Числа Гурвица. Лекция 5. 19.02.21.

Ур-е транспозиции.

$$H^\circ(u;p_1,p_2,\dots) = \sum_{m=0}^{\infty}\sum_{\mu} h^\circ_{m,\mu}\, p_{\mu_1}p_{\mu_2}\dots \frac{u^m}{m!}$$ — ряд для несвязных чисел

$$H(u;p_1,p_2,\dots) = \sum_{m=0}^{\infty}\sum_{\mu} h_{m,\mu}\, p_{\mu_1}p_{\mu_2}\dots \frac{u^m}{m!}$$

Теор $$\frac{\partial H^\circ}{\partial u} = \underbrace{\frac{1}{2}\sum_{n=1}^{\infty}\sum_{i+j=n}\left((i+j)p_ip_j\frac{\partial}{\partial p_{i+j}} + ij\,p_{i+j}\frac{\partial^2}{\partial p_i\partial p_j}\right)}_{W} H^\circ$$

Обозначение $$H^\circ(u;p_1,p_2,\dots) = \sum_{m=0}^{\infty} H^\circ_m(p_1,p_2,\dots)\frac{u^m}{m!}$$

Следствие $$W H^\circ_m(p_1,p_2,\dots) = H^\circ_{m+1}(p_1,p_2,\dots)$$

Д-во $$\frac{\partial H^\circ}{\partial u} = WH^\circ = W(H^\circ_0(p_1,p_2,\dots)) + W(H^\circ_1(p_1,p_2,\dots))\frac{u^1}{1!} + W(H^\circ_2(p_1,p_2,\dots))\frac{u^2}{2!} + \dots$$

$$= \frac{\partial}{\partial u}\left(H^\circ_0(p_1,p_2,\dots) + H^\circ_1(p_1,p_2,\dots)\frac{u^1}{1!} + H^\circ_2(p_1,p_2,\dots)\frac{u^2}{2!} + \dots\right) = H^\circ_1(p_1,p_2,\dots) + H^\circ_2(p,\dots)\frac{u^1}{1!} + H^\circ_3(p_1,p_2,\dots)\frac{u^2}{2!} + \dots$$

$$W = \frac{1}{2}\sum_{n=1}^{\infty}\sum_{i+j=n}\left((i+j)p_i p_j \frac{\partial}{\partial p_{i+j}} + ij\, p_{i+j}\frac{\partial^2}{\partial p_i \partial p_j}\right)$$

$$H_1^0(p_1, p_2 \dots) = \sum_{\mu} h^0_{1,\mu}\, p_{\mu_1} p_{\mu_2} \dots$$

$$\boxed{H_0^0(p_1, p_2 \dots) = e^{p_1} = 1 + p_1 + \frac{p_1^2}{2!} + \dots}$$

$m = 0$ транспозиций $\quad \mu = 1^n \quad h^0_{0,1^n} = \frac{1}{n!}$

$$H_1^0(p_1, p_2, \dots) = W H_0^0(p_1, p_2, \dots)$$

$$W = \frac{1}{2}\Big(\underbrace{2p_1^2 \frac{\partial}{\partial p_2}}_{n=2,\ i=j=1} + \underbrace{p_2 \frac{\partial^2}{\partial p_1^2}}\Big) + 2\cdot\frac{1}{2}\Big(\overbrace{3p_1 p_2 \frac{\partial}{\partial p_3} + 1\cdot 2\, p_3 \frac{\partial^2}{\partial p_1 \partial p_2}}^{n=3}\Big) + \dots$$

$\boxed{n = 1 \rightarrow \text{ничего нет, т.к. нет } p_0}$

$$H_1^0(p_1, p_2, \dots) = \frac{1}{2}p_2\left(\frac{\partial^2}{\partial p_1^2} e^{p_1}\right) = \frac{1}{2}p_2 e^{p_1} = \frac{1}{2}p_2\left(1 + p_1 + \frac{p_1^2}{2!} + \dots\right) = \underbrace{\frac{1}{2}p_2} + \frac{1}{2}p_1 p_2 + \dots$$

$m = 1$

$$\frac{h^0_{1,2^1}}{2!} = \frac{1}{2}$$

$$\frac{1}{2} = \frac{h^0_{1,1^1 2^1}}{2!}$$

Док-во $\frac{\partial H^0}{\partial u} = \frac{1}{2} \sum_{n=1}^{\infty} \sum_{i+j=n} \left((i+j)\, p_i p_j \frac{\partial}{\partial p_{i+j}} + ij\, p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j} \right) H^0$

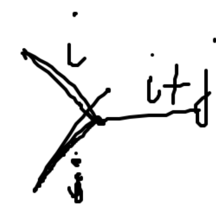
$$H^0 = \sum_{m=0}^{\infty} \sum_{\mu} \boxed{h^0_{m,\mu}\, p_{\mu_1} p_{\mu_2} \cdots} \frac{u^m}{m!}$$

$$\frac{\partial H^0}{\partial u} = \sum_{m=0}^{\infty} \sum_{\mu} \boxed{h^0_{m+1,\mu}\, p_{\mu_1} p_{\mu_2} \cdots} \frac{u^m}{m!}$$

Хотим в правой части при $\frac{u^m}{m!}$ вместо $h^0_{m,\mu}$ получить $h^0_{m+1,\mu}$.

$$\sigma = \tau_m \circ \ldots \circ \tau_1$$

$$\cancel{\tau_{m+1} \circ \sigma = \tau_{m+1} \circ \tau_m \circ \ldots \circ \tau_1}$$



Если циклы длин i, j склеились

$$p_i \cdot p_j \longrightarrow p_{i+j}$$

Если цикл длины $i+j$ разрезается

$$p_{i+j} \longrightarrow p_i p_j$$

$H^0 = \exp(H)$ $\qquad$ $H^0 := H^0(u; p_1, p_2 \ldots)$ $\qquad$ $e^t = 1 + t + \frac{t^2}{2!} + \frac{t^3}{3!} + \ldots$

$H := H(u; p_1, p_2 \ldots)$

$$\frac{\partial \exp(H)}{\partial u} = \frac{1}{2}\sum_{n=1}^{\infty} \sum_{i+j=n} \left((i+j)\, p_i p_j \frac{\partial}{\partial p_{i+j}} + ij\, p_{i+j} \frac{\partial^2}{\partial p_i \partial p_j}\right) \exp(H)$$

$$\|$$

1) $\exp(H) \dfrac{\partial H}{\partial u}$

$$H(u, p_1, p_2, \ldots) = \sum_{m=0}^{\infty} H_m(p_1, p_2 \ldots) \frac{u^m}{m!}$$

$$\frac{\partial}{\partial u}\exp(H) = 1 + H + \frac{H^2}{2!} + \ldots \qquad \frac{\partial}{\partial u}\underbrace{H \cdot H \cdots H}_{k} = k H^{k-1} \frac{\partial H}{\partial u}$$

2) $\dfrac{\partial \exp(H)}{\partial p_k} = \dfrac{\partial H}{\partial p_k} \exp(H)$

3) $\dfrac{\partial^2 \exp(H)}{\partial p_i \partial p_j} = \dfrac{\partial}{\partial p_i}\left(\dfrac{\partial H}{\partial p_j} \exp(H)\right) = \left(\dfrac{\partial^2 H}{\partial p_i \partial p_j} + \dfrac{\partial H}{\partial p_i} \cdot \dfrac{\partial H}{\partial p_j}\right) \exp(H)$

$$\frac{\partial H}{\partial u} = \frac{1}{2}\sum_{n=1}^{\infty} \sum_{i+j}^{n} \left((i+j)\, p_i p_j \frac{\partial H}{\partial p_{i+j}} + ij\, p_{i+j}\left(\frac{\partial^2 H}{\partial p_i \partial p_j} + \frac{\partial H}{\partial p_i} \frac{\partial H}{\partial p_j}\right)\right)$$

Формула Гурвица (1893)

$$h_{n-2+\ell(\mu),\mu} = \frac{(n-2+\ell(\mu))!}{|\mathrm{Aut}\,\mu|}\, n^{\ell(\mu)-3} \prod_{i=1}^{\ell(\mu)} \frac{\mu_i^{\mu_i}}{\mu_i!}$$

$\ell(\mu)$ – число циклов в μ

$n-2+\ell(\mu)$ – это минимальное число транспозиций, чтобы получить связное число.

$$\mu = \mu_1, \mu_2, \dots, \mu_{\ell(\mu)}$$

$$\mu = 2,2,1,1 \Rightarrow |\mathrm{Aut}\,\mu| = 2!\cdot 2!$$

$$n-\underline{1}+\ell(\mu)-\underline{1}$$