# Elementary Introduction to the Theory of Automorphic forms. Lecture 1

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# Double periodicity.

The theory of automorphic starts as some auxiliary part of the theory of double-periodic functions.
For function of complex variable one can consider double periodic functions: $f(z+\omega_1)=f(z)$, $f(z+\omega_2)=f(z)$, for function of real variable this notion is not reasonable, as if $\omega_1/\omega_2 \in \mathbb{Q}$, one has just periodic function with some period $\omega$: $\mathbb{Z}\omega_1+\mathbb{Z}\omega_2=\mathbb{Z}\omega$; if $\omega_1/\omega_2 \in \mathbb{R}\setminus\mathbb{Q}$ the set $\mathbb{Z}\omega=\mathbb{Z}\omega_1+\mathbb{Z}\omega_2$ is dense in $\mathbb{R}$.

# Parameters of lattices.

In the theory of periodic functions the period itself is not important parameter as it evidently can be changed (and normalised to,say, 1) by rescaling of the argument. For double periodicity the ratio $\tau = \omega_1/\omega_2$ is true parameter of the periodicity. By change of the sign of $\omega_1$ one can make the imaginary part of $\tau$ positive
At the other hand, the periodicity depends not from periods $\omega_1$ and $\omega_2$ but from the full lattice $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$. the system of generators $\omega_1$, $\omega_2$ and $\tilde{\omega}_1$, $\tilde{\omega}_2$ of $L$ are related by integer unimodular matrix:

$$\begin{pmatrix} \tilde{\omega}_1 \\ \tilde{\omega}_2 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}.$$

Hence, $\tilde{\tau} = \frac{\tilde{\omega}_1}{\tilde{\omega}_2} = \frac{a\tau+b}{c\tau+d}$

# Automorphic Objects.

The previous speculations correspond to invariant with respect to dilatation function on the set of embedded in $\mathbb{C}$ free abelean 2-groups $\mathbb{Z} \oplus \mathbb{Z}$ a function on upper half-plane $\mathbb{H} = \{\tau | \Im(\tau > 0\}$ which is invariant with respect to action of $\mathrm{SL}_2(\mathbb{Z})$ by fractional linear transformations.

It is reasonable to consider less restrictive condition to function, namely, one can study the functions which are *homogeneous* with respect to dilatations : $F(\lambda L \subset \mathbb{C}) = \lambda^{-k} F(L \subset \mathbb{C})$. Invariant functions can be realised as quotients of the homogeneous.

The homogeneous function produces a function on the upper half plane such that

$$f(\frac{a\tau + b}{c\tau + d}) = (c\tau + d)^k f(\tau).$$

Objects which have such a transformation law are known as *automorphic*. The integer number $k$ is the weight of the object.

# Eisenstein Series

We shall demonstrate existence of automorphic objects.

## Proposition

*For $k > 2$ the series $\sum_{l \in L \setminus 0} l^{-k}$ absolutely converges and defines a homogeneous function of the lattice $L \subset \mathbb{C}$*

Convergence follows from the integral criteria: consider the parallelogram $\Pi_0 = \{\alpha_1\omega_1 + \alpha_2\omega_2|, |\alpha_i| \leq \frac{1}{2}\}$ and put $\Pi_l = l + \Pi_0$.
Evidently $\cup_{l \in L} \Pi_l = \mathbb{C}$ For $l$ such that
$|l| > r = \max_{\Pi_0} |z| = \frac{1}{2}\max(|\omega_1 \pm \omega_2|)$.
For rather small $\rho$ the disc $D_\rho = \{Z||z| < \rho\}$ belongs to $\Pi_0$ , hence
$\cup_{l \in L \setminus 0} \Pi_l \subset \mathbb{C} \setminus D_\rho = \{z||z| \geq \rho\}$.
For $|l| > \frac{1}{2}\max_{\Pi_l} |z|$ by triangle inequality , hence
$|l|^{-k} < 2^k \min_{\Pi_l} |z|^{-k}$. The square of $\Pi_l$ equals
$2|\Im(\omega_1\bar{\omega_2} - \omega_2\bar{\omega_1})|$. So

$$|l|^{-k} < \frac{1}{2|\Im(\omega_1\bar{\omega_2} - \omega_2\bar{\omega_1})|} \int_{\Pi_l} \frac{2^k d\,x d\,y}{|z|^k}$$

.

The property to be homogeneous is evident.

Split a partial sum $S_I = \sum_{l \in I} |l|^{-k}$ over $I$ for $I \subset L \setminus 0$ in two terms: $S_I^0 = \sum_{l \in I || l| \leq r} l^{-k}$ and $S_I^\infty = \sum_{l \in I || l| > r} l^{-k}$
The first term is a finite sum bounded by finite sum $S^0 = \sum_{l \in L \setminus 0 || l| \leq r} l^{-k}$. The second can be bounded by application of the the estimate of any term by the integral:

$$\sum_{l \in I || l| > r} l^{-k} < \sum_{l \in I || l| > r} \frac{1}{2|\Im(\omega_1 \bar{\omega_2} - \omega_2 \bar{\omega_1})|} \int_{\Pi_l} \frac{2^k d\, x d\, y}{|z|^k} =$$

$$\frac{1}{2|\Im(\omega_1 \bar{\omega_2} - \omega_2 \bar{\omega_1})|} \int_{\Phi_I} \frac{2^k d\, x d\, y}{|z|^k},$$

where $\Phi_I = \cup_{l \in I || l| > r} \Pi_l$. As $\Phi_I \subset \cup_{l \in L \setminus 0} \Pi_l \subset \{z || z| \geq \rho\}$, the integral is bounded by the integral over domain $\{z || z| \geq \rho\}$ The last integral converges at infinity, so partial sums are bounded, hence the series $\sum_{l \in L \setminus 0} |l|^{-k}\}$ converges.

# The Limit of the Eisenstein Series at Infinity.

The corresponding function on the upper half plane evidently has the form

$$e_k(\tau) = \sum_{(m,n)\neq(0,0)} (m\tau + n)^{-k}.$$

We shall demonstrate that for even $k$ the Eisenstein series is not identically zero (for odd $k$ they evidently vanish). For this we show that the limit for $\tau$ to $i\infty$ does not vanish.

## Proposition

*For even $k \geq 4$*

$$\lim_{t\to\infty} e_k(it) = \lim_{t\to\infty} \sum_{(m,n)\neq(0,0)} (mit + n)^{-k} = \sum_{n\neq 0} n^{-k} = 2\zeta(k) \neq 0$$

*Proof.*

for $m, n \neq 0$

$$|(mit+n)^{-k}| = |(mit+n)^{-2}| \times |(mit+n)^{2-k}| < t^{-2}m^{-2} \times n^{2-k}$$

Hence

$$\left| \sum_{(m,n)\neq(0,0)} (mit+n)^{-k} - \sum_{n\neq 0} n^{-k} \right| = \left| \sum_{(m,n),m\neq 0} (mit+n)^{-k} \right| <$$

$$< \sum_{(m,n),m\neq 0,n\neq 0} (mit+n)^{-k} + |t|^{-k} \sum_{m,m\neq 0} |m|^{-k} <$$

$$< \sum_{(m,n),m\neq 0,n\neq 0} t^{-2}m^{-2} \times n^{2-k} + |t|^{-k} \sum_{m,m\neq 0} |m|^{-k} = O(t^{-2}),$$

as for $k \geq 4$ all series converges.

# Automorphic functions

The automorphic functions can be realised as quotients of automorphic forms, f.e. $(e_4)^3/(e_6)^2$. There is a preferable choice. Put $\Delta = \frac{1}{1728}\left((60e_4)^3 - (140e_6)^2\right)$. Further we show that this automorphic form of weight 12 has not vanishes at the upper half-plane. The standard automorphic function is $j = \frac{(60e_4)^3}{\Delta}$.

Literature to next lecture: Elliptic Functions according to Eisenstein and Kronecker Andre Weil Chapter II