$$\begin{cases} u_{tt} - a^2 u_{xx} = f(t,x) \\ u(t,0) = u(t,l) = 0 \\ u(0,x) = u_0(x) \\ u_t(0,x) = v_0(x) \end{cases}$$

$$u(t,x) = \sum_{k=1}^{\infty} f_k(t) \sin\frac{k\pi x}{l}$$

надо найти.

$$f(t,x) = \sum_{k=1}^{\infty} f_k(t) \sin\frac{k\pi x}{l}, \quad f_k(t) = \frac{2}{l}\int_0^l f(t,s)\sin\frac{k\pi s}{l}\,ds$$

известно

$$\sum_{k=1}^{\infty}\left(p_k''(t)+\frac{a^2k^2\pi^2}{l^2}p_k(t)\right)\sin\frac{k\pi x}{l}=\sum_{k=1}^{\infty}f_k(t)\sin\frac{k\pi x}{l}$$

(1) $$p_k''(t)+\frac{a^2k^2\pi^2}{l^2}p_k(t)=f_k(t) \qquad k=1,2,3,\dots$$

$$u_0(x)=u(0,x)=\sum_{k=1}^{\infty}p_k(0)\sin\frac{k\pi x}{l}\Rightarrow$$

(2) $$\Rightarrow\quad p_k(0)=\frac{2}{l}\int_0^l u_0(s)\sin\frac{k\pi s}{l}\,ds$$

$$V_0(x) = u_t(0,x) = \sum_{k=1}^{\infty} p_k'(0) \sin\frac{k\pi x}{l}$$

(3) $$p_k'(0) = \frac{2}{l}\int_0^l V_0(s)\sin\frac{k\pi s}{l}\,ds$$

<u>Итого</u>: надо $\forall k = 1, 2, 3, \ldots$

ур-е (1) с нач. усл. (2) + (3)

(это задача Коши для ОДУ 2 порядка, квадр.)

Как ее решать?

Интересное частн. случай (упражнение).

Упр. 1 $\left\{ \begin{array}{l} f(t,x) = \cos\omega t \sin\frac{2\pi x}{l}. \\ \\ V_0 = 0 \qquad u_0(x) = 3\sin\frac{4\pi x}{l} - \sin\frac{5\pi x}{l}. \end{array} \right.$

$$\left\{ \begin{array}{l} p_k'' + \frac{a^2k^2\pi^2}{l^2} = \begin{cases} 0, & k \neq 2. \\ \cos\omega t, & k = 2. \end{cases} \\ \\ p_k(0) = \begin{cases} 3, & k = 4. \\ -1, & k = 5 \\ 0, & \text{иначе}. \end{cases}, \quad p_k'(0) = 0. \end{array} \right.$$

(A_2) $\begin{cases} p_2'' + \dfrac{4a^2\pi^2}{\ell^2}\, p_2 = \cos \text{[illegible]}\, t. \\ p_2(0) = 0 \\ p_2'(0) = 0. \end{cases}$

$k = 2$

(A_4) $\begin{cases} p_4^{4} + \dfrac{16a^2\pi^2}{\ell^2}\, p_4 = 0 \\ p_2(0) = 3 \\ p_2'(0) = 0 \end{cases}$

$k = 4$

(A_5) $\begin{cases} p_5'' + \dfrac{2k_5 a^2 \bar{u}^2}{e^2} p_5 = 0 \\ p_5(0) = -1 \\ p_5'(0) = 0 \end{cases}$

(A_k) $\begin{cases} p_k'' + \dfrac{a^2 k^2 \bar{u}^2}{e^2} p_k = 0 \\ p_k(0) = 0 \\ p_k'(0) = 0 . \end{cases}$

$k \neq 2, 4, 5$

Решение (A_k) : $p_k(t) = ~~\alpha_k \cos \frac{ak\bar{u}}{e} t + \beta_k \sin \frac{ak\bar{u}}{e} t~~$.

$0 = p_k(0) = \alpha_k$, $0 = p_k'(0) = \beta_k \frac{ak\bar{u}}{e} \Rightarrow$

$\Rightarrow \beta_k = 0$

Итог: "k" у задачи (A_k), $k \neq 2, 4, 5$

решение $p_k \equiv 0$

$$(A_5): \begin{cases} p_5'' + \dfrac{25a^2\pi^2}{l^2} p_5 = 0, \\ p_5(0) = -1, \quad p_5'(0) = 0. \end{cases}$$

$$p_5(t) = \underset{-1}{\cancel{\alpha_5}} \cos\frac{5a\pi t}{l} + \underset{0}{\cancel{\beta_5}} \sin\frac{5a\pi t}{l}$$

$$-1 = p_5(0) = \alpha_5, \quad 0 = p_5'(0) = -1 \cdot \cancel{\frac{5a\pi}{l} \sin \frac{5a\pi}{l} t}\Big|_{t=0} + \frac{5a\pi}{l} \beta_5 \cdot 1$$

Итог "5": Реш. (A_5): $p_5(t) = -\cos\frac{5a\pi t}{l}$

Реш. (A_4).

$$\begin{cases} p_4'' + \frac{16a^2\pi^2}{l^2} p_4 = 0 \\ p_4(0) = 3 \\ p_4'(0) = 0. \end{cases}$$

$\Rightarrow p_4(t) = \alpha_4 \cos\frac{4\pi a t}{l} + \beta_4 \sin\frac{4a\pi t}{l}$ (crossed out; α_4 replaced by 3)

$3 = p_4(0) = \alpha_4$

$0 = p_4'(0) = \frac{4a\pi}{l}\beta_4 \cdot 1 \Rightarrow$

$\Rightarrow \beta_4 = 0$

Итог "4" | Реш. $(A_1) \doteq P_1(t) = \xi \cos \frac{4a\pi t}{l}$.

Реш. (A_2)

$$\begin{cases} P_2'' + \frac{4a^2\pi^2}{l^2} P_2 = \cos \omega t \\ P_2(0) = P_2'(0) = 0. \end{cases}$$

$$P_2(t) = \alpha_2 \sin \frac{2a\pi}{l} t + \beta_2 \cos \frac{2a\pi}{l} t + A \cos \omega t$$

$$A\left(-\omega^2 + \frac{4a^2\pi^2}{l^2}\right) \cos \omega t = \cos \omega t$$

$$\omega \neq \frac{2a\pi}{l}$$

$$A = \frac{1}{\frac{4a^2\pi^2}{l^2} - \omega^2}$$

$$\beta_2(t) = \alpha_2 \sin\frac{2\pi a t}{\ell} + \beta_2 \cos\frac{2\pi a t}{\ell} + \frac{1}{\frac{4a^2\pi^2}{\ell^2} - \omega^2}\cos\omega t$$

$$0 = \beta_2(0) = \beta_2 + \frac{1}{\frac{4a^2\pi^2}{\ell^2} - \omega^2} \Rightarrow \beta_2 = -\frac{1}{\frac{4a^2\pi^2}{\ell^2} - \omega^2}$$

$$0 = \beta_2'(0) = \frac{2\pi a}{\ell}\alpha_2 \cdot 1 \Rightarrow \alpha_2 = 0$$

Uron "2" | $\beta_2(t) = -\frac{1}{\frac{4a^2\pi^2}{\ell^2} - \omega^2}\cos\frac{2\pi a t}{\ell} + \frac{\cos\omega t}{\frac{4a^2\pi^2}{\ell^2} - \omega^2}$

$$u(t,x) = \sum_{k=1}^{\infty} p_k(t) \sin\frac{k\pi x}{l} =$$

$$= p_2(t)\sin\frac{2\pi x}{l} + p_4(t)\sin\frac{4\pi x}{l} + p_5(t)\sin\frac{5\pi x}{l} =$$

$$= \left(-\frac{1}{\frac{4a^2\pi^2}{l^2} - \omega^2}\cos\frac{2a\pi t}{l} + \frac{1}{\frac{4a^2\pi^2}{l^2} - \omega^2}\cos\omega t \right)\sin\frac{2\pi x}{l} +$$

$$+ 3\cos\frac{4a\pi t}{l}\sin\frac{4\pi x}{l} - \cos\frac{5a\pi t}{l}\sin\frac{5\pi x}{l}$$

Если $\omega \approx \frac{2\pi a}{l}$ $\quad \left|\frac{1}{\frac{4\pi^2 a^2}{l^2} - \omega^2}\right| \gg 1.$

Слагаемые при этом очень крупные (с очень большой амплитудой)

колеб. с частотами $\frac{2\pi a}{l}$ $\quad \omega \approx \frac{2\pi a}{l}.$

$\frac{k\pi a}{l}$ — частота собств. колеб. струны ("собств. частоты")

Резонанс (энергетич. резонанс).

При $\omega \to$ С. ч. $\Rightarrow$ Амплитуда колебаний $\to \infty$.

Напоминание $\Omega \subset \mathbb{R}^n$

$u_k \in \mathcal{D}'(\Omega)$ $\quad u_k \xrightarrow[k\to\infty]{} u \in \mathcal{D}'(\Omega)$

(слабо в смысле обобщ. ф-ций)

если $\langle \varphi, u_k \rangle \xrightarrow[k\to\infty]{} \langle \varphi, u \rangle$, $\varphi \in \mathcal{D}(\Omega)$

Тогда, в частности, оказывается, что $D^\alpha u_k \longrightarrow D^\alpha u$

Случай $\omega = \frac{2a\pi}{l}$.

$$p_2(t) = \alpha_1 \sin\frac{2a\pi t}{l} + \alpha_2 \cos\frac{2a\pi t}{l} + At\cos\omega t.$$

↑ подобрал.

<u>Упр. 2.</u> $f(t,x) = \cos\omega t$, $V_0 = 0$, $u_0 = 0$.

$$f_k(t) = \frac{2}{l}\int_0^l \cos\omega t \sin\frac{k\pi s}{l}\,ds = \frac{2}{l}\cos\omega t \int_0^l \sin\frac{k\pi s}{l}\,ds =$$

$$= \frac{2}{l} \cos\omega t \left(-\cos\frac{k\pi s}{l} \cdot \frac{l}{k\pi} \right)\Bigg|_0^l =$$

$$= -\frac{2}{k\pi}\left(\cos k\pi - 1\right)\cos\omega t = -\frac{2}{k\pi}\left((-1)^k - 1\right)\cos\omega t =$$

$$= \frac{2}{k\pi}\left(1 + (-1)^{k+1}\right)\cos\omega t = \cos\omega t \begin{cases} \frac{4}{k\pi}, & k = 2m-1 \\ 0 & k = 2m \end{cases}$$

$m \in 1, 2, \ldots$

$$\begin{cases} p_{2m}'' + \dfrac{4m^2 a^2 \pi^2}{\ell^2} p_{2m} = 0 & m = 1, 2, 3, \dots \\ p_{2m-1}'' + \dfrac{(2m-1)^2 a^2 \pi^2}{\ell^2} p_{2m-1} = \dfrac{4 \cos \omega t}{(2m-1)\pi}, & m = 1, 2, 3, \dots \\ p_k(0) = p_k'(0) = 0. \end{cases}$$

$\underline{k = 2m}$ |.

$$\begin{cases} p_{2m}'' + \dfrac{4m^2 a^2 \pi^2}{\ell^2} p_{2m} = 0 \\ p_{2m}(0) = p_{2m}'(0) = 0 \end{cases} \Rightarrow p_{2m}(t) = 0,$$

$\underline{k = 2m - 1}$ |

$$\begin{cases} P''_{2m-1} + \dfrac{(2m-1)^2 a^2 \pi^2}{l^2} P_{2m-1} = \dfrac{4}{(2m-1)\pi} \cos \omega t \\ P_{2m-1}(0) = P'_{2m-1}(0) = 0. \end{cases}$$

Нерезонансный случай

$$\omega \neq \frac{(2m-1) a\pi}{l}$$

$$m = 1, 2, 3, \ldots$$

$$P_{2m-1}(t) = \alpha_{2m-1} \sin \frac{(2m-1) a\pi t}{l} + \beta_{2m-1} \cos \frac{(2m-1) a\pi t}{l} + A_{2m-1} \cos \omega t.$$

$$= -A_{2m-1}$$

A_{2m-1} — посчитать.

$$A_{2m-1}\left(-\omega^2+\frac{(2m-1)^2a^2\pi^2}{l^2}\right)\cos\omega t=\frac{4}{(2m-1)\pi}\cos\omega t$$

$$A_{2m-1}=\frac{4}{\pi}\,\frac{1}{(2m-1)\left(\frac{(2m-1)^2a^2\pi^2}{l^2}-\omega^2\right)}$$

Подберем α_k, β_k из нач. усл.

$$0=p_{2m-1}(0)=\beta_{2m-1}+A_{2m-1}\Rightarrow\beta_{2m-1}=-A_{2m-1}$$

$$0=p_{2m-1}'(0)=\alpha_{2m-1}\frac{(2m-1)a\pi}{l}\cdot 1\Rightarrow\alpha_{2m-1}=0.$$

Итак:

$$p_{2m-1}(t) = \frac{4}{\pi}\,\frac{1}{(2m-1)\left(\frac{(2m-1)^2 a^2 \pi^2}{l^2} - \omega^2\right)}\left(\cos\omega t - \cos\frac{(2m-1)\pi a t}{l}\right)$$

$$p_{2m}(t) = 0.$$

$$u(t,x) = \frac{4}{\pi}\sum_{m=1}^{\infty}\frac{\sin (2m-1)\pi x/l}{(2m-1)\left(\frac{(2m-1)^2 a^2 \pi^2}{l^2} - \omega^2\right)}\left(\cos\omega t - \cos\frac{(2m-1)a\pi t}{l}\right)$$