Elementary Introduction to the Theory of Automorphic forms. Lecture 2.

Andrey Levin

NRU HSE Faculty of Mathematics

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Eisenstein Series

We shall demonstrate existence of automorphic objects.

Proposition

For $k > 2$ the series $\sum_{l \in L \setminus 0} l^{-k}$ absolutely converges and defines a homogeneous function of the lattice $L \subset \mathbb{C}$

Convergence follows from the integral criteria: consider the parallelogram $\Pi_0 = \{\alpha_1\omega_1 + \alpha_2\omega_2|, |\alpha_i| \leq \frac{1}{2}\}$ and put $\Pi_l = l + \Pi_0$.
Evidently $\cup_{l \in L} \Pi_l = \mathbb{C}$ For l such that
$|l| > r = \max_{\Pi_0} |z| = \frac{1}{2} \max(|\omega_1 \pm \omega_2|)$.
For rather small ρ the disc $D_\rho = \{Z||z| < \rho\}$ belongs to Π_0 , hence
$\cup_{l \in L \setminus 0} \Pi_l \subset \mathbb{C} \setminus D_\rho = \{z||z| \geq \rho\}$.
For $|l| > \frac{1}{2} \max_{\Pi_l} |z|$ by triangle inequality , hence
$|l|^{-k} < 2^k \min_{\Pi_l} |z|^{-k}$. The square of Π_l equals
$2|\Im(\omega_1\bar{\omega_2} - \omega_2\bar{\omega_1})|$. So

$$|l|^{-k} < \frac{1}{2|\Im(\omega_1\bar{\omega_2} - \omega_2\bar{\omega_1})|} \int_{\Pi_l} \frac{2^k d\,x d\,y}{|z|^k}$$

.

The property to be homogeneous is evident.

Split a partial sum $S_I = \sum_{l \in I} |l|^{-k}$ over I for $I \subset L \setminus 0$ in two terms: $S_I^0 = \sum_{l \in I || l | \leq r} l^{-k}$ and $S_I^\infty = \sum_{l \in I || l | > r} l^{-k}$
The first term is a finite sum bounded by finite sum $S^0 = \sum_{l \in L \setminus 0 || l | \leq r} l^{-k}$. The second can be bounded by application of the the estimate of any term by the integral:

$$\sum_{l \in I || l | > r} l^{-k} < \sum_{l \in I || l | > r} \frac{1}{2|\Im(\omega_1 \bar{\omega_2} - \omega_2 \bar{\omega_1})|} \int_{\Pi_l} \frac{2^k d\,x d\,y}{|z|^k} =$$

$$\frac{1}{2|\Im(\omega_1 \bar{\omega_2} - \omega_2 \bar{\omega_1})|} \int_{\Phi_l} \frac{2^k d\,x d\,y}{|z|^k},$$

where $\Phi_I = \cup_{l \in I || l | > r} \Pi_l$. As $\Phi_I \subset \cup_{l \in L \setminus 0} \Pi_l \subset \{z || z| \geq \rho\}$, the integral is bounded by the integral over domain $\{z || z| \geq \rho\}$ The last integral converges at infinity, so partial sums are bounded, hence the series $\sum_{l \in L \setminus 0} |l|^{-k}\}$ converges.

General Remarks.

We will construct periodic function by application the following general

Fact. Let a group G acts on vector space V. Then the sum $\sum_{g \in G} g(v)$ is G-invariant.

For finite group this argument is very useful, but for infinite group the convergence of the required series should be proved.

We demonstrate this for the case $G = \mathbb{Z}$, V be the space of functions on $\mathbb{R}$ and action is the shift of the argument: $\mu \in \mathbb{Z}$ maps a function $f(x)$ to $f(x - \mu)$.The simplest function ($=$ a vector in the vector space of function) is the monomial x^n.

For $k \geq 0$ the series is very divergent (the terms do not tend to 0).

For $k < -1$ the series is absolutely uniformly convergent on compact sets, this following from comparison with sum $\sum_{\mu=1}^{\infty} \mu^k$.

From such shape of convergence follow periodicity (as $\mathbb{Z}$ invariance).

For $k = -1$ the series in standard treating as $\lim_{M \to \infty, N \to \infty} \sum_{-M}^{N}$ diverges according to asymptotic $(x + \mu)^{-1} \sim \mu^{-1}$ and divergence of the harmonic series $\sum \mu^{-1}$.

Eisenstein Summation.

Eisenstein proposed a way of redefining of summation over all integers $\mathbb{Z}$ as $\sum_e = \lim_{M\to\infty} \sum_{-M}^{M}$. In this sense the series for $k = -1$ converges as

$$\sum_{\mu=-M}^{M} \frac{1}{x-\mu} = \frac{1}{x} + \sum_{\mu=1}^{M} \left(\frac{1}{x-\mu} + \frac{1}{x+\mu} \right),$$

$$\frac{1}{x-\mu} + \frac{1}{x+\mu} = \frac{2x}{x^2 - mu^2} \sim \frac{-2x}{\mu^2}.$$

This series absolutely uniformly convergent on compact sets. Note that such a regularization breaks the translation invariance, hence for periodicity of the regularized series we shall apply extra arguments.

Trigonometric Functions according to Eisenstein

Definition

The Eisenstein trigonometric functions are the following series:
$\varepsilon_n(x) = \sum_{-\infty}^{\infty}(x+\mu)^{-n}$ *for* $n > 1$; $\varepsilon_1(x) = \sum_e (x+\mu)^{-1}$

Lemma

a) for $n > 1$ *the function* $\varepsilon_n(x)$ *is periodic:* $\varepsilon_n(x+1) = \varepsilon_n(x)$
b) $\varepsilon_n(-x) = (-1)^n \varepsilon_n(x)$
c)$\varepsilon_n(x)' = -n\varepsilon_{n+1}(x)$
d) the function $\varepsilon_1(x)$ *is periodic:* $\varepsilon_1(x+1) = \varepsilon_1(x)$

Draft of a proof. First we prove absolute uniform convergence of series on the bounded sets. The last can be proved by the following consideration. Let $|x| < K \in$ then for $\mu > K$
$|x+\mu|^{-n} < |\mu - K|^{-n}$ and $\sum_{\mu=K+2}^{\infty} |x+m|^{-n} \leq \sum_{\nu=2}^{\infty} \nu^{-n}$;
$\nu^{-n} < \max_{[\nu-1,\nu]} |x|^{-n} \Rightarrow \nu^{-n} < \int_{[\nu-1,\nu]} |x|^{-n} d\,x \Rightarrow \sum_{\nu=2}^{\infty} \nu^{-n} <$
$\int_{[1,\infty]} |x|^{-n} d\,x = \frac{1}{n-1}$.

a), b) for $n > 1$ and c) follow immediately from the absolute convergence and standard theorems of the Calculus. I leave this as the homework. Convergence of the Eisenstein summation for ε_1 is also the homework.
b) for $n = 1$ is evident as the Eisenstein summation is invariant with respect to change of sign. A Proof of d)

$$\varepsilon_1(x+1) - \varepsilon_1(x) = \lim_{M\to\infty} \left(\sum_{\mu=-M}^{M} \frac{1}{x+1-\mu} - \sum_{\mu=-M}^{M} \frac{1}{x-\mu} \right) =$$

$$= \lim_{M\to\infty} \left(\frac{1}{x+1+M} - \frac{1}{x-M} \right) = 0$$

Laurent series at the point 0

Put $\gamma_m = \sum_{\mu \neq 0} \mu^{-m}$. This series converges for $m \geq 2$

Lemma

For $|x| < 1$ $\varepsilon_1(x) = x^{-1} - \sum_{m=2}^{\infty} \gamma_m x^{m-1}$

A proof is based on the formula for geometric series: for $|s| < 1$ $\frac{1}{1-s} = \sum_{j=0}^{\infty} s^j$, so $\frac{1}{x-\mu} = -\frac{\mu^{-1}}{1-x/\mu} = -\sum_{m=1}^{\infty} \mu^{-m} x^{m-1}$ Hence

$$\varepsilon_1(x) = \frac{1}{x} + \sum_{\mu=1}^{\infty} \left(\frac{1}{x-\mu} + \frac{1}{x+\mu} \right) =$$

$$= \frac{1}{x} + \sum_{\mu=1}^{\infty} \left(-\sum_{m=1}^{\infty} \mu^{-m} x^{m-1} - \sum_{m=1}^{\infty} (-\mu)^{-m} x^{m-1} \right) =$$

$$\frac{1}{x} - \sum_{\mu \neq 0}^{\infty} \sum_{m=1}^{\infty} \mu^{-m} x^{m-1} \stackrel{?}{=} \frac{1}{x} - \sum_{m=2}^{\infty} \sum_{\mu \neq 0}^{\infty} \mu^{-m} x^{m-1} =$$

$$\frac{1}{x} - \sum_{m=2}^{\infty} \gamma_m x^{m-1}.$$

The prelast equality is changing of order of summation, so this should be justified. This is based on the bound $\mu^{-2l} |x|^{2l-1} \leq \mu^{-2} |x|^{2l-1}$, so the sum $\sum_l \sum_\mu \mu^{-2l} |x|^{2l-1}$ is bounded by $\sum_l \sum_\mu \mu^{-2} |x|^{2l-1}$ which factorise into product $\sum_\mu \mu^{-2} \sum_l |x|^{2l-2}$ of convergent series. I left details as a homework. The term by term differentiation produces the expansion for all ε's

Addition Formulas.

Consider the identity $x + y - \nu = (x - \mu) + (y - \nu + \mu)$ and divide it by $(x - \mu)(y - \nu + \mu)(x + y - \nu)$

Perform first the Eisenstein summations over μ and second over ν:

$$
\begin{array}{ccc}
\frac{1}{x-\mu}\frac{1}{y-\nu+\mu} & = & \frac{1}{x+y-\nu}\left(\frac{1}{x-\mu}+\frac{1}{y-\nu+\mu}\right) \\
 & \sum_\mu & \\
\sum_\mu \frac{1}{x-\mu}\frac{1}{y-\nu+\mu} & = & \frac{1}{x+y-\nu}\left(\varepsilon_1(x)+\varepsilon_1(y-\nu)\right) \\
\| & & \| \\
\sum_\mu \frac{1}{x-\mu}\frac{1}{y-\nu+\mu} & = & \frac{1}{x+y-\nu}\left(\varepsilon_1(x)+\varepsilon_1(y)\right) \\
 & \sum_\nu & \\
\sum_\nu \sum_\mu \frac{1}{x-\mu}\frac{1}{y-\nu+\mu} & = & \sum_\nu \frac{1}{x+y-\nu}\left(\varepsilon_1(x)+\varepsilon_1(y-\nu)\right) \\
\nparallel & & \| \\
\sum_\kappa \sum_\mu \frac{1}{x-\mu}\frac{1}{y-\kappa} & \neq & \varepsilon_1(x+y)\left(\varepsilon_1(x)+\varepsilon_1(y)\right) \\
\| & & \| \\
\varepsilon_1(x)\varepsilon_1(y) & \neq & \varepsilon_1(x+y)\left(\varepsilon_1(x)+\varepsilon_1(\nu)\right)
\end{array}
$$

In this calculation we change variable of summation $\kappa = \nu - \mu$, hence change order of summation which is not legal.

We resolve this trouble by passing to the equality

$$\frac{1}{(x-\mu)^2}\frac{1}{(y-\nu+\mu)^2} =$$

$$= \frac{1}{(x+y-\nu)^2}\left(\frac{1}{(x-\mu)^2}+\frac{1}{(y-\nu+\mu)^2}\right)+$$
$$+\frac{2}{(x+y-\nu)^3}\left(\frac{1}{(x-\mu)}+\frac{1}{(y-\nu+\mu)}\right).$$

This identity is $\frac{d}{dx}\frac{d}{dy}$ of the initial identity. For average of this identity the change the order of summation is legal as series is absolutely convergent. This is homework also.
As result we get

$$\varepsilon_2(x)\varepsilon_2(y) = \varepsilon_2(x+y)\left(\varepsilon_2(x)+\varepsilon_2(y)\right)+2\varepsilon_3(x+y)\left(\varepsilon_1(x)+\varepsilon_1(y)\right)$$

Consider the expansion of this identity near $y = 0$

$$\varepsilon_2(x)\left(\frac{1}{y^2} + \gamma_2 + \cdots\right) =$$

$$= \left(\varepsilon_2(x) + \varepsilon_2(x)'y + \frac{1}{2}\varepsilon_2(x)''y^2 + \cdots\right)\left(\frac{1}{y^2} + \gamma_2 + \cdots\right) +$$

$$= \left(\varepsilon_3(x) + \varepsilon_3(x)'y + \cdots\right)\left(\frac{1}{y} + \varepsilon_1(x) + \cdots\right)$$

The comparison of the constants in y get some quadratic relation. The expansion near $x + y = 0$ produces another relation. After some elementary calculations (See Weil Chapter II) one get $d\varepsilon_1(x)/dx = -\varepsilon_1(x)^2 + -3\gamma_2$.
Put $\tilde{\pi} = \sqrt{3\gamma_2}$ and $\mathrm{e}(x) = (\varepsilon_1(x) + \tilde{\pi})/(\varepsilon_1(x) - \tilde{\pi})$. Then $\mathrm{e}(0) = 1$ and $\mathrm{e}(x)' = 2\tilde{\pi}i\mathrm{e}(x)$. Hence the Tailor series of this function at 0 coincides with series for $\exp(2\tilde{\pi}ix)$.