Elementary Introduction to the Theory of Automorphic forms The Lectures 4-5

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Siegel Forms

We start from two classical generalization of the previous constructions.

The symplectic group Sp_g is the group of $2g \times 2g$ matrices h the shape $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$ such that $\begin{pmatrix} D^t & -B^t \\ -C^t & A^t \end{pmatrix}$ is inverse of h. So, $\mathrm{SL}_2 = \mathrm{Sp}_1$.

The Siegel upper half-space $\mathbb{H}_g$ is the set of complex symmetric $g \times g$ matrices Ω with positive defined imaginary part, $\Omega^t = \Omega, \frac{\Omega - \overline{\Omega}}{2i} \gg 0$.

The group $\mathrm{Sp}_g(\mathbb{R})$ acts on $\mathbb{H}_g$ by the rule $\Omega \mapsto (A\Omega + B)(C\Omega + D)^{-1}$. This is the homework.

A geometric interpretation of this object is done in the book "Tata Lectures on Theta"of D.Mumford, Chapter II Section 4. The automorphic factor should be choosen as $\det(C\Omega + D)$.

For this case one can easily repeat constructions and considerations above, including the group-approach construction of the Eisenstein series.

Hilbert Forms

The Hilbert construction is more arithmetic. We will discuss in details the simplest case of degree two.

Let integer $\Delta > 0$. Put $\mathcal{O}_{4\Delta} = \mathbb{Z}[\Delta]$ and $\mathcal{O}_{\Delta} = \mathbb{Z}[\frac{1+\sqrt{\Delta}}{2}]$ if $\Delta \equiv 1 \mod 4$. In general we shall consider *totally real order $\mathcal{O}$ of some degree r*.

The ring $\mathcal{O}_{\Delta}$ is naturally embedded in the ring $\mathbb{R} \oplus \mathbb{R}$ by the rule $\nu = r + s\sqrt{\Delta} \in \mathcal{O}_{\Delta}$ maps to (ν_+, ν_-), $\nu_{\pm} = r \pm s\sqrt{\Delta} \in \mathbb{R}$, so $\mathrm{SL}_2(\mathcal{O}_{\Delta})$ embeds into $\mathrm{SL}_2(\mathbb{R}) \times \mathrm{SL}_2(\mathbb{R})$ hence acts on $\mathbb{H} \times \mathbb{H}$ with coordinates (τ_+, τ_-).

In this case one has to basic automorphic factors $\gamma_{\pm}\tau_{\pm} + \delta_{\pm}$ for $\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_{\Delta})$, so the weight is a pair (k_+, k_-).

For general totally real order $\mathcal{O}$ of some degree r the group $\mathrm{SL}_2(\mathcal{O})$ acts on on $\mathbb{H}^{\times r}$, and the weight has r components.

For this case also one can easily repeat constructions and considerations above, including the group-approach construction of the Eisenstein series.

Arithmetical Reminder

An *Order* $\mathcal{O}$ is a finitely generated as a group subring of the field of the algebraic numbers. The rank r of this free abelian group called the degree. Elements of an order are integer algebraical numbers. The field of fractions $\mathcal{O} \otimes \mathbb{Q}$ is a number field of the same degree as the order. As finite extension, it is generated by one algebraic number: $\mathcal{O} \otimes \mathbb{Q} = \mathbb{Q}[\alpha]$. If all roots of the minimal polynom F of α are real numbers, the field $\mathcal{O} \otimes \mathbb{Q} = \mathbb{Q}[\alpha]$ and the order $\mathcal{O}$ are called *totally real*. Any real root $\tilde{\alpha}_j$ $(j = 1, 2, \cdots, r)$ of the polynom F determines an embedding of the ring $\mathcal{O} \otimes \mathbb{Q}$, hence an embedding of the ring $\mathcal{O}$ itself, into $\mathbb{R}$ by sending α to $\tilde{\alpha}_j$. The direct sum of this maps produces required map $\mathcal{O} \to \mathbb{R}^{\oplus r}$.

the Spinor Construction.

We present some occasional realisation of quadratic forms in small dimensions

The first is case of dimensions 3 and 4

i) The set $\mathrm{Mat}_{2\times 2}$ of 2×2 matrices is equipped by quadratic form determinant of signature $(2,2)$. Left and right multiplications $M \mapsto gMh$ by unimodular matrices g and h preserve this form. Hence we have the map $\mathrm{SL}_2 \times \mathrm{SL}_2 \mapsto \mathrm{SO}(2,2)$

ii) Variants

a. If we put in the formula of action $h = g^{-1}$ the set of the traceless matrices is preserved by such action, the signature of the determinant is equal to $(1, 2)$. Hence we have the map $\mathrm{SL}_2 \mapsto \mathrm{SO}(1, 2)$.

b. For case of complex coefficients put in the formula of action $h = \bar{g}^t$, restrict this action on anti-hermitian matrices $M = -\bar{M}^t$, the signature became $(3, 1)$, so we get a map $\mathrm{SL}_2(\mathbb{C}) \mapsto \mathrm{SO}(3, 1)$.By using both restrictions $h = g^{-1} = \bar{g}^t$ and the space of the traceless anti-hermitian matrices one get $\mathrm{SU}_2 \mapsto \mathrm{SO}(3)$

The prelast construction is known as The Dirac spinors and the last one is known as Pauli spinors.

The Upper Halfplane and the Signature $(1, 2)$

We follow to the Variant a from the previous slide. Consider the integer quadratic form

$$-j^2 - kl = \det \begin{pmatrix} j & k \\ l & -j \end{pmatrix}$$

of signature $(1, 2)$
Respond to a point τ at the upper half-plane the matrix

$$V_\tau = \begin{pmatrix} \tau \\ 1 \end{pmatrix} \times (-1\ \tau), \mathrm{tr}\, V_\tau = 0$$

This matrix is degenerate, hence is isotropic with respect to the quadratic form determinant The condition on the imaginary part can be interpret in the following way. Any real quadratic form determines either complex quadratic form or hermitian form. The last is equal to $P(\overline{v}, v)$, where P denotes the polarization of the initial quadratic form.
For traceless matrices $\det M = -\frac{1}{2}\mathrm{tr}(M^2)$, so
$P_{\det}(\overline{V}_\tau, V_\tau) = -\mathrm{tr}(\overline{V}_\tau V_\tau)/2 = (\tau - \overline{\tau})^2$ is negative.

Conversely, an isotropic with respect to determinant complex matrix V is degenerated, hence is equal to product of column and row:

$$V = \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \times (-\tilde{\omega}_2 \ \tilde{\omega}_1), \mathrm{tr}(V) = \mathrm{tr}\left(\begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix} \times (-\tilde{\omega}_2 \ \tilde{\omega}_1)\right) =$$

$$\mathrm{tr}\left((-\tilde{\omega}_2 \ \tilde{\omega}_1) \times \begin{pmatrix} \omega_1 \\ \omega_2 \end{pmatrix}\right) = -\tilde{\omega}_2\omega_1 + \tilde{\omega}_1\omega_2$$

So the rows $(\omega_2 \ \omega_1)$ and $(-\tilde{\omega}_2 \ \tilde{\omega}_1)$ are proportional and V is equal to $\lambda \begin{pmatrix} \mu_1 \\ \mu_2 \end{pmatrix} \times (-\mu_2 \ \mu_1)$.

$$-2P_{\mathrm{det}}(V, \overline{V}) = \mathrm{tr}(V\overline{V}) = -|\lambda|^2(-|\mu_2\bar{\mu}_1 + \mu_1\bar{\mu}_2)^2.$$

Hence $\mu_2 \neq 0$ and the quotient $\tau = \mu_1/\mu_2$ is not real). The action of $\mathrm{SL}_2(\mathbb{Z})$ by conjugation produce the required action on ω's

The Real Quadratic Orders and the Signature $(2, 2)$

Let $\mathcal{O}$ be a real quadratic order. The map $\sigma : r + s\sqrt{\Delta} \to r - s\sqrt{\Delta}$ is isomorphism of $\mathcal{O}$ and $\mathrm{N}\nu = \sigma(\nu)\nu$ is an integer quadratic form of the signature $(1, 1)$ in contrast with *imaginry* quadratic extension $\mathbb{C}$ of $\mathbb{R}$. Consider the abelian group with respect to addition of σ-antihermitian matrices: $M^t = (M)$ equipped by natural action of $\mathrm{SL}_2(\mathcal{O})$:
$M \to hM\sigma(h^t), h \in \mathrm{SL}_2(\mathcal{O})$. The determinant on such matrices is an integer quadratic form of the signature $(2, 2)$. The matrix

$$V(\tau_+, \tau_-) = \begin{pmatrix} \tau_+ \\ 1 \end{pmatrix} \times (-1 \ \tau_-)$$

belongs to the complexification of this abelian group and $V(\tau_+, \tau_-)$ is isotropic with respect to the determinant.
The determinant of 2×2 can be expressed as
$\det M = \frac{1}{2}((\mathrm{tr} M)^2 - \mathrm{tr}(M^2))$. So the polarisation equals to
$P_{\det}(M_1, M_2) = \frac{1}{2}((\mathrm{tr} M_1)\mathrm{tr} M_2) - \mathrm{tr}(M_1 M_2))$.
$P_{\det}(\overline{V(\tau_+, \tau_-)}, V(\tau_+, \tau_-)) = (\tau_+ - \overline{\tau}_+)(\tau_- \overline{\tau}_-)$ is negative.

The Pluecker Quadric, the Siegel 3-Fold and the Signature $(3,2)$

Let M be a 2×4 matrix. Consider 6 its 2×2 minors a_{ij}, $1\le i<j\le 4$. Then they are subject of equation
$a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23}=0$
The prove is based on the technique of anti-commutative polynoms. Let ξ_1,ξ_2,ξ_3,ξ_4 anticommute $\xi_i\wedge\xi_j=-\xi_j\wedge\xi_i$. correspond to M a pair of linear polynomds l_1 and l_2:
$(l_1l_2)=X(\xi_1,\xi_2,\xi_3\xi_4)$. Then $\nu=l_1\wedge l_2=\sum a_{ij}\xi_i\wedge\xi_j$,
$(a_{12}a_{34}-a_{13}a_{24}+a_{14}a_{23})\xi_1\wedge\xi_2\wedge\xi_3\wedge\xi_4=\nu\wedge\nu=$
$l_1\wedge l_2\wedge l_1\wedge l_2=-l_1\wedge l_2\wedge l_2\wedge l_1=0$
This quadratic form has signature $(3,3)$.

We shall apply this speculation to the matrix $X=\begin{pmatrix}\Omega\\ 1_2\end{pmatrix}$, $\Omega\in\mathbb{H}_2$. As Ω is symmetric, $a_{24}+a_{13}=0$ and the Pluecker quadratic form reduces to $a_{12}a_{34}+a_{13}^2+a_{14}a_{23}$ of signature $(3,2)$.

Integer Hyper-Lorentzial Quadratic Forms.

Let q be an integer quadratic form on free abelian group L of rank $n+2$ such that its signature as real quadratic form on $L_{\mathbb{R}} = L \otimes \mathbb{R}$ equals $(n, 2)$. Put $\Omega(q)$ be a part of complex isotropic cone $\{v \in L_{\mathbb{C}} = L \otimes \mathbb{C} | q(v) = 0\}$ define by positivity of the Hermitian continuation of q to $L_{\mathbb{C}}$: $b_q(v, \overline{v}) > 0$ where b_q denote the polarization of q: $b_q(v, v) = q(v)$. $\Omega(q)$ is disjoint union of two components $\Omega(q) = \Omega(q)^+ \cup \Omega(q)^-$

Evidently the group $\mathrm{SO}(q)$ of automorphisms of the integer form q acts on $\Omega(q)$, denote by $\mathrm{SO}(q)^+$ the stabilizer of the component $\Omega(q)^+$. So $\mathrm{SO}(q)^+$ acts on the cone $\Omega(q)^+$. We can consider homogeneous functions on this cone $\Omega(q)^+$, invariant with respect to action of the group $\mathrm{SO}(q)^+$.