums

$$\frac{S_n - ES_n}{\sqrt{\mathrm{Var}\ S_n}},$$

where $S_n = X_1 + \cdots + X_n$, to the normal distribution. This is the reason why the normal distributions plays an important role in probability theory and mathematical statistics.

Changing the sum S_n by the maxima $M_n = \max\{X_1, X_2, \ldots, X_n\}$, $n = 1, 2, \ldots$, we get a different limiting scheme with the extreme value distributions (see Chapter 21) determining the asymptotic behavior of the normalized random variable M_n. It should be mentioned here that if $X_k \sim N(0, 1)$, $k = 1, 2, \ldots$, are independent random variables, then

$$P\left\{\frac{M_n - a_n}{b_n} < x\right\} \to e^{-e^{-x}} \tag{23.66}$$

as $n \to \infty$, where

$$a_n = \sqrt{2 \log n} - \frac{\log \log n + \log 4\pi}{2\sqrt{2 \log n}} \tag{23.67}$$

and

It is easy to see that

$$\tan 2\pi U \stackrel{d}{=} \tan \pi \left(U - \frac{1}{2}\right)$$

and so

$$Z \stackrel{d}{=} \tan \pi \left(U - \frac{1}{2}\right).$$

Comparing (23.78) with (12.5), we readily see that Z has the standard $C(0,1)$ distribution.

Next, let us consider the random variables

$$Y_1 = \frac{2VW}{\sqrt{V^2+W^2}} \quad \text{and} \quad Y_2 = \frac{W^2-V^2}{\sqrt{V^2+W^2}}. \qquad ($$

Shepp (1964) proved that Y_1 and Y_2 are independent random variables h standard normal distribution. This result can easily be checked by (23.74). We have

$$\begin{aligned}(Y_1, Y_2) &\stackrel{d}{=} \left(2\sqrt{2X}\sin 2\pi U \cos 2\pi U,\ 2\sqrt{2X}(\cos^2 2\pi U - \sin^2 2\pi U)\right) \\ &\stackrel{d}{=} \left(\sqrt{2X}\sin 4\pi U,\ \sqrt{2X}\cos 4\pi U\right). \qquad (23\end{aligned}$$

Taking into account that

$$(\sin 4\pi U,\ \cos 4\pi U) \stackrel{d}{=} (\sin 2\pi U,\ \cos 2\pi U)$$

and hence

$$(Y_1, Y_2) \stackrel{d}{=} \left(\sqrt{2X}\sin 2\pi U,\ \sqrt{2X}\cos 2\pi U\right),$$

we immediately obtain from (23.74) that

$$(Y_1, Y_2) \stackrel{d}{=} (V, W),$$

which proves Shepp's (1964) result.

Bansal et al. (1999) proved the following converse of Shepp's result. Let and W be independent and identically distributed random variables, and let Y_1 and Y_2 be as defined in (23.79). If there exist real a and b with $a^2+b^2=$ such that $aY_1 + bY_2$ has the standard normal distribution, then V and are standard normal $N(0,1)$ random variables. Now, summarizing the res given above, we have the following characterization of the normal distri "Let V and W be independent and identically distributed Then, $V \sim N(0,1)$ and W

$$\frac{1}{\sqrt{\pi\, x}}\, e^{-x}, \qquad x > 0.$$

Then, the squared standard normal variable V^2 has $\Gamma\left(\frac{1}{2}, 0, 2\right)$ with pdf

$$\frac{1}{\sqrt{2\pi x}}\, e^{-x/2}, \qquad x > 0.$$

If we now consider the sum

$$S_n = \sum_{k=1}^{n} V_k^2, \tag{23.81}$$

where $V_1, V_2, \ldots, V_n$ are i.i.d. normal $N(0,1)$ random variables, then the reproductive property of gamma distributions readily reveals that S_n has $\Gamma(n/2, 0, 2)$ distribution. In Chapter 20 we mentioned that this special case of $\Gamma(n/2, 0, 2)$ distributions, where n is an integer, is called a *chi-square* (χ^2) *distribution* with n degrees of freedom. Based on (20.24), the following result is then valid: For any $k = 1, 2, \ldots, n$, the random variables

$$T_{k,n} = \frac{V_k^2}{S_n}$$

and S_n are independent, and $T_{k,n}$ has the standard beta$\left(\frac{1}{2}, \frac{n-1}{2}\right)$ distribution. In particular, if $n = 2$,

$$T_{1,2} = \frac{V_1^2}{V_1^2 + V_2^2} \quad \text{and} \quad T_{2,2} = \frac{V_2^2}{V_1^2 + V_2^2}$$

have the standard arcsine distribution.

Exercise 23.8 Show that the quotient $(V_1^2 + V_2^2)/S_4$ has the standard uniform $U(0,1)$ distribution.

Exercise 23.9 Show that the pdf of

$$\chi_n = \sqrt{S_n} = \left(\sum_{k=1}^{n} V_k^2\right)^{1/2} \tag{23.82}$$

is given by

$$p_n(x) = \frac{1}{\Gamma(n/2)\; 2^{(n-2)/2}}\, x^{n-1} e^{-x^2/2}, \qquad x \geq 0. \tag{23.83}$$

The distribution with pdf (23.83) is called as *chi distribution* (χ distrib tion) with n degrees of freedom. Note that (23.83), when $n = 2$, correspon to the Rayleigh density in (23.69). The case when $n = 3$ with pdf

$$p_3(x) = \sqrt{\frac{2}{\pi}}\, x^2 e^{-x^2/2}, \qquad x > 0, \tag{23.84}$$

is called the *standard Maxwell distribution.*

Consider now the quotient

$$T = \frac{V}{\sqrt{\sum_{k=1}^{n} V_k^2/n}} = \frac{V}{\sqrt{S_n/n}}, \tag{23.85}$$

where $V, V_1, \ldots, V_n$ are all independent random variables having standard normal distribution. The numerator and denominator of (23.85) are independent and have normal distribution and chi distribution with n degrees of freedom, respectively.

Exercise 23.10 Show that the pdf of T is given by

$$p_T(t) = \frac{\Gamma\left(\frac{n+1}{2}\right)}{\sqrt{n\pi}\,\Gamma(n/2)}\left(1 + \frac{t^2}{n}\right)^{-(n+1)/2}, \qquad -\infty < t < \infty. \tag{23.86}$$

The distribution with pdf (23.86) is called *Student's t distribution with n degrees of freedom.* Note that Student's t distribution with one degree of freedom (i.e., case $n = 1$) is just the standard Cauchy distribution.

Exercise 23.11 As $n \to \infty$, show that the t density in (23.86) converges to the standard normal density function.

Now, let $Y = 1/V^2$ be the reciprocal of a squared standard normal $N(0, 1)$ random variable. Then, it can be shown that the pdf of Y is

$$p_Y(x) = \frac{1}{\sqrt{2\pi}}\, x^{-3/2} \exp\left\{-\frac{1}{2x}\right\}, \qquad x > 0. \tag{23.87}$$

The lognormal distribution is also sometimes called the *Cobb-Douglas distribution* in the economics literature.

Exercise 23.12 Using the relationship $X = (\log Y - m)/\sigma$, where X is a standard normal variable, show that the kth moment of Y is given by

$$E(Y^k) = E\left(e^{k(m+\sigma X)}\right) = e^{km+\frac{1}{2}k^2\sigma^2}. \tag{23.94}$$

Then, deduce that

$$EY = e^m \sqrt{\omega} \quad \text{and} \quad \operatorname{Var} Y = e^{2m}\omega(\omega - 1),$$

where $\omega = e^{\sigma^2}$.

Lognormal distributions possess many interesting properties and have also found important applications in diverse fields. A detailed account of these developments on lognormal distributions can be found in the books of Aitchison and Brown (1957) and Crow and Shimizu (1988).

Along the same lines, Johnson (1949) considered the following transformations:

Johnson distrib

$$X = a + b\log\left(\frac{Y}{1-Y}\right) \quad \text{and} \quad X = a + b\sinh^{-1} Y, \tag{23.95}$$

where X is once again distributed as standard normal. The distributions of Y in these two cases are called *Johnson's S_B and S_U distributions*, respectively. These distributions have been studied rather extensively in both the statistical and applied literature .

Exercise 23.13 By transformation of variables, derive the densities of Johnson's S_B and S_U distributions.

Exercise 23.14 For the S_U distribution, show that the mean and variance are

$$\sqrt{e^{1/b^2}}\sinh\left(\frac{a}{b}\right) \quad \text{and} \quad \frac{1}{2}\left(e^{1/b^2} - 1\right)\left\{e^{1/b^2}\cosh\left(\frac{2a}{b}\right) + 1\right\}, \tag{23.96}$$

in a simple analytical form: normal, Cauchy, and (23.87). Of course, Y as well as any other stable random variable has an infinitely divisible distribution. 233

We may observe that many distributions related to the normal distribution are infinitely divisible. However, in order to show that not all distributions related closely to normal are infinitely divisible or even decomposable, we will give the following example. It is easy to check that if a random variable X with a characteristic function $f(t)$ and a pdf $p(x)$ has a finite second moment α_2, then $f^*(t) = \dfrac{f^{(2)}(t)}{f^{(2)}(0)}$ is indeed a characteristic function which corresponds to the pdf

$$p^*(x) = \frac{x^2 p(x)}{\alpha_2}\,.$$

Now, let $X \sim N(0,1)$. Then,

$$f(t) = e^{-t^2/2} \quad \text{and} \quad p(x) = \frac{1}{\sqrt{2\pi}}\, e^{-x^2/2};$$

consequently,

$$f^*(t) = (1-t^2)e^{-t^2/2} \tag{23.89}$$

is the characteristic function of a random variable with pdf

$$p^*(x) = x^2 p(x) = \frac{1}{\sqrt{2\pi}}\, x^2 e^{-x^2/2}. \tag{23.90}$$

The characteristic function in (23.89) and hence the distribution in (23.90) are indecomposable.

Let $X \sim N(0,1)$ and $X = a + b\log Y$. Then, Y is said to have a *lognormal distribution* with parameters a and b. By a simple transformation of random variables, we find the pdf of Y as

$$p_Y(y) = \frac{b}{\sqrt{2\pi}\,y} \exp\left\{-\frac{1}{2}(a + b\log y)^2\right\}, \qquad y > 0. \tag{23.91}$$

We may take b to be positive without loss of any generality, since $-X$ has the same distribution as X. An alternative reparametrization is obtained by replacing the parameters a and b by the mean m and standard deviation σ of the random variable $\log Y$. Then, the two sets of parameters satisfy the relationships

$$m = -\frac{a}{b} \quad \text{and} \quad \sigma = \frac{1}{b}, \tag{23.92}$$

so that we have $X = (\log Y - m)/\sigma$, and the lognormal pdf under this reparametrization is given by

$$p_Y(y) = \frac{1}{\sqrt{2\pi}\,\sigma y} \exp\left\{-\frac{1}{2}\frac{(\log y - m)^2}{\sigma^2}\right\}, \qquad y > 0. \tag{23.93}$$