Elementary Introduction to the Theory of Automorphic forms
Lecture7

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θ-Sums.

We shall illustrate another very important source of the modular forms by example. Put

$$\theta'_{11}(\tau) = \sum_{k\in\mathbb{Z}} i^{(2k+1)}\pi i(2k+1)\exp 2\pi i\left(\frac{(2k+1)^2}{8}\tau\right)$$

Then $\theta'_{11}(\frac{a\tau+b}{c\tau+d}) = \zeta_8(c\tau+d)^{\frac{3}{2}}\theta'_{11}(\tau)$, where ζ_8 is some root of unity of order 8, $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$ is choosen such that $c > 0$, if $c = 0$, then $d > 0$, hence $\Im(c\tau + d) > 0$ and one have the standard brunch of square root.

We deduce this result from modular property of more general object.

Define *the theta function* $\theta_{11}(\xi;\tau)$ as the following convergent series

$$\theta_{11}(\xi;\tau) = \sum_{k\in\mathbb{Z}} \exp\left(2\pi i\left(\frac{(2k+1)^2}{8}\tau + \frac{(2k+1)}{2}(\xi+\frac{1}{2})\right)\right).$$

Evidently $\theta'_{11}(\tau) = \partial\theta_{11}(\xi;\tau)/\partial\xi|_{\xi=0}$
Straightforward properties of θ: It is odd as function in ξ,

$$\theta(\xi + m\tau + n; \tau) = (-1)^{m+n} \exp\left(-2\pi i \left(\frac{m^2}{2}\tau + m\xi\right)\right) \theta_{11}(\xi;\tau),$$

$$2\pi i \frac{\partial}{\partial\tau}\theta_{11}(\xi;\tau) = \frac{1}{2}\frac{\partial^2}{\partial^2\xi}\theta_{11}(\xi;\tau)$$

Less evident properties of θ: It does not vanishes identically. Indeed, restrict it to the real axe $\Im(\xi) = 0$.Then the integral

$$\int_0^1 \theta_{11}(\xi;\tau) \exp\left(2\pi i\left(-\frac{(2k+1)}{2}(\xi+\frac{1}{2})\right)\right) d\xi =$$

$$\exp\left(2\pi i\left(\frac{(2k+1)^2}{8}\tau\right)\right) \neq 0$$

The order zero of θ at $\xi = 0$ equals 1 and all zeros of θ are $\xi = m\tau + n$

Proof. As odd function θ vanishes at $\xi = 0$. Let there is a zero ξ_0 different from $m\tau + n$ then $\xi_0 - m\tau - n$ is zero also, there is a zero in the paralellogram $\{r\tau + s| - \frac{1}{2} \leq r, s \leq \frac{1}{2}\}$. By the argument principle the number of zeros can be calculated by integrating over the boundary, from qusi-periodicity this integral equals $2\pi i$.

Modular Properties of the θ-Function.

$$\theta_{11}(\frac{\xi}{c\tau+d};\frac{a\tau+b}{c\tau+d}) = \zeta_8\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(c\tau{+}d)^{\frac{1}{2}} exp\left(\frac{\pi i c\xi^2}{(c\tau+d)}\right)\theta_{11}(\xi;$$

Proof. Straightforward calculation calculation shows that

$$\exp\left(-\frac{\pi i c\xi^2}{(c\tau+d)}\right)\theta_{11}(\frac{\xi}{c\tau+d};\frac{a\tau+b}{c\tau+d})/\theta_{11}(\xi;\tau)$$

is double-periodic and holomorphic in ξ, hence is function in τ only. Now apply to

$$\theta_{11}(\frac{\xi}{c\tau+d};\frac{a\tau+b}{c\tau+d}) = F(\tau) exp\left(\frac{\pi i c\xi^2}{(c\tau+d)}\right)\theta_{11}(\xi;\tau)$$

the operator $2\pi\frac{\partial}{\partial\tau}\frac{1}{2}\frac{\partial^2}{\partial^2\xi}$. We get $\frac{\partial F}{\partial\tau} = \frac{1}{2}\frac{c}{c\tau+d}F$. So $F(\tau) = (c\tau+d)^{\frac{1}{2}} C\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ for some constant $C\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ We shall prove that the constant $C\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)$ is a 8th root of unity

Note that up to ± 1, which came from the square root of the automorphic factor, $C(M)$ form a cocycle: $C(M_1 M_2) = \pm C(M_1)C(M_2)$. So, it is sufficient to evaluate C at generators of the group. Therefore, a proof splits in two parts. First we determine the generators of the group $\mathrm{SL}_2(\mathbb{Z})$. Second we check the statement for generators.

Lemma

The group $\mathrm{SL}_2(\mathbb{Z})$ *is generated by matrices* $T = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ *and* $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$

Sketch of a proof. The multiplication by T and S add to the top element of the column the lower one or interchange elements. Hence one can perform the Euclidean algorithm. The elements of first column of unimodular matrix are coprime, hence this Euclidean algorithm reduce it to upper-triangular.

Now calculate for generators. For T from the congruence $(2k+1)^2 = 4k^2 + 4k + 1 = 4k(k+1) + 1 \equiv 1 \mathrm{mod} 8$ we get $C(T) = \exp\left(\frac{2\pi i}{8}\right)$
For S we apply the following consideration. Substitute $\tau = i$ i is a fixed point of S, so in the formula we have θ-functions of the same argument $\tau = i$

$$\theta_{11}(\frac{\xi}{i}; i) = C(S)(1i+0)^{\frac{1}{2}} exp\left(\frac{\pi i \xi^2}{i}\right)\theta_{11}(\xi;\tau)$$

and comparison of the limits as $\xi \to 0$ get $C(S)\sqrt{i} = i^{-1}$
$C(T) = \exp\left(\frac{-6\pi i}{8}\right)$, .

More θs

I present more theta functions, which I will use later.

$$\theta_{00}(\xi;\tau) = \sum_{k\in\mathbb{Z}} \exp\left(2\pi i\left(\frac{k^2}{2}\tau + k\xi\right)\right).$$

Easily $\theta_{00}(\xi + \frac{\tau}{2} + \frac{1}{2};\tau) = \exp\left(-2\pi i\left(\frac{1}{8}\tau + \frac{1}{2}(\xi + \frac{1}{2})\right)\right)\theta_{11}(\xi;\tau)$.
Note that the quasiperiodicity of θ can be reinterpret in the following way: For integer m, n put

$$T_{m,n}(f(\xi;\tau)) = \exp\left(2\pi i\left(\frac{m^2}{2}\tau + m(\xi+n)\right)\right) f(\xi + m\tau + n;\tau)$$

. Then $T_{m,n}(\theta_{11}(\xi;\tau)) = (-1)^{(m+n}\theta_{11}(\xi;\tau)$ and $T_{m,n}(\theta_{00}(\xi;\tau)) = \theta_{00}(\xi;\tau)$.
One can interpolate the operator T_{**} to the rational indices, then $\theta_{11}(\xi;\tau) = T_{\frac{1}{2},\frac{1}{2}}\theta_{00}(\xi;\tau)$. For any pair α,β, α,β are either 0 or 1 put $\theta_{\alpha\beta}(\xi;\tau) = T_{\frac{\alpha}{2},\frac{\beta}{2}}\theta_{00}(\xi;\tau)$. For more details see Mumford.