HW2

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Readable solutions of the problems will be sent as photo or scan or pdf-file 16.03 to alevin57@gmail.com before 17.00

The symplectic group Sp_g is the group of $2g \times 2g$ matrices h the shape $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$ is inverse of h. So, $\mathrm{SL}_2 = \mathrm{Sp}_1$.

The Siegel upper half-space H_g is the set of complex symmetric $g \times g$ matrices Ω with positive defined imaginary part, $\Omega^t = \Omega$, $\frac{\Omega - \overline{\Omega}}{2i} \gg 0$, a real matrix B is positive defined $B \gg 0$ if for any real nonzero ccolumn v $v^t B v > 0$

1) Prove that if B is positive defined ($v^t B v > 0$, $v \neq 0$), then for any complex nonzero column v $\bar{v}^t B v > 0$.

2)a.Prove that $(\overline{\Omega} C^t + D^t)(A\Omega + B) - (\overline{\Omega} A^t + B^t)(C\Omega + D) = \Omega - \overline{\Omega}$ by analysis of the 5-tiple product

$$\begin{pmatrix} \overline{\Omega} & 1_g \end{pmatrix} \begin{pmatrix} A^t & C^t \\ B^t & D^t \end{pmatrix} \begin{pmatrix} 0_g & -1_g \\ 1_g & 0_g \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} \Omega \\ 1_g \end{pmatrix}.$$

b. Deduce from the previous formula that $C\Omega + D$ is invertible.

The group $\mathrm{Sp}_g(R)$ acts on H_g by the rule $\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}$. b.

3) Check of validity of the definition of action:

a. Prove that $(A\Omega + B)(C\Omega + D)^{-1}$ is symmetric.

b. Check the identity

$$\Im\left((A\Omega + B)(C\Omega + D)^{-1}\right) = (\overline{\Omega} C^t + D)^{-1} \Im(\Omega)(C\Omega + D)^{-1},$$

Prove that the heft hand side, hence the right hand side, is positive definite.

4)Prove the cocycle condition:

$$\det\left((A\Omega + B)(C\Omega + D)^{-1}\right) =$$

$$= \det\left((A_1\left((A_2\Omega + B_2)(C_2\Omega + D)^{-1}\right) + B_1) \times\right.$$

$$\left.\times (C_1\left((A_2\Omega + B_2)(C_2\Omega + D)^{-1}\right) + D_1)^{-1}\right) \det\left((A_2\Omega + B_2)(C_2\Omega + D)^{-1}\right).$$

where

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} A_1 & B_1 \\ C_1 & D_1 \end{pmatrix} \begin{pmatrix} A_2 & B_2 \\ C_2 & D_2 \end{pmatrix}.$$

5)a.Prove that there is a matrix over the ring $Z[2\cos\phi]$ which is conjugate to the matrix of rotation by angle ϕ.

b.Construct elements in $\mathrm{SL}_2(Z)$ of orders 3, 4 and 6. What are their fixed points in the upper half plane?

c) .Construct an element in $\mathrm{SL}_2(Z[\sqrt{2})$ of order 8. What are their fixed points in the square of upper half plane?