Elementary Introduction to the Theory of Automorphic forms
Lecture 6

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March 12, 2021

The Structure Theorem for Modular Forms in One Variable.

Remind

$\Im\tau > 0$. q denotes $\exp(2\pi i\tau)$, so $0 < |q| < 1$.

Definition

A function $f(\tau)$ on the upper half-plane is called a modular form of the weight k if it is

a) automorphic of the weight k:

$f((a\tau + b)/(c\tau + d)) = (c\tau + d))^k f(\tau)$ for $ad - bc = 1$,

$a, b, c, d \in \mathbb{Z}$, The expression $(c\tau + d)$ is known as automorphic factor;

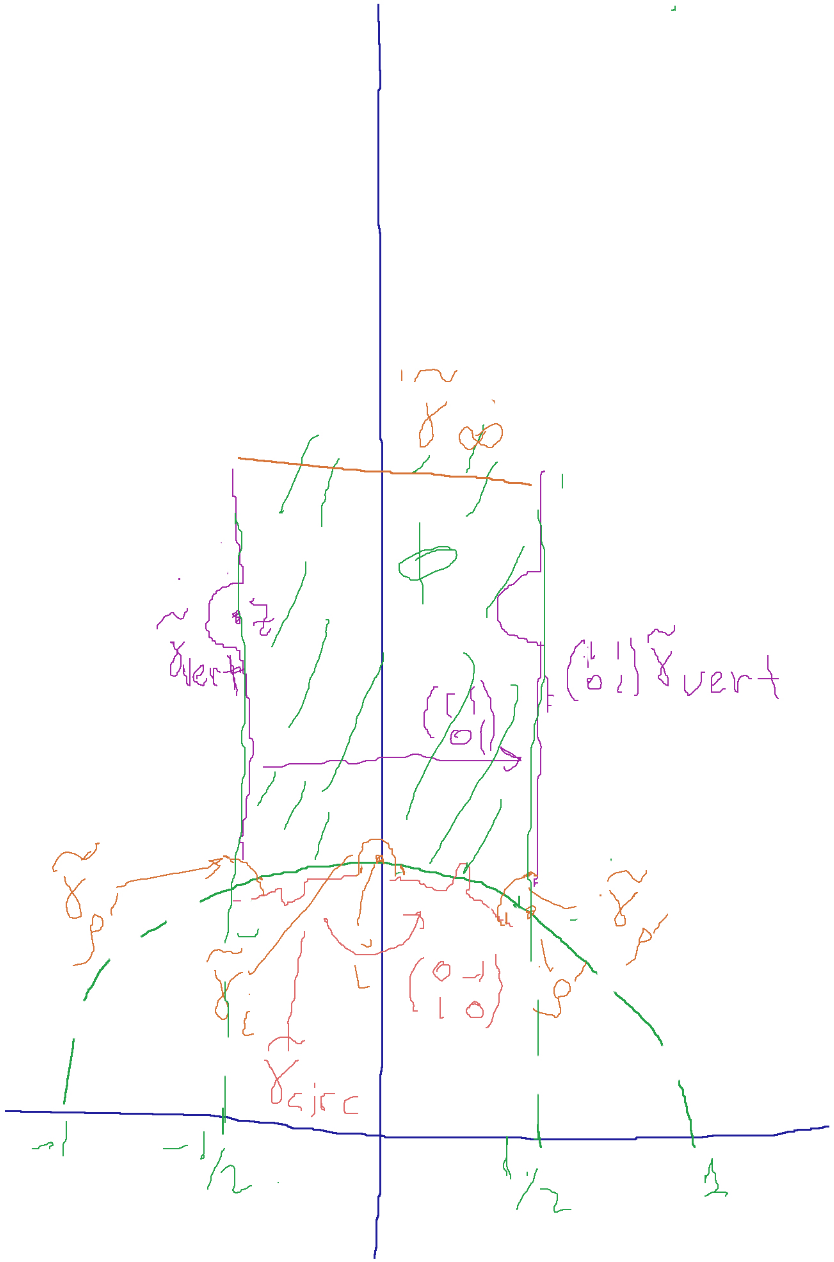
b) It is holomorphic function at the unit disc as function in q:

Definition

The modular figure is the subset $\overline{\Phi}$ of the upper half-plane defined by inequalities $|\tau|^2 \geq 1, -\frac{1}{2} \leq \Re\tau \leq \frac{1}{2}$.

Lemma

Any $\mathrm{SL}_2(\mathbb{Z})$-orbit in $\mathbb{H}$ intersect $\overline{\Phi}$.


γvert
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-1
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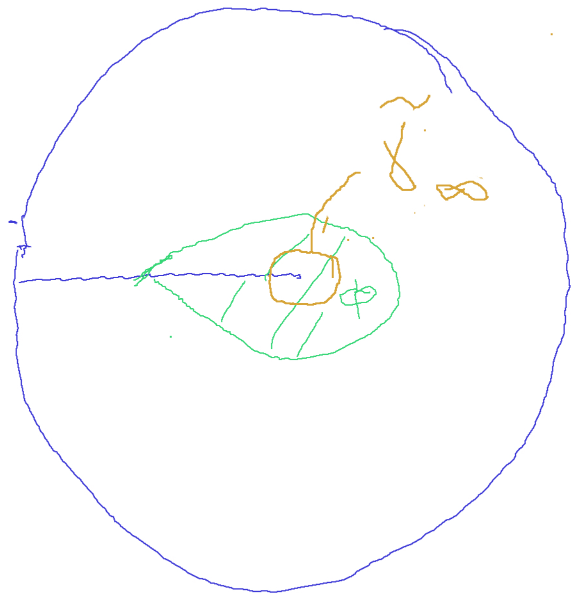

Proof. The number τ responds to the system of generators of a lattice $L \subset \mathbb{C}$ by the rule $L = \mathbb{Z}\omega_1 + \mathbb{Z}\omega_2$ $\tau = \omega_1/\omega_2$. The orbit corresponds to the lattice itself.Pick the system of generators as ω_2 be the shortest vector of $L$and ω_1 be the second in length non-collinear to Ω_2 in contrclockwise direction. Evidently $|\tau \geq 1$ and if $\pm\Re\tau > \frac{1}{2}$, $\omega_1 \mp \omega 2$ is shorter that ω_1.
Let Φ^0 denotes the interior of the $\overline{\Phi}$, $\gamma_{\rm vert} = \{\rho + it, t > 0\}$, $\gamma_{\rm circ} = \{\exp(2\pi i s), \frac{1}{4} < s < \frac{1}{3}\}$and $\Phi = \Phi^0 \cup \gamma_{\rm vert} \cup \gamma_{\rm cyrc}$
Any modular form has finitely many zeros in Φ. Indeed, the exponential function $\exp 2\pi i$ embed Φn in the unit disc.
Put $\rho = -\frac{1}{2} + \frac{\sqrt{3}}{2}i$. For any holomorphic function f on $\mathbb{H}$ denote by $\nu_\tau(f)$ the order of zero of f at τ. For $\mathbb{Z}$ periodic function denote by ν_∞ the order of the zero at $q = 0$.

Theorem

Let f be a nonzero modular form of the weight k Then

$$\frac{k}{12} = \sum_{\tau \in \Phi} \nu_\tau(f) + \nu_\infty(f) + \frac{1}{2}\nu_i(f) + \frac{1}{3}\nu_\rho(f)$$

A proof is based on the Cauchy's argument principle, we take a contour $\tilde{\gamma}^{\delta}$ which encircle zeroes of f in Φ, then the integral of $d \log f$ over $\tilde{\gamma}^{\delta}$ equals to the sum of orderes of zeros up to factor $2\pi i$. We choose $^{a} sdeformation of the boundary of \overline{\Phi}$ which preserve the following property of this boundary. It is the union of paths γ's and there images under action of elements of $\mathrm{SL}_2(\mathbb{Z})$ with opposite orientation. If the zero of f belongs to γ we replace small neighborhood of this zero in γ by arc outside Φ^0.

Formally:

Put Z be the union of set of all zeros of f in Φ with points i and ρ; choose $r < \frac{1}{3}|z_{\alpha} - z_{\beta}|; z_{\alpha}, z_{\beta} \in Z$ and $M > r + \Im z, z \in Z$. Define two families of paths $\tilde{\gamma}^{\delta}_{\mathrm{vert}}$ and $\tilde{\gamma}^{\delta}_{\mathrm{circ}}(\delta < r)$.Put $\rho' = \frac{1}{2} + \frac{\sqrt{3}}{2} i$ denote $D_{\rho,\delta} = \{\tau || \tau - \rho| < \delta\}$, $S_{\rho,\delta}\{\tau || \tau - \rho| = \delta\}$ and the same for points ρ' and i Consider the set $Z_{\mathrm{vert/circ}}$ of zeroes of f on $\gamma_{\mathrm{vert/circ}}$.Denote by $D_{\mathrm{vert/circ}}$ the union of discs $\bigcup_{z \in Z_{\mathrm{vert/circ}}} \{\tau || \tau - z| < r\}$ and by $S_{\mathrm{vert/circ}}$ the union of circles $\bigcup_{z \in Z_{\mathrm{vert/circ}}} \{\tau || \tau - z| = r\}$.

Put

$$\tilde{\gamma}^{\delta}_{\mathrm{vert}} = (\gamma_{\mathrm{vert}} \setminus (D_{\mathrm{vert}} \cup D_{\rho,\delta} \cup \{\tau | \Im\tau > M\})) \cup \left(S_{\mathrm{vert}} \setminus \Phi^0\right),$$

$$\tilde{\gamma}^{\delta}_{\mathrm{circ}} = (\gamma_{\mathrm{circ}} \setminus (D_{\mathrm{circ}} \cup D_{\rho,\delta} \cup D_{i,\delta})) \cup \left(S_{\mathrm{circ}} \setminus \Phi^0\right),$$

$$\tilde{\gamma}_{\infty} = \{iM + \frac{1}{2} - s | 0 \le s \le 1\}, \tilde{\gamma}^{\delta}_{*} = S_{*,\delta} \cap \overline{\Phi}, * = \rho, \rho', i.$$

The chain of paths

$$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \tilde{\gamma}^{\delta}_{\mathrm{vert}}, \tilde{\gamma}_{\infty}, \left(\tilde{\gamma}^{\delta}_{\mathrm{vert}}\right)^{-1}, \tilde{\gamma}^{\delta}_{\rho}, \left(\tilde{\gamma}^{\delta}_{\mathrm{circ}}\right)^{-1}, \tilde{\gamma}^{\delta}_{i}, \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \tilde{\gamma}^{\delta}_{\mathrm{circ}}, \tilde{\gamma}^{\delta}_{\rho'}$$

forms a loop $\tilde{\gamma}^{\delta}$ encircling required zeroes f.

As f is a modular form of the weight k,
$f\left(\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\tau\right) = (0\tau+1)^k f(\tau)$ and contributions of
$\begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}\tilde{\gamma}^{\delta}_{\mathrm{vert}}$ and $\left(\tilde{\gamma}^{\delta}_{\mathrm{vert}}\right)^{-1}$ cancel;
$f\left(\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\tau\right) = (1\tau+0)^k f(\tau)$, hence the restriction of $d\log f$ to $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\tilde{\gamma}^{\delta}_{\mathrm{circ}}$ equals to the restriction $d\log f + kd\log\tau$ to $\tilde{\gamma}^{\delta}_{\mathrm{circ}}$ and contribution of the integrals over these to part is opposite to integral of $d\log f$ over $\tilde{\gamma}^{\delta}_{\mathrm{circ}}$. The limit of this integral at $\delta \to 0$ equals to integral of $d\log\tau$ over γ_{circ}. The last integral equal to $-\frac{2\pi i}{12}$ as γ_{circ} is one twelfth of the circle.
The lengths of arcs $\tilde{\gamma}^{\delta}_{\rho}$ and $\tilde{\gamma}^{\delta}_{\rho'}$ behave as $\frac{2\pi\delta}{6} + o(\delta)$ and length of $\tilde{\gamma}^{\delta}_{i}$ is equal to $\pi\delta + o(\delta)$. This produces required prefactors of orders in this points.
For evaluating integral over $\tilde{\gamma}_{\infty}$ apply the exponential map. this path transforms to the cirle around $q=0$, hence the integral equals to $-2\pi i\nu_{\infty}(f)$.

Theorem

a) a modular form of zeroth weight is a constant.
b) there are not modular forms of the weight 2.
c) a modular form of the weight k, $k = 4, 6, 8, 10$ *is proportional to the Eisenstein series* $e_k(\tau)$ *(variant* e_4, e_6, e_4^2 $e_4 e_6$*).*
d) $e_4(i) \neq 0$, $e_6(\rho) \neq 0$. *So, any polynom in* e_4 *and* e_6, *f.e. the discriminant* $\Delta = \frac{1}{1728}\left((60e_4)^3 - (140e_6)^2\right)$, *is not zeroth function.*
e) The discriminant Δ *has no zeros at* $\mathbb{H}$ *and the order of its zero at* ∞ *equals* 1.

Proof a, c is based on subtraction from a mudular form the expectable expression in such way that the difference vanishes at infinity. According to the formula for orders of zeroes this difference should vanishes as $k < 12$. Other statements follows from this formula immediately.

Lemma

Any natural number $k \geq 14$ *equals to sum of* $4th$ *and* $6th$ *:*
$k = 4\lambda + 6\mu$, $\lambda, \mu \in \mathbb{Z}$ $\lambda, \mu \geq 0$

Theorem

Any modular form can be expressed in unique way as polynom in e_4 and e_6 with complex coefficients.

Proof. For small weights this is proven in c) of the previous Theorem. Let the weight k of the form is not less that 12 for $k = 4\lambda + 6\mu$ for some constant C $\tilde{f} = f - Ce_4^{\lambda} e_6^{\mu}$ vanishes at infinity and $\tilde{f}/\Delta$ is modular form of the weight of $k - 12$. Induction in the weight.

Overview of the Elliptic Functions.

Definition

An elliptic function is a function which is
1.holomorphic on the complement to the finite union sets of the shape $\{\eta + +m\tau + n | m, n \in \mathbb{Z}\}$
2.is double periodic with periods 1*,* τ
3. all its singularities a poles of finite order.

Theorem

Entire elliptic function is a constant.

Definitions-Examples:

$$\wp(\xi;\tau) = \frac{1}{\xi^2} + \sum_{(m,n)\neq(0,0)} \left(\frac{1}{(\xi + m\tau + n)^2} - \frac{1}{(m\tau + n)^2} \right)$$

$$\wp'(\xi;\tau) = \sum_{(m,n)} \frac{-2}{(\xi + m\tau + n)^3}$$

These series absolutely uniformly converge, hence they are elliptic functions and $\wp'(\xi;\tau)$ is the derivative of $\wp(\xi;\tau)$.
Near $\xi = 0$

$$\wp(\xi;\tau) = \frac{1}{\xi^2} + \sum_{k=1}^{\infty}(2n+1)e_{2k+2}(\tau)\xi^{2k}$$

The Weirstraß equation

$$\wp'^2 = 4\wp^3 - 60e_4\wp - 140e_6$$

Proof. The difference of the lhs and the rhs is elliptic and its value at $\xi = 0$ is zero.
The discriminant of the qubic equation in the rhs up to factor 1278 coincides with Δ $\wp'$ is odd, so it vanishes at semi-periods $\xi = \frac{\tau+1}{2}, \frac{\tau}{2}, \frac{1}{2}$. As this function has "unique" (up to shifts) pole of the third order, there are no more that three zeroes up to shifts.
Hence semi-periods are all zeroes of $\wp$" and values of $\wp$ at semi-periods are roots of the equation $4x3 - 60e_4x - 140e_6$. They are mutually different. So
$4x3 - 60e_4x - 140e_6 = 4\left(x - \wp(\frac{\tau+1}{2})\right)\left(x - \wp(\frac{\tau}{2})\right)\left(x - \wp(\frac{1}{2})\right)$