Задача 1.

$u \equiv u(x) \in \mathbb{R}$, $x \in \mathbb{R}$

$$(1)\quad \begin{cases} u' = \sqrt{u}\,(u-1) \\ u(x_0) = u_0 \end{cases}$$

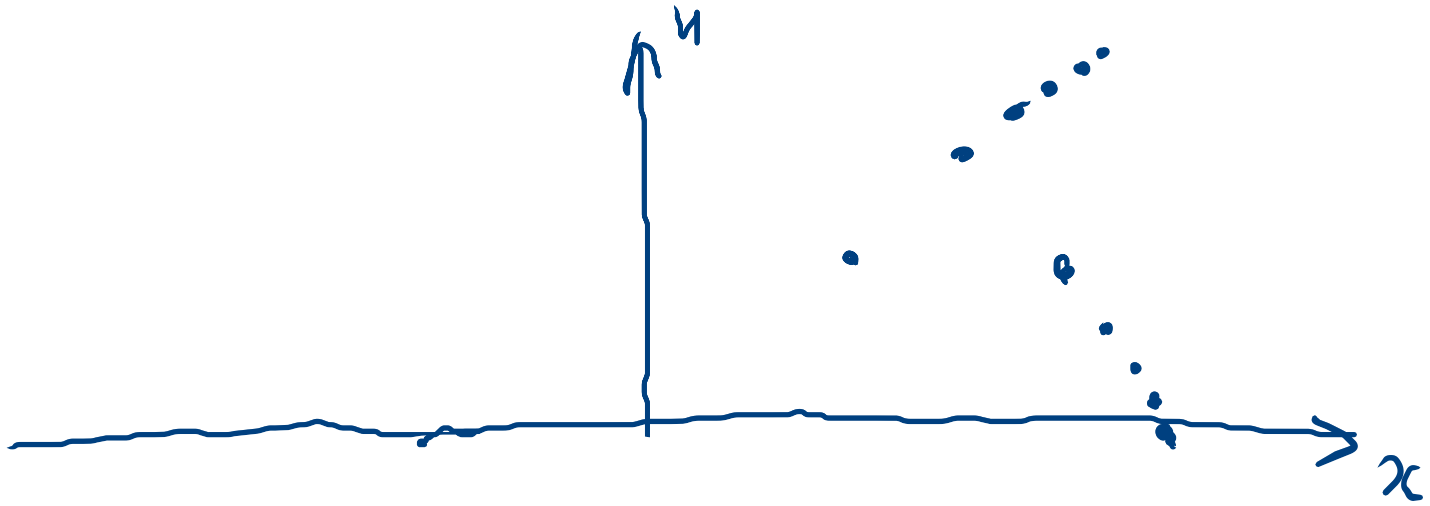
$f(x,u) = \sqrt{u}\,(u-1)$

$D = \mathbb{R} \times \{u \geq 0\}$ замкн

$z_n = (x_n, u_n) \to z = (x, u)$

$x_n \to x$

$u_n \to u$.



$x_0 \in \mathbb{R}$, $u_0 \geq 0$.

1) Стац. реш. (точки покоя)

$u \equiv C$. $\qquad u' = 0 = \sqrt{C}\,(C-1)$

$$\Downarrow \left[\begin{array}{l} C = 0 \\ C = 1. \end{array}\right.$$

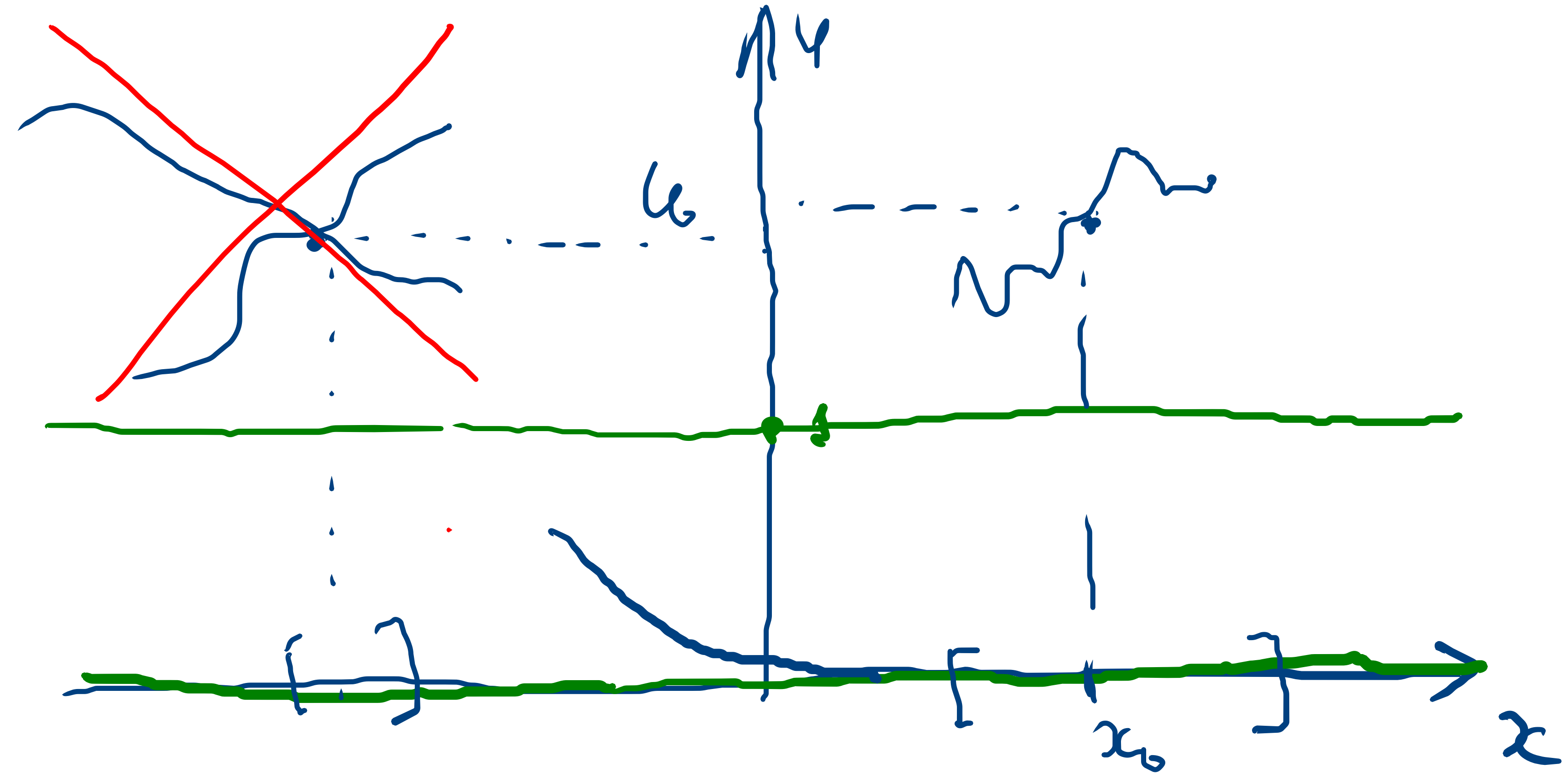
$u \equiv 0$, $u \equiv 1$.

2). $\exists!$ реш.

$$f(x,u) = \sqrt{u}\,(u-1)$$

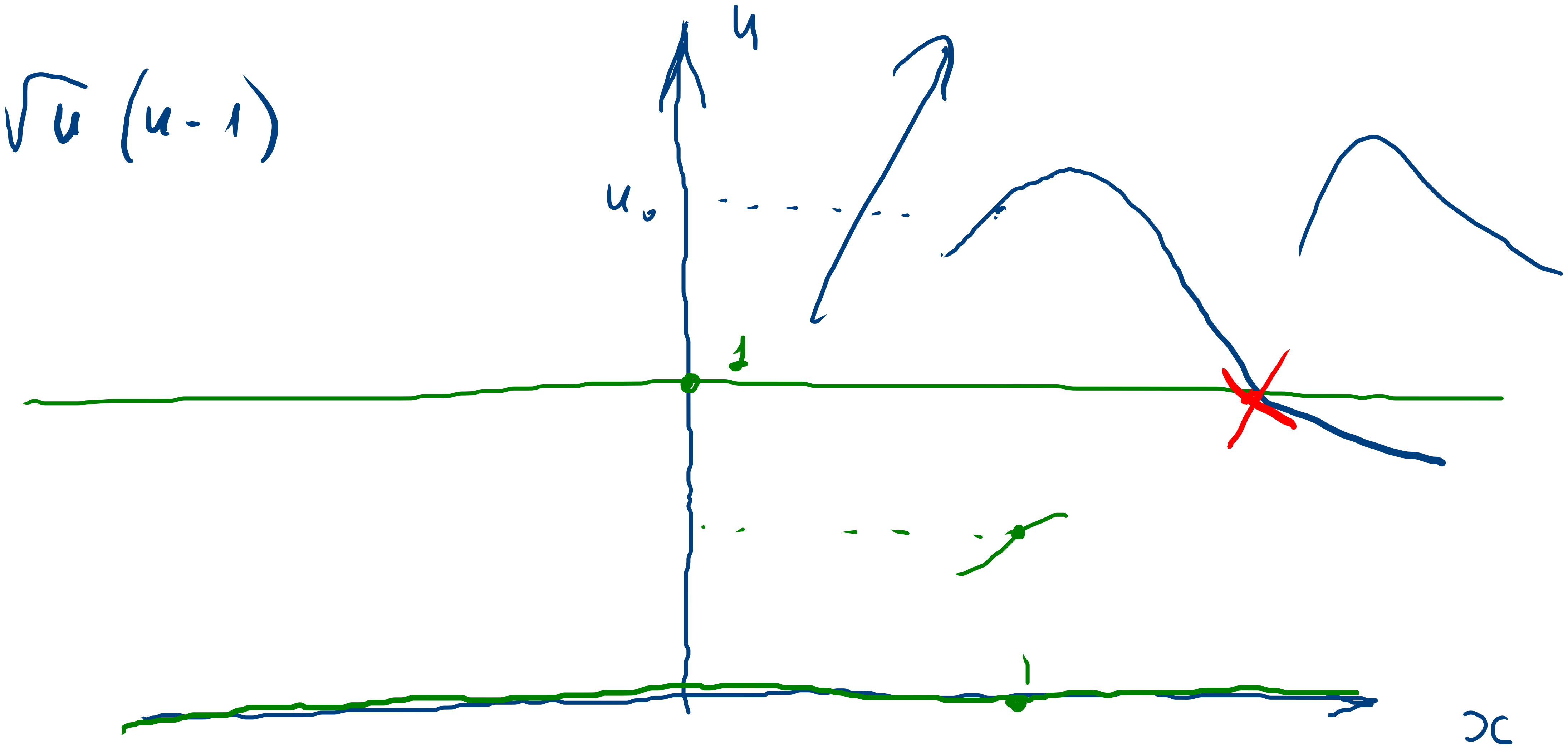
$$\frac{df}{du} = \frac{1}{2\sqrt{u}}(u-1) + \sqrt{u}.$$

непр. на $\{u > 0\}$.



$\Rightarrow \forall\,(x_0, u_0)$: $u_0 > 0$ задача Коши (1) может иметь не более одного реш. и локально реш. $\exists$.

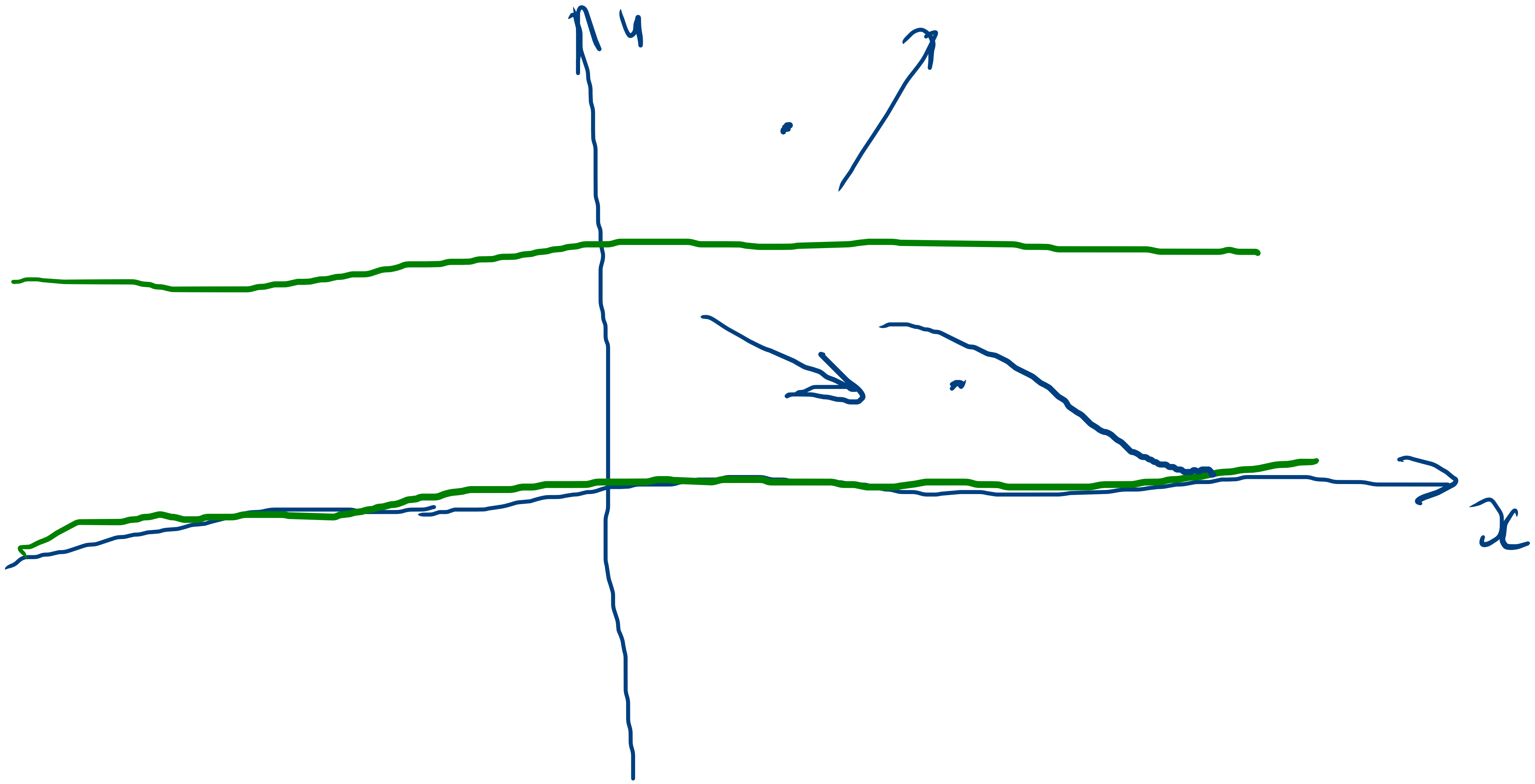
3). $u' = \sqrt{u}\,(u-1)$



3.1). $x_0 \in \mathbb{R},\ u_0 > 1 \Rightarrow u = u(0) > 1$.

$\Rightarrow \sqrt{u}\,(u-1) > 0 \Rightarrow u' > 0$.

$\Rightarrow u$ – возраст.



3.2) $0 < u_0 < 1 \Rightarrow 0 \leq u(x) < 1$

$f(x,u) = \sqrt{u}\,(u-1) < 0,$

пока $u > 0,$

3.3) $u_0 = 1 \Rightarrow u(x) \equiv 1.$

Ex. 3.1.) $u_0 > 1 \implies u > 1$, $u \uparrow$

$$u' = \sqrt{u}\,(u-1) \qquad \int_{u_0}^{u} \frac{dv}{\sqrt{v}\,(v-1)} = \int_{x_0}^{x} dx$$

$$\int_{u_0}^{u} \frac{dv}{\sqrt{v}\,(v-1)} = 2\int_{\sqrt{u_0}}^{\sqrt{u}} \frac{\cancel{s}\,ds}{\cancel{s}\,(s^2-1)} = \int_{\sqrt{u_0}}^{\sqrt{u}} \left(-\frac{1}{s+1} + \frac{1}{s-1}\right) ds =$$

$\sqrt{v} = s$

$$\boxed{v = s^2 \qquad dv = 2s\,ds}$$

$$= \ln\left|\frac{s-1}{s+1}\right|\Bigg|_{\sqrt{u_0}}^{\sqrt{u}} = \ln\left(\frac{\sqrt{u}-1}{\sqrt{u}+1}\right) - \ln\left(\frac{\sqrt{u_0}-1}{\sqrt{u_0}+1}\right)$$

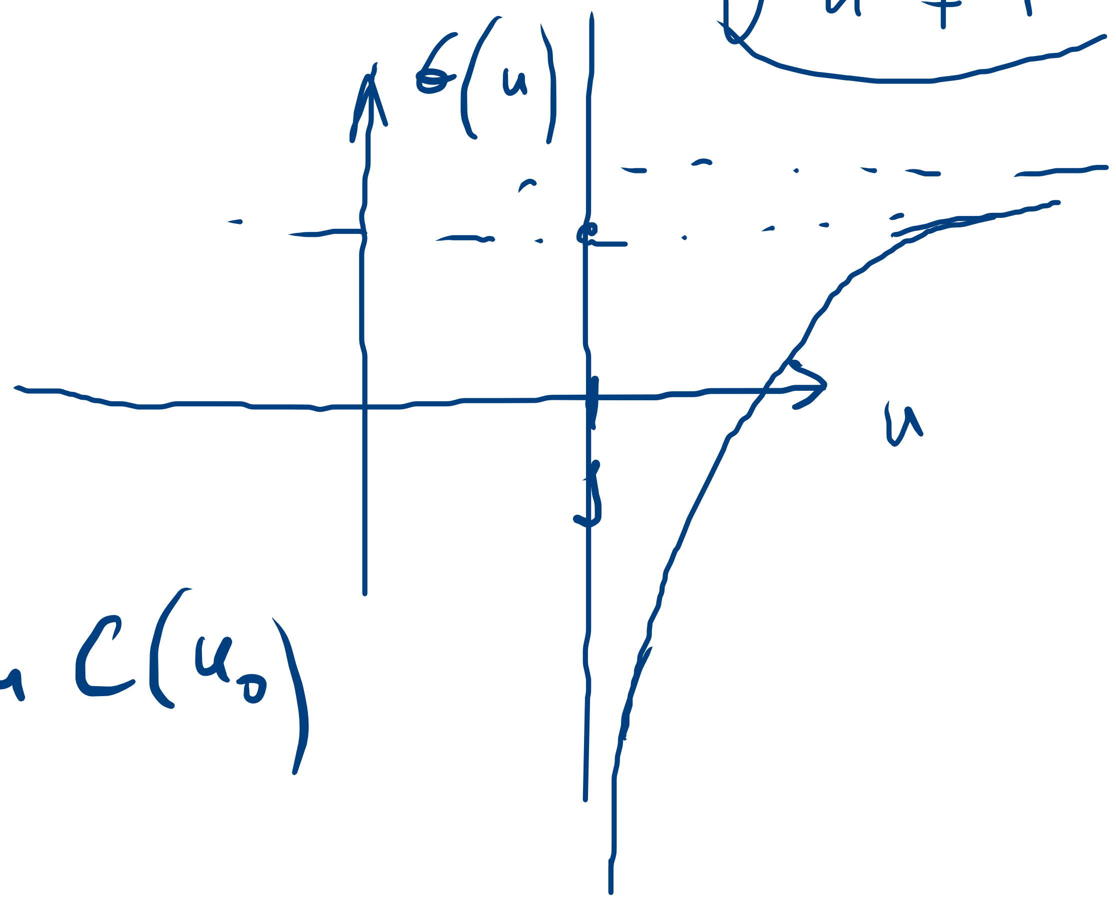
$$\ln\left(\frac{\sqrt{u}-1}{\sqrt{u}+1}\right) - \ln\left(\frac{\sqrt{u_0}-1}{\sqrt{u_0}+1}\right) = x - x_0$$

$G(u) = F(x)$

$$\ln\left(\underbrace{\frac{\sqrt{u_0}+1}{\sqrt{u_0}-1}}_{C(u_0)>0} \cdot \frac{\sqrt{u}-1}{\sqrt{u}+1}\right) = x - x_0$$

$$\boxed{\ln\left(C(u_0)\,\frac{\sqrt{u}-1}{\sqrt{u}+1}\right) = x - x_0.}$$

$$G(u) = \ln\left(C\,\frac{\sqrt{u}-1}{\sqrt{u}+1}\right) \qquad G:(1,+\infty)\to\mathbb{R}$$



$$\lim_{u\to+\infty} G(u) = \ln C$$

$$\lim_{u\to 1+0} G(u) = -\infty$$

$$G(u) = x - x_0$$

$G\uparrow$ (проверить!)

$$x - x_0 < \ln C(u_0)$$

$$x < x_0 + \ln C(u_0)$$

Интерв. $\exists$ реш. $\quad (-\infty,\ x_0 + \ln C(u_0))$

$$\lim_{x\to x_0+\ln C(u_0)-0} u(x) = +\infty.$$

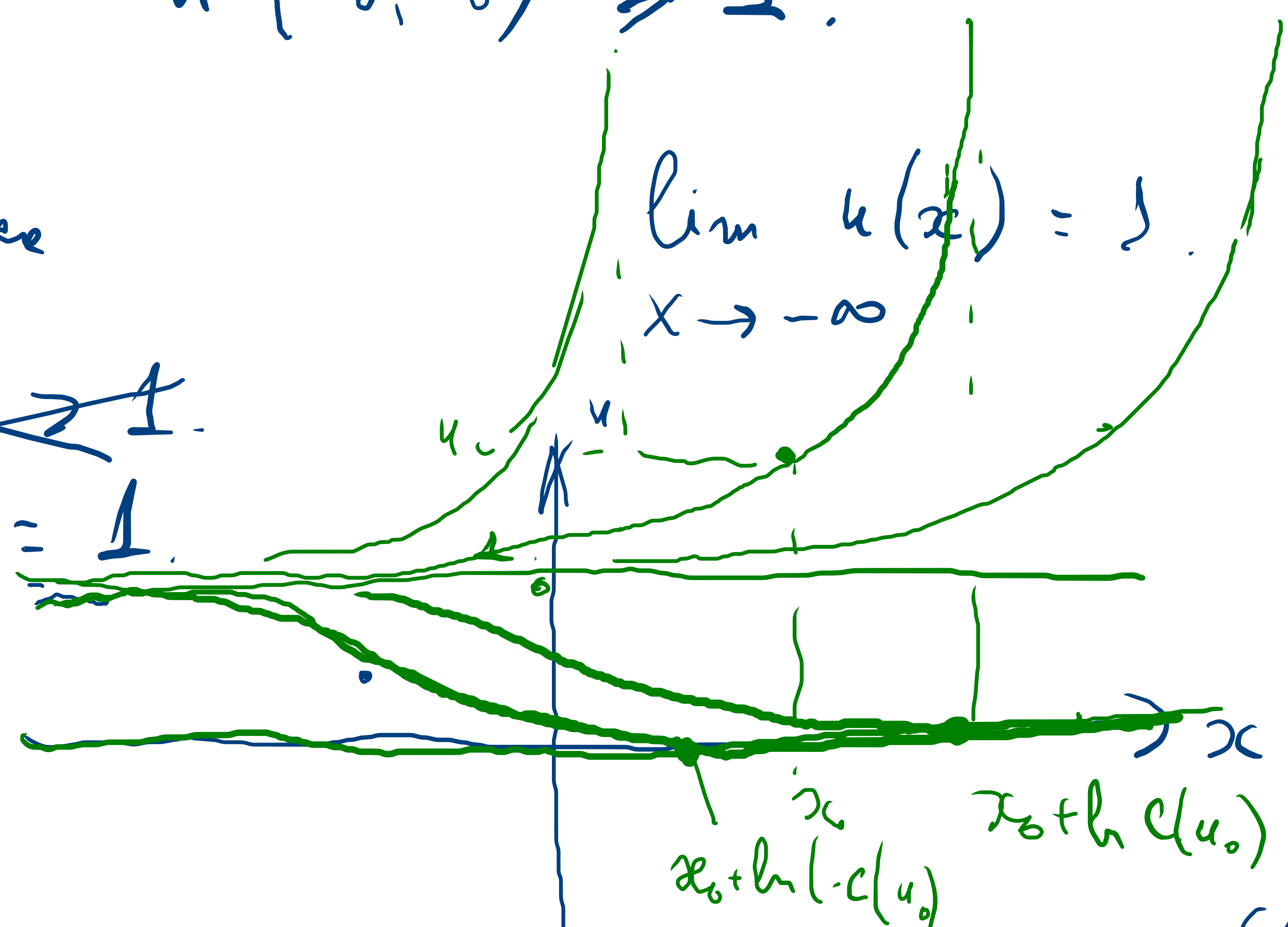
$$\lim_{x\to-\infty} u(x) = \ ?$$

$$\exists \lim_{x\to-\infty} u(x) = \tilde{u}(u_0, x_0) \geq 1.$$

по теор. Вейерштрасса

~~либо~~ $\cancel{\tilde{u} > 1}$.

~~либо~~ $\tilde{u} = 1$.

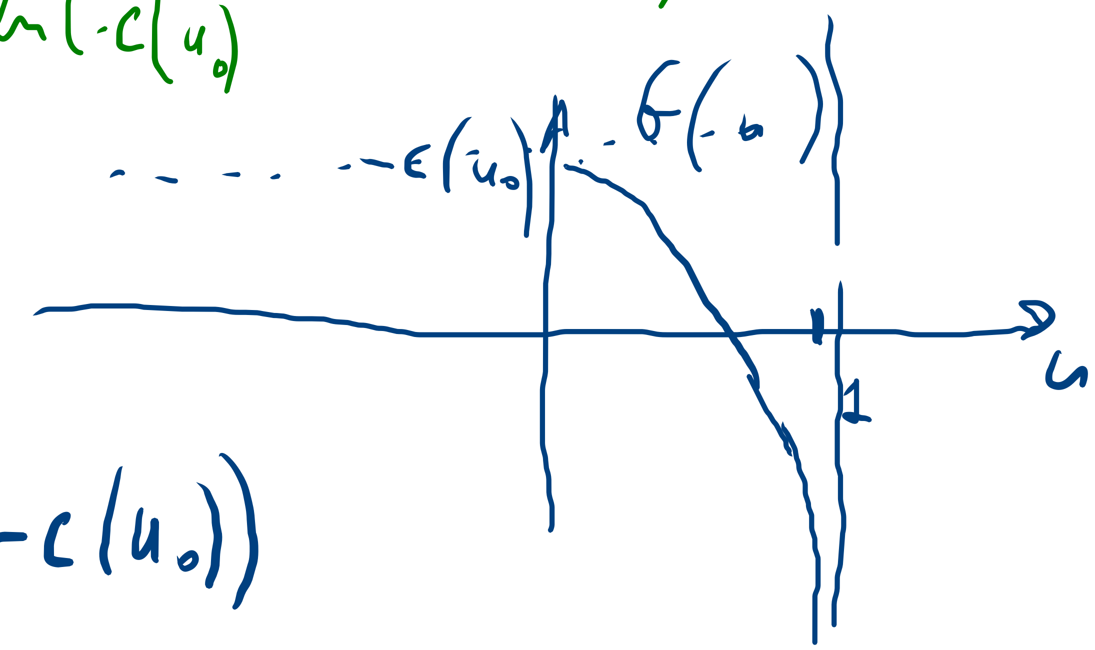


3.2) $0 < u_0 < 1$.

$G(u) = x - x_0$ $\qquad c(u_0) < 0$.

$$\lim_{u\to 1-0} G(u) = -\infty$$

$$\lim_{u\to 0} G(u) = \ln(-c(u_0))$$



$$\cancel{(-\infty,\ x_0+\ln(-c(u_0)))} \qquad \underline{\mathbb{R}}$$

$$\lim_{x\to-\infty} u(x) = 1.$$

$$\lim_{x\to x_0+\ln(-c(u_0))} u(x) = 0.$$

$$\lim_{x\to x_0+\ln(-c(u_0))} u'(x) = \lim_{x\to\dots} \sqrt{u(x)}\,(u(x)-1) = 0.$$

$$u(x) = \begin{cases} \text{из формулы } G(u) = x - x_0, & \text{если } x < x_0+\ln(-c(u_0)) \\ 0, & \text{если } x \geqslant \underbrace{\qquad}_{\dots} \end{cases}$$

$$\frac{dy}{dx} = f(x, y(x)), \quad y(x_0) = y_0 \qquad f : \mathbb{R}\times\mathbb{R} \to R$$

(*) $$|f(x,y_1) - f(x,y_2)| \le C\,|y_1 - y_2| \quad \text{— в окр. } (x_0, y_0)$$

$$C > 0$$

! Если $\frac{\partial f}{\partial y}$ непр. в окрестности (x_0, y_0), то (*) выполнено

Почему:

$$f(x,y_1) - f(x,y_2) = \frac{\partial f}{\partial y}(x,\xi)\,(y_1 - y_2)$$

$$\xi \in (y_2, y_1)$$

$$\left|\frac{\partial f}{\partial y}(x,\xi)\right| \le C.$$

$$|f(x,y_1) - f(x,y_2)| \le C\,|y_1 - y_2|$$

$$\begin{cases} \dot{y} = \sin(y-t) \\ y(t_0) = y_0 \end{cases}$$

$y(\cdot) \in \mathbb{R}$