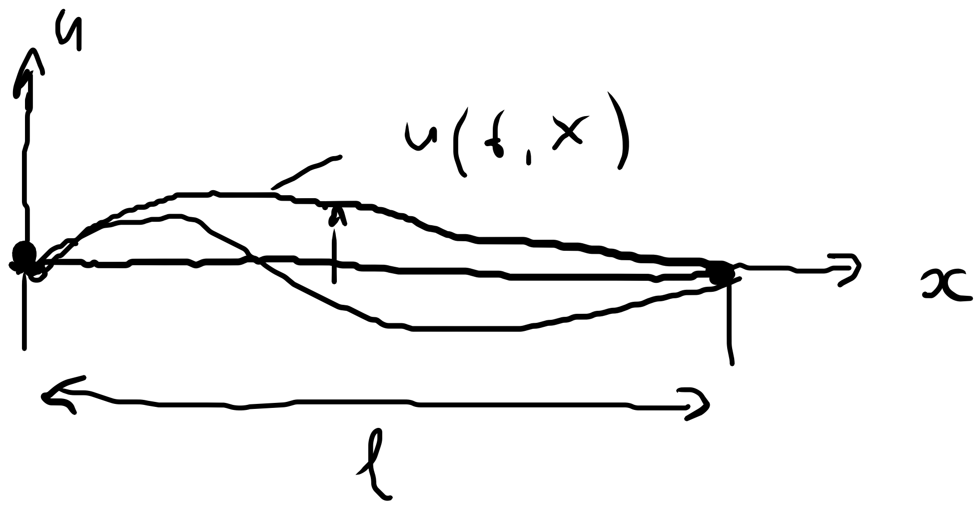
Колебания струны.

$\Box_a u := u_{tt} - a^2 u_{xx}$



D'Alembert

$$u = u(t, x)$$

Волновое уравнение (ур-е колебаний)

$$\begin{cases} u_{tt} - a^2 u_{xx} = 0. \\ u(t, 0) = u(t, l) = 0 \\ u(0, x) = u_0(x) \\ u_t(0, x) = v_0(x) \end{cases}$$

$t > 0, \quad x \in (0, l).$

жесткое закрепление

Для простоты $v_0 \equiv 0.$

Метод разделения переменных = Метод Фурье.

$$u_{tt} - a^2 u_{xx} = 0 \qquad u(t,0) = u(t,l) = 0$$

$$\cancel{T(t)}X(0) = \cancel{T(t)}X(l) = 0$$

$$\forall t > 0.$$

$$u(t,x) = T(t)X(x)$$

$$T_{tt}X - a^2 T X_{xx} = 0.$$

$$\frac{T_{tt}}{a^2 T}(t) = \frac{X_{xx}}{X}(x) = \lambda \qquad t > 0, \quad x \in (0,l)$$

$$T_{tt} - \lambda a^2 T = 0$$

$$\begin{cases} X_{xx} - \lambda X = 0 \\ X(0) = X(l) = 0 \end{cases}$$

(частн. сл. зад. Штурма – Лиувилля)

$$\begin{cases} \bar{X}_{xx} - \underbrace{\lambda}_{} \bar{X} = 0 \\ \bar{X}(0) = \bar{X}(l) = 0 \end{cases}$$

λ – с.з. соотв.

задача Штурма –

– Лиувилля

1). $\lambda = 0$ $\qquad X_{xx} = 0 \Rightarrow \bar{X}(x) = Ax + B$

$$\Downarrow$$

$$\bar{X} \equiv 0$$

2). $\lambda > 0$ $\qquad \bar{X}(x) = C_1 e^{\sqrt{\lambda}x} + C_2 e^{-\sqrt{\lambda}x}$

$= A\,\mathrm{sh}\sqrt{\lambda}x + \cancel{B\,\mathrm{ch}}\sqrt{\lambda}x.$

$\underset{\sim}{0} = X(0) = A\cdot 0 + B\cdot 1 = \underset{\sim}{B}$

$0 = X(l) = A\,\mathrm{sh}\sqrt{\lambda}l \Rightarrow \left[\begin{array}{l} A = 0 \\ \cancel{\mathrm{sh}\sqrt{\lambda}l} = 0. \end{array}\right.$

3). $\lambda < 0$.

$$X(x) = A \sin\sqrt{-\lambda}\, x + \cancel{B \cos\sqrt{-\lambda}\, x}.$$

$$0 = X(0) = A \cdot 0 + B \cdot 1 = B$$

$$0 = X(l) = A \sin\sqrt{-\lambda}\, l \Rightarrow$$

$A = 0$

$\sin\sqrt{-\lambda}\, l = 0$

$$\sqrt{-\lambda}\, l = k\pi, \quad k = 1, 2, 3, 4 \dots$$

$$\boxed{\lambda_k = -\frac{k^2\pi^2}{l^2}, \quad X_k(x) = A \sin\frac{k\pi x}{l}}$$

$$\lambda = -\frac{k^2\pi^2}{l^2}, \qquad k = 1, 2, 3, \dots$$

$$T_{tt} + \frac{k^2\pi^2 a^2}{l^2} T = 0.$$

$$T_k(t) = \alpha_k \sin\frac{k\pi a t}{l} + \beta_k \cos\frac{k\pi a t}{l}, \quad X_k = A_k \sin\frac{k\pi x}{l}$$

$$u_k(t, x) = \left(\alpha_k \sin\frac{k\pi a t}{l} + \beta_k \cos\frac{k\pi a t}{l}\right) A_k \sin\frac{k\pi x}{l} =$$

$$= \left(a_k \sin\frac{k\pi a t}{l} + b_k \cos\frac{k\pi a t}{l}\right) \sin\frac{k\pi x}{l}$$

$$u(t,x) = \sum_{k=1}^{\infty} \left(\alpha_k \sin\frac{k\pi a t}{l} + \beta_k \cos\frac{k\pi a t}{l}\right)\sin\frac{k\pi x}{l}$$

$$u_0(x) = u(0,x) = \sum_{k=1}^{\infty} \beta_k \sin\frac{k\pi x}{l}$$

ряд Фурье ф-ции u_0 по синусам.

$$0 = u_t(0,x) = \sum_{k=1}^{\infty} \frac{k\pi a}{l}\alpha_k \sin\frac{k\pi x}{l} = \frac{\pi a}{l}\sum_{k=1}^{\infty} k\alpha_k \sin\frac{k\pi x}{l}$$

ряд Фурье ф-ции 0 по синусам.

$$\alpha_k = 0, \quad \beta_k:$$

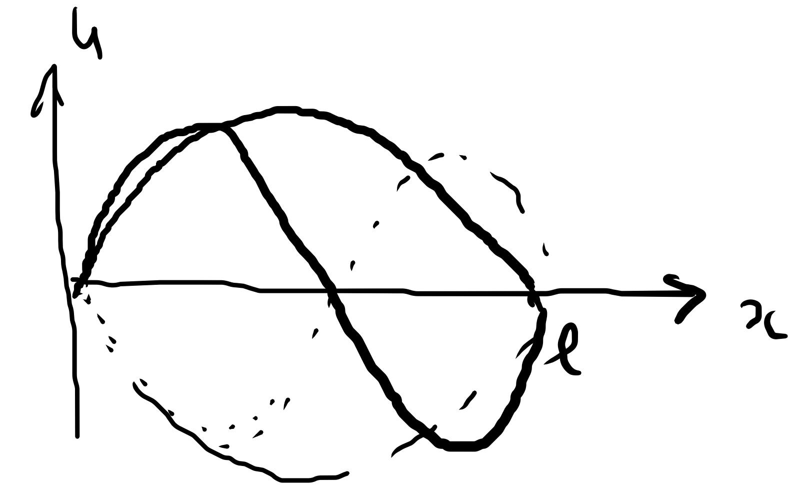
$$\int_0^l u_0(x)\sin\frac{k\pi x}{l}\,dx = \sum_{j=1}^{\infty} \beta_j \underbrace{\int_0^l \sin\frac{j\pi x}{l}\sin\frac{k\pi x}{l}\,dx}_{=0,\ k\neq j}$$

$$\int_0^l u_0(x)\sin\frac{k\pi x}{l}\,dx = \beta_k \int_0^l \sin^2\frac{k\pi x}{l}\,dx = \beta_k\frac{l}{2}$$

Итого:

$$\alpha_k = 0 \qquad \boxed{\beta_k = \frac{2}{l}\int_0^l u_0(x)\sin\frac{k\pi x}{l}\,dx}$$

$$u(t,x) = \sum_{k=1}^{\infty} \beta_k \underbrace{\cos\frac{k\pi a t}{l}\sin\frac{k\pi x}{l}}_{\text{гармоники}}$$



$k=1 \quad \nu_1 = a/l$

$k=2 \quad \nu_2 = 2a/l = 2\nu_1, \quad \nu_3 = 3\nu_1, \quad \nu_4 = 4\nu_1$