Задача 1

$$(1)\quad \begin{cases} \dfrac{dy}{dx} = \dfrac{x+y}{x-y} \\[2ex] y(x_0) = y_0 \end{cases}$$

$y(\cdot) \in \mathbb{R}, \quad x \in \mathbb{R}$

$$f(x,y) := \frac{x+y}{x-y}$$

$$f : \underbrace{\mathbb{R}^2 \setminus \{(x,y) : x = y\}}_{D} \to \mathbb{R}$$

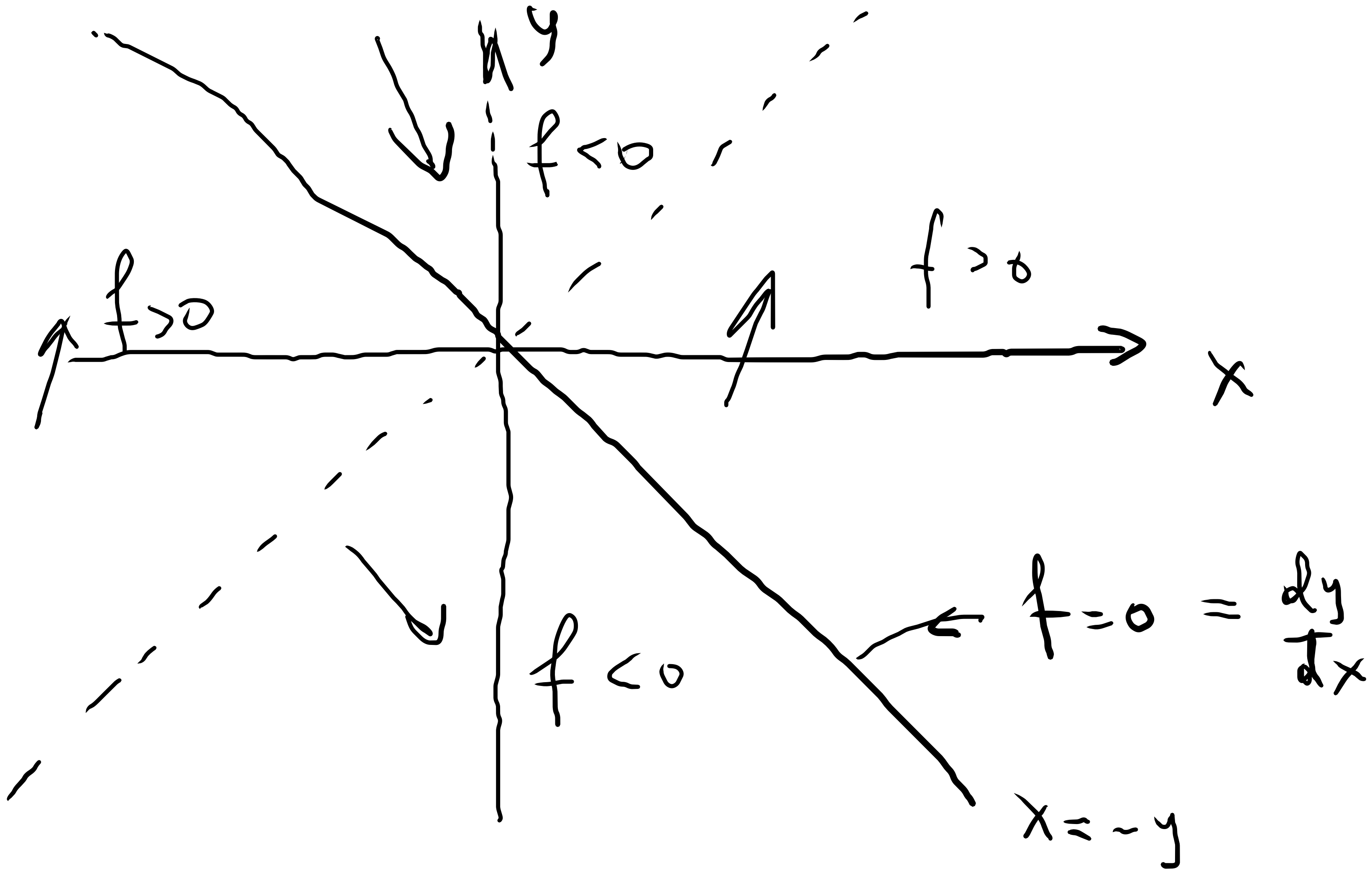
$$(x_0, y_0) \in D \Rightarrow \boxed{x_0 \neq y_0}$$

Решение 1) f – непр. $\Rightarrow$ реш. $\exists$ локально $\forall (x_0, y_0) : x_0 \neq y_0$

2). $$\frac{\partial f}{\partial y}(x,y) := \frac{1 \cdot (x-y) + (x+y)}{(x-y)^2} = \frac{2x}{(x-y)^2}$$ непр. на D

$\Rightarrow$ в окр. каждой $(x_0, y_0) \in D$ вып. усл. теор. единств. $\Rightarrow$ решения (1) единств.

3).



$$f(x,y) = \frac{x+y}{x-y}$$

4). $z(x) := -y(-x)$

$$z'(x) = y'(-x) = f(-x, y(-x)) = \frac{-x + y(-x)}{-x - \underbrace{y(-x)}_{z(x)}} = \frac{-x - z(x)}{-x + z(x)} = \frac{x+z}{x-z}$$

$\Rightarrow$ Если $y(\cdot)$ – решение ур-я $\Rightarrow$ $z(x) := -y(-x)$ – тоже решение.

$\Rightarrow$ картина траекторий симм. отн. точки 0.

4). • $\dfrac{dy}{dx} = \dfrac{x+y}{x-y} = \dfrac{1+\frac{y}{x}}{1-\frac{y}{x}}$ $:x$ $(x \neq 0)$

• $u(x) := y(x)/x$

$\rightarrow$ $y(x) = x\,u(x)$

$y'(x) = 1 \cdot u(x) + x u'(x) = \dfrac{1+u}{1-u}$

$u + x u' = \dfrac{1+u}{1-u}$

$x u' = \left(\dfrac{1+u}{1-u} - u\right) = \dfrac{1+u^2}{1-u}$

(если $x=0$, то $0 = \dfrac{1+u^2}{1-u}$ — этого не может быть $\Rightarrow x \neq 0$).

ур-е с разделением перем.

$\dfrac{dy}{dx} = f_1(y) f_2(x)$

$\Updownarrow$

$\displaystyle\int_{y_0}^{y} \frac{dy}{f_1(y)} = \int_{x_0}^{x} \frac{dx}{f_2(x)}$

$F(y) = G(x)$

$$\begin{cases} u' = \dfrac{1}{x} \dfrac{1+u^2}{1-u} \\ u(x_0) = u_0 \end{cases}$$

$u_0 \neq 1$. $\quad x_0 \neq 0$.

$(u \neq 1) \quad (x \neq 0)$

$$\frac{(1-u)}{1+u^2}\, du = \frac{dx}{x}$$

$$\int_{u_0}^{u} \frac{1-v}{1+v^2}\, dv = \int_{x_0}^{x} \frac{ds}{s}$$

$$\Updownarrow$$

$$\int_{u_0}^{u} \frac{dv}{1+v^2} - \int_{v_0}^{v} \frac{\overbrace{v\,dv}^{\frac{1}{2}d(1+v^2)}}{1+v^2} = \ln|s| \Big|_{x_0}^{x} = \ln(|x|/|x_0|) = \ln\left|\frac{x}{x_0}\right|$$

$$\Updownarrow$$

$$\operatorname{arctg} v \Big|_{u_0}^{u} - \frac{1}{2} \ln(1+v^2) \Big|_{v_0}^{v}$$

$$G(v) := \operatorname{arctg} v - \frac{1}{2}\ln(1+v^2) \qquad G'(u) = \frac{1-u}{1+u^2}$$

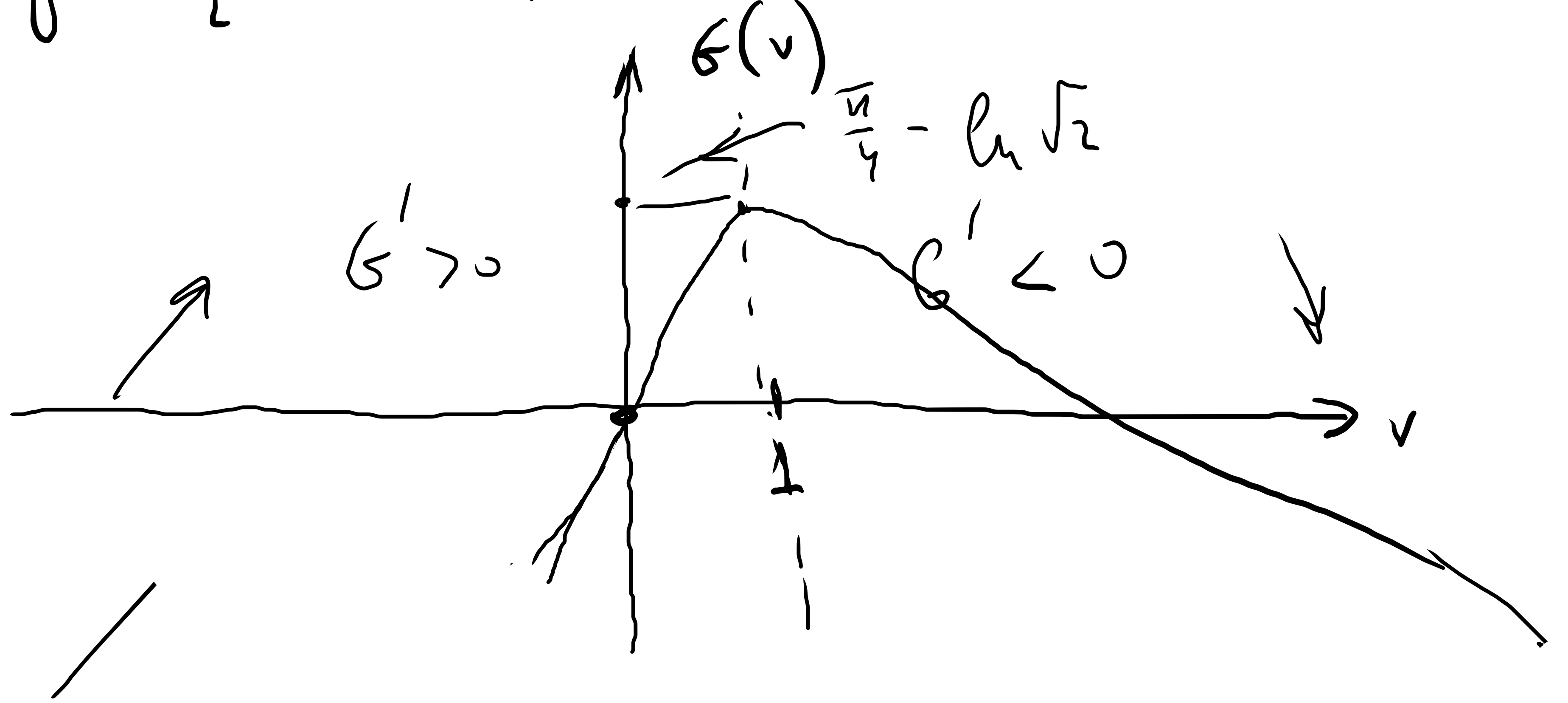
$$G(u) - G(u_0) = \ln|x| - \ln|x_0|$$

$$\boxed{G(u) = \ln C|x|} \qquad \ln C = \underbrace{G(u_0) - \ln|x_0|}$$

$$C = \exp(\ \dots\ ) > 0$$

$$C = C(x_0, u_0) > 0$$

$$G(1) = \operatorname{arctg} 1 - \frac{1}{2}\ln 2 = \frac{\pi}{4} - \ln\sqrt{2}$$



<u>Сл. 1.</u> $u_0 > 1 \Rightarrow u(x) > 1 \quad \forall x$

$$G(u) = \ln C|x|$$

<u>Сл. 2.</u> $u_0 < 1 \Rightarrow u(x) < 1 \quad \forall x$

$$G(u) = \ln C|x|.$$

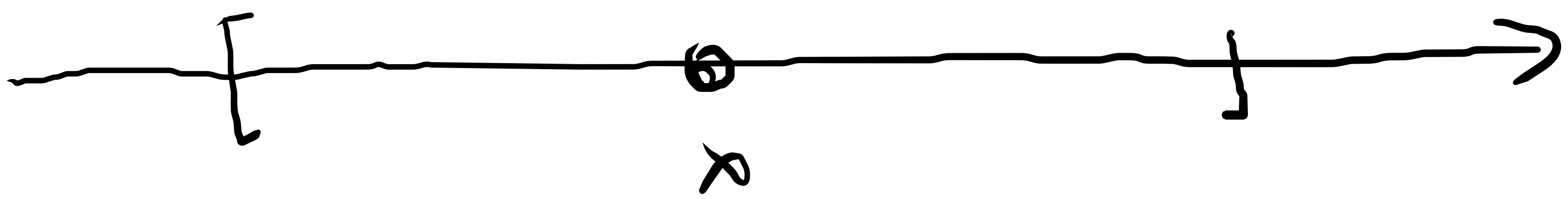
$$\ln C|x| \leq \frac{\pi}{4} - \ln\sqrt{2}.$$

$$|x| \leq \frac{1}{C} e^{\pi/4 - \ln\sqrt{2}}$$ $\Rightarrow$ решение $\exists$ только на огр. интервале по x

Хочется сказать, что реш. $\exists$ при

$$\underbrace{-\frac{1}{C} e^{\pi/4 - \ln\sqrt{2}}}_{a^-(x_0,\,y_0)} \le x \le \underbrace{\frac{1}{C} e^{\pi/4 - \ln\sqrt{2}}}_{a^+(x_0,\,y_0)}$$

← формально неверно



$$x \to a^+ - 0 \qquad\Longrightarrow\qquad \underset{\ln C|x|}{\overset{\shortparallel}{G(u)}} \to \pi/4 - \ln\sqrt{2}$$

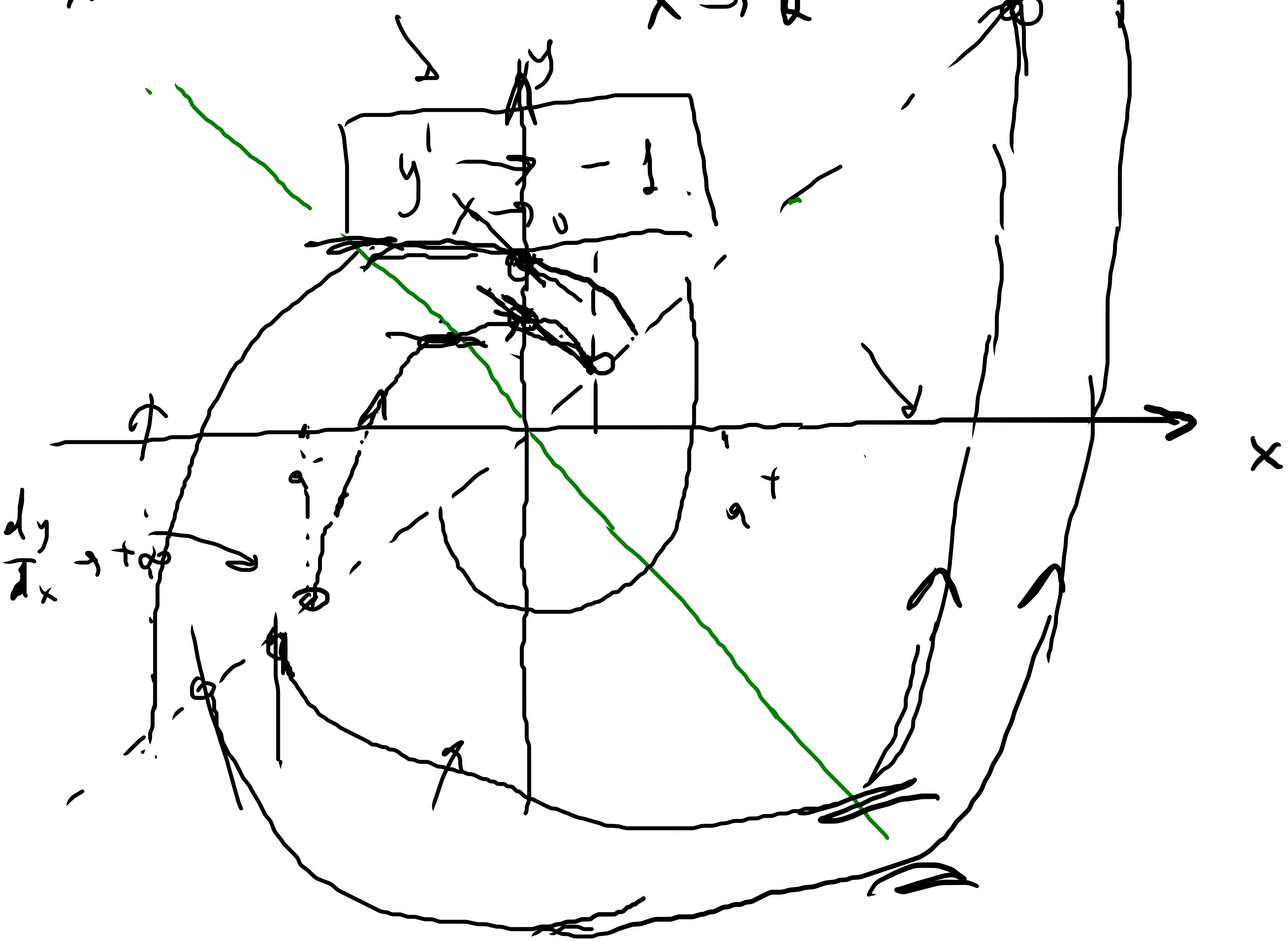
$$x \to a^- \ne 0$$

$$\Rightarrow \quad u \to 1.$$

$$u(x) = \frac{y(x)}{x} \Rightarrow y(x) = \underset{\downarrow\, a^\pm}{x}\;\underset{\downarrow\, 1}{u(x)} \to a^\pm$$

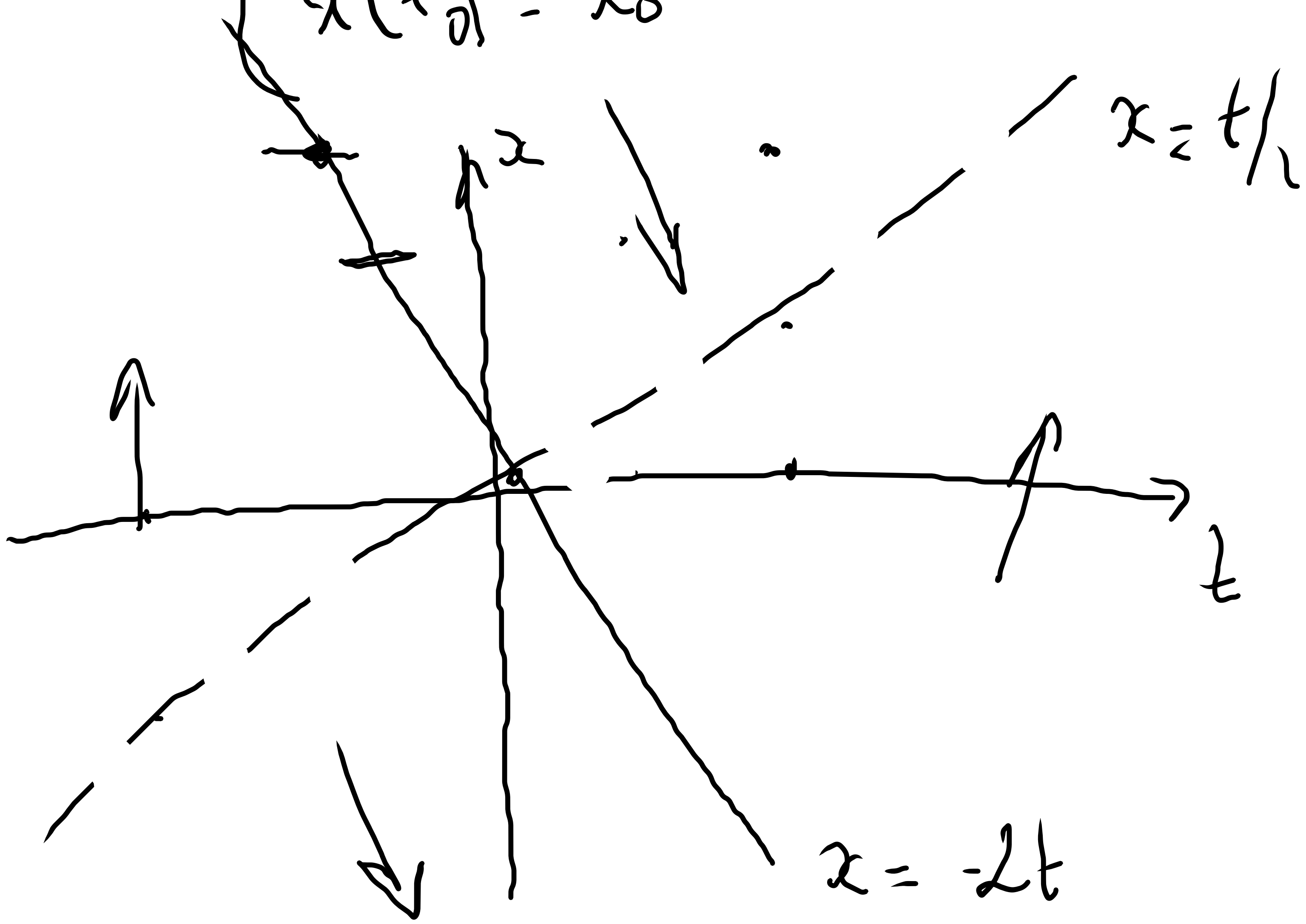
$$y'(x) = \frac{x+y}{x-y}$$

$$\lim_{x \to a^+} y'(x) = \lim_{x \to a^+} \frac{\overset{\to a^+}{x} + \overset{\to a^+}{y(x)}}{x - y(x)} = \infty$$



Задача 2

$$\begin{cases} \dot{x} = \dfrac{2t+x}{t-2x} \\ x(t_0) = x_0 \end{cases}$$



$x = x(t) \in \mathbb{R}$

$t \in \mathbb{R}$

$$f(t, x) := \frac{2t+x}{t-2x}$$

$\mathrm{Dom}\, f := \{(t, x) : t \neq 2x\}$

Упр. $y(t) := -x(-t)$

$$\dot{y}(t) \underset{\text{д-ть}}{=} \frac{2t+y}{t-2y}.$$

$\Rightarrow$ симм. отн. точки O.

$x(t) = t\,u(t)$

$$\dot{x} = u + t\dot{u} = \frac{2t + x}{t - 2x} = \frac{2 + u}{1 - 2u}$$

$(t \neq 0)$

(t = 0 ← рассм отдельно)

$$t\dot{u} = \frac{2 + u}{1 - 2u} - u = 2\,\frac{1 + u^2}{1 - 2u} \qquad u \neq \frac{1}{2}.$$

$$\frac{dt}{t} = \frac{1}{2}\,du\left(\frac{1 - 2u}{1 + u^2}\right)$$

$$\frac{2\,dt}{t} = \frac{du}{1 + u^2}(1 - 2u) = d\left(\operatorname{arctg} u - \ln(1 + u^2)\right)$$

$$\operatorname{arctg} u - \ln(1 + u^2) = \ln C t^2$$

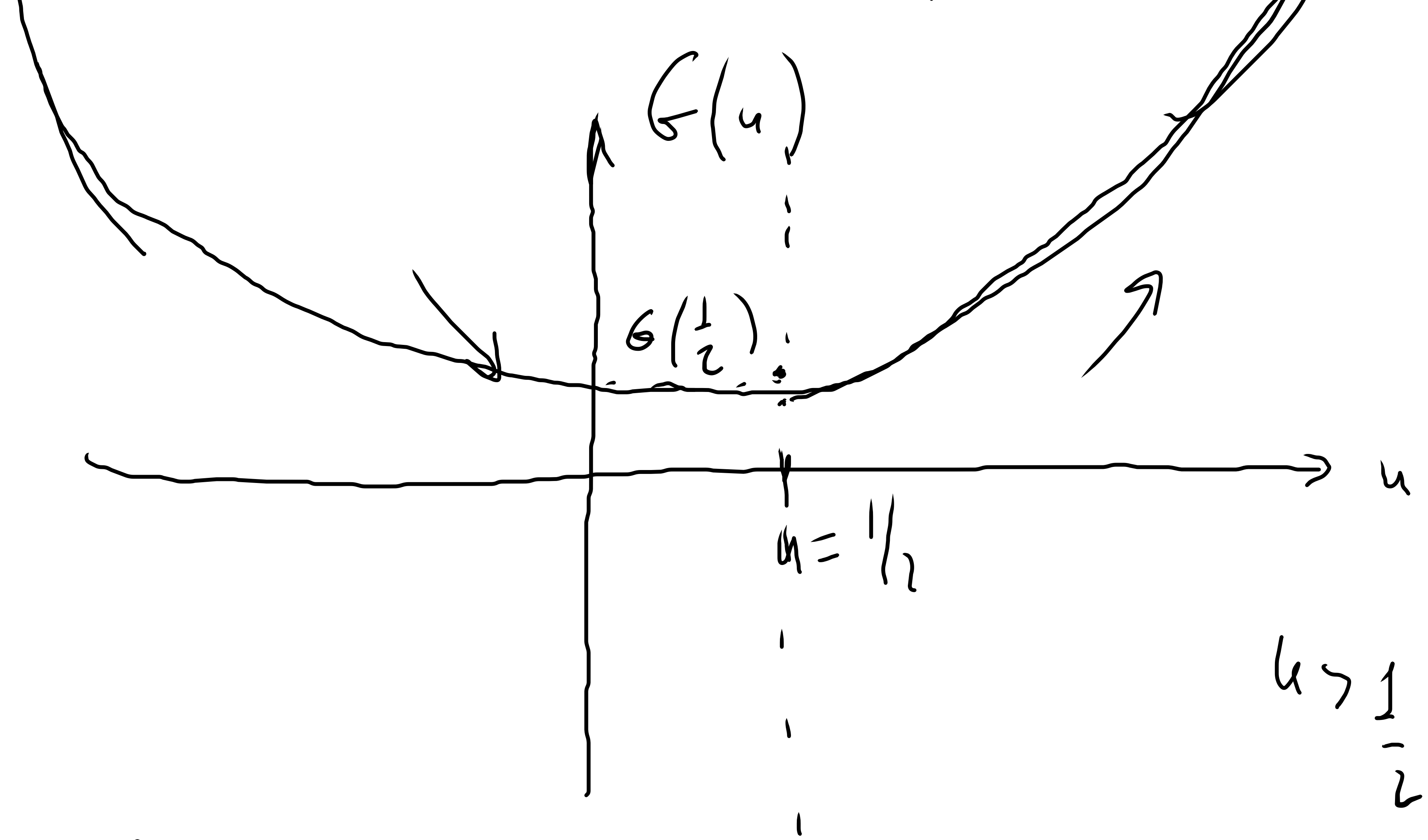
$C = C(u_0, t_0) = C(x_0, t_0) > 0$

?? ..

$$G(u) = (1+u^2)\, e^{\operatorname{arctg} u} = Ct^2 \qquad \underline{C = C(t_0, x_0) > 0}$$

$$\Downarrow$$

$$u = u(t).$$



$$G'(u) = \frac{1-2u}{1+u^2}$$

$u > \frac{1}{2}$ $\qquad u = G^{-1}(Ct^2)$

$u < \frac{1}{2}$ $\qquad u = G^{-1}(Ct^2)$

$$G\left(\tfrac{1}{2}\right) < Ct^2$$