q-Expansion of the Eisenstein Series. Modular forms.

Put $\tilde{\pi} = \sqrt{3\gamma_2}$ and $\mathrm{e}(x) = (\varepsilon_1(x) + \tilde{\pi} i)/(\varepsilon_1(x) - \tilde{\pi} i)$. Then $\mathrm{e}(0) = 1$ and $\mathrm{e}(x)' = 2\tilde{\pi} i \mathrm{e}(x)$. Hence the Tailor series of this function at 0 coincides with series for $\exp(2\tilde{\pi} i x)$. Conversely

$$\varepsilon_1(x) = 2\tilde{\pi} i \frac{(\mathrm{e}(x)+1)}{(\mathrm{e}(x)-1)} = \begin{cases} \tilde{\pi} i - 2\tilde{\pi} i \sum_{l=0}^{\infty} \mathrm{e}(lx) & \Im x > 0 \\ -\tilde{\pi} i + 2\tilde{\pi} i \sum_{l=0}^{\infty} \mathrm{e}(-lx) & \Im x < 0 \end{cases}$$

Hence for $m > 1$

$$\varepsilon_m(x) == \begin{cases} 2\tilde{\pi} i \frac{(-1)^m}{(m-1)!} \sum_{l=0}^{\infty} (2\tilde{\pi} i l)^{m-1} \mathrm{e}(lx)) & \Im x > 0 \\ 2\tilde{\pi} i \frac{1}{(m-1)!} \sum_{l=0}^{\infty} (2\tilde{\pi} i l)^{m-1} \mathrm{e}(-lx) & \Im x < 0 \end{cases}$$

The Eisenstein series $e_k(\tau) = \sum_{(m,n)\neq(0,0)} (m\tau + n)^{-k}.$, $k \geq 4$ can be rewrote as:
$e_k(\tau) = \sum_{(0,n)\neq(0,0)} (m\tau + n)^{-k} + \sum_{(m,n),m>0} (m\tau + n)^{-k} + \sum_{(m,n),m<0} (m\tau + n)^{-k} = \gamma_k + \sum_{m>0} \varepsilon_k(m\tau) + \sum_{m<0} \varepsilon_k(m\tau)$.

Denote $\mathrm{e}(\tau)$ by q. As $\Im\tau > 0$ for even k we have (for odd k the Eisenstein series evidently vanishes)

$$e_k(\tau) = \gamma_k + 2\frac{(2\tilde{\pi}i)^k}{(k-1)!}\sum_{m>0}\sum_{l\geq 0} l^{k-1}q^{ml}.$$

This series absolutely converges for $|q| < 1$, we left this as a homework. Hence it is holomorphic function in the unit disc. this motivate the following definition:

Definition

A function $f(\tau)$ on the upper half-plane is called a modular form of the weight k if it is

a) automorphic of the weight k:

$f((a\tau + b)/(c\tau + d)) = (c\tau + d))^k f(\tau)$ for $ad - bc = 1$, $a, b, c, d \in \mathbb{Z}$, The expression $(c\tau + d)$ is known as automorphic factor;

b) a Laurent power series in $q = \exp(2\tilde{\pi}i\tau)$:

$$f(\tau) = \sum_{n=0}^{\infty} a_n q^n.$$

Unformal Remark.

The conditions a) an b) are almost complement. The common part of this conditions reflects that the matrix $\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ acts on τ as the shift $\tau \mapsto \tau + 1$ and the automorphic factor is equal to 1, so automorphic form should depends in τ through q. At the other hand, the expression f as function in q is not adopted to the automorphy condition and regularity in $q = 0$ is not natural for a function in τ.

The automorphic factor is the subject of so-called cocycle condition which get compatibility of the automorphy condition with multiplication of matricis:

$$(cA + dC)\tau + (cB + dD) = \left(c\frac{A\tau + B}{C\tau + D} + d \right)(C\tau + D).$$

The Group Theoretical Approach.

The fraction-linear action of $\mathrm{SL}_2(\mathbb{R})$ on the upper half-plane $\mathbb{H}$ is transitive, indeed $\tau = x + yi = \frac{(x/\sqrt{y})i - \sqrt{y}}{((1/\sqrt{y})i+0)}$. The stabiliser of the point i equals to

$\mathrm{SO}_2(\mathbb{R}) = \left\{ \begin{pmatrix} a & b \\ -b & a \end{pmatrix} | a^2 + b^2 = 1 \right\} = \{z = a - bi | |z| = 1\}.$

So $\mathbb{H} = \mathrm{SL}_2(\mathbb{R})/\mathrm{SO}_2(\mathbb{R})$. Note that for $\mathrm{SO}_2(\mathbb{R})$ the automorphic $(-bi + a)$ factor equals z. Correspond to a function f on the upper Half-plane a function $\check{f}$ on $\mathrm{SL}_2(\mathbb{R})$ by the rule

$$\check{f}\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = (ci + d)^{-k} f\left(\frac{ai + b}{ci + d}\right).$$

Accordig to the cocycle condition for automorphic factor, this function transform under right action of $\mathrm{SO}_2(\mathbb{R}) = \{z | |z| = 1\}$. by character $z \mapsto z^{-k}$, so it belongs to representation of $\mathrm{SL}_2(\mathbb{R})$, induced from this character of $\mathrm{SO}_2(\mathbb{R})$. At the other hand, according to the cocycle condition for automorphic factor, the automorphy condition for f is equivalent to invariance of $\check{f}$ with respect to left action of $\mathrm{SL}_2(\mathbb{Z})$.

The automorphic factor$(c\tau + d)$ is trivial for unipotent subgroup $\mathrm{N} = \left\{ \begin{pmatrix} 1 & b \\ 0 & 1 \end{pmatrix} \right\}$, hence the average any power of automorphic factor over set of cosets $\Gamma^\infty \setminus \mathrm{SL}_2(\mathbb{Z}), (\Gamma^\infty = \mathrm{SL}_2(\mathbb{Z}) \cap \mathrm{N})$ is $\mathrm{SL}_2(\mathbb{Z})$-invariant.

Lemma

The set of cosets $\Gamma^\infty \setminus \mathrm{SL}_2(\mathbb{Z})$ is equal to the set of coprime pairs (c, d).

The map is evident $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto (c, d)$. The Γ^∞-coset is equal to $\left\{ \begin{pmatrix} a + jc & b + jd \\ c & d \end{pmatrix} | j \in \mathbb{Z} \right\}$, so the map is well define on the coset. c and d are coprime as the matrix is unimodular. Conversely, according to the Euclid algorithm, the grater common divisor of c and d, which is equal to 1, has the linear presentation $1 = xc + yd$, and (c, d) is the image of unimodular matrix $\begin{pmatrix} y & -x \\ c & d \end{pmatrix}$ and the map is surjective.

If matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $\begin{pmatrix} a' & b' \\ c & d \end{pmatrix}$ both are unimodular, $(a' - a)d - (b' - b)c = 0$; c and d are coprime, so $(a' - a)$ is divisible by c, say $a' - a = jc$ then $b' - b = jd$, hence this matrices lie in one coset. We prove that the map is injective. in this way we get the series

$$\sum_{c,\, d \text{ coprime}} (c\tau + d)^{-k}.$$

Essentially this is the Eisnstein series as for any nonzero pair one have unique decomposition $(m, n) = f(c, d)$, where f is gcd of m and n, so

$$\sum_{(m,n)\neq(0,0)} (m\tau + n)^{-k} = \sum_{c,\, d \text{ coprime}, f>0} (f(c\tau + d))^{-k} =$$

$$= \sum_{f=1}^{\infty} f^{-k} \times \sum_{c,\, d \text{ coprime}} (c\tau + d)^{-k}.$$

So, we present the Eisenstein series in Group theoretical approach.